Stat 218 Introduction to Stochastic Processes

Homework 6

Posted: 5/17/2007 *Due: 5/24/2007*

Let $\{B(t) : t \geq 0\}$ be a standard Brownian motion. Recall that the Ornstein-Uhlenbeck process $\{X(t) : t \geq 0\}$ is defined as

$$X(t) \equiv e^{-\alpha t/2} B(\alpha e^{\alpha t}) \,, \tag{1}$$

for some $\alpha > 0$.

1. Compute $\mathbb{P}\{X(t+h) \in [x, x + \mathrm{d}x) | X(t) = x_0\}$.
2. Let $p_t(x|x_0)$ be the density of $X(t)$ given $X(0) = x_0$. Use conditioning to show that it satisfy the partial differential equation
   [Previous version contained 2 misprints: thanks to Amrita for pointing this out]
   $$\frac{\partial p_t}{\partial t}(x|y) = \frac{1}{2}\alpha p_t(x|y) + \frac{1}{2}\alpha x \frac{\partial p_t}{\partial x}(x|y) + \frac{1}{2}\alpha^2 \frac{\partial^2 p_t}{\partial x^2}(x|y) \tag{2}$$
3. Consider now the generalized process $X(t) \equiv e^{-\beta t} B(\alpha e^{\alpha t})$. For which values of α and β is this process stationary?
4. Write a PDE for the density of the new process.