STATS 218 Homework 6 Solutions.

Question 1.

Since the Brownian process $B(t)$ is a Gaussian process and $X(t)$ is a scaled time-transformed version of $B(t)$, $X(t)$ is clearly also a Gaussian process. $E[X(t)] = 0$ and, for $s \le t$:

$$\mathrm{Cov}(X(s), X(t)) = e^{-\alpha\frac{s+t}{2}} \cdot \alpha e^{\alpha s} = \alpha e^{-\alpha\frac{t-s}{2}}. \tag{1}$$

Note that this is a stationary process, as the covariance only depends on $t - s$. So $(X(t), X(t+h))$ has a bivariate Gaussian distribution with means 0, variances α and covariance $\alpha e^{-\alpha h/2}$. Thus the conditional distribution follows:

$$X(t+h)|X(t) = x_0 \sim N\Big(x_0 e^{-\alpha h/2}, \alpha\big(1 - e^{-\alpha h}\big)\Big). \tag{2}$$

Therefore we get the density

$$P(X(t+h) \in [x, x+dx)|X(t) = x_0) = \frac{1}{\sqrt{2\pi\alpha(1-e^{-\alpha h})}} \exp\left(-\frac{(x - x_0 e^{-\alpha h/2})^2}{2\alpha(1-e^{-\alpha h})}\right) dx. \tag{3}$$

Note that this density is independent of t.

Question 2.

We mimic the derivation of Ross (8.5.2), which was done for Brownian motion. We condition on $X(t-h)$. By the stationary property of the process, we can write

$$p_t(x|y) = \int p_h(x|a) p_{t-h}(a|y)\ da \tag{4}$$

and perform a power series expansion of $p_{t-h}(a|y)$ about $p_t(x|y)$:

$$\begin{aligned} p_t(x|y) &= \int \Big[p_t(x|y) + (a-x)\frac{\delta}{\delta x}p_t(x|y) - h\frac{\delta}{\delta t}p_t(x|y) \\ &\quad + \frac{(a-x)^2}{2}\frac{\delta^2}{\delta x^2}p_t(x|y) + o(h)\Big] p_h(x|a)\ da. \end{aligned} \tag{5, 6}$$

The higher degree terms in h are clearly $o(h)$, and the higher degree terms in x are constant multiples of normal (third and higher) moments about a term that has $O(h)$ rate convergence to the normal mean, and so these terms are $o(h)$ as well.

Using (3), it is easy to show that $\int p_h(x|a)\ da = e^{\alpha h/2}$. We also get

$$\int (a-x) p_h(x|a)\ da = e^{\alpha h/2} x (e^{\alpha h/2} - 1), \tag{7}$$

$$\int (a-x)^2 p_h(x|a)\ da = \alpha e^{\alpha h/2}(e^{\alpha h} - 1) + x^2(e^{3\alpha h/2} - 2e^{\alpha h} + e^{\alpha h/2}). \tag{8}$$

Note that the coefficient of x^2 in this latter equation is $o(h)$. As $e^{mh} - 1 = mh + o(h)$, we simplify (5) to get

$$p_t(x|y) \cdot \frac{-\alpha h}{2} + o(h) = \frac{\delta}{\delta t}p_t(x|y) \cdot (-h) + \frac{\delta}{\delta x}p_t(x|y) \cdot \frac{\alpha h x}{2} + \frac{\delta^2}{\delta x^2}p_t(x|y) \cdot \frac{\alpha^2 h}{2} + o(h). \tag{9}$$

For the equation to be true for all h, the coefficients of the terms linear in h must match, such that

$$\frac{\delta}{\delta t}p_t(x|y) = \frac{1}{2}\alpha p_t(x|y) + \frac{1}{2}\alpha x \frac{\delta}{\delta x}p_t(x|y) + \frac{1}{2}\alpha^2 \frac{\delta^2}{\delta x^2}p_t(x|y). \tag{10}$$

Question 3.

Again this process is a Gaussian process with mean 0 for all t. For $s \leq t$, the covariance function is now given as

$$\text{Cov}(X(s), X(t)) = e^{-\beta(s+t)} \cdot \alpha e^{\alpha s} = \alpha e^{(\alpha-\beta)s-\beta t}. \tag{11}$$

This is a function of $t - s$ iff $2\beta = \alpha$, and so this is the necessary and sufficient condition that the process is stationary.

Question 4.

Working from the conditional distribution

$$X(t)|X(t-h) = x_0 \sim N\Big(x_0 e^{-\beta h}, \alpha e^{-2\beta t}\big(e^{\alpha t} - e^{\alpha(t-h)}\big)\Big), \tag{12}$$

we can figure densities $p_{t|s}(x|y)$, the density of $X(t)$ at x, given $X(s) = y$ (for $s \leq t$). If s is not specified, it is assumed to be 0 (keeping with the notation in question 2). Again using the framework above, we perform a power series expansion of the density $p_{t-h}(a|y)$ in the expression

$$p_t(x|y) = \int p_{t|t-h}(x|a) p_{t-h}(a|y)\ da. \tag{13}$$

Solving for the linear terms in h (just as is done in question 2, but with different density functions), the following equation emerges:

$$\frac{\delta}{\delta t} p_t(x|y) = \beta p_t(x|y) + \beta x \frac{\delta}{\delta x} p_t(x|y) + \frac{1}{2}\alpha^2 e^{(\alpha-2\beta)t} \frac{\delta^2}{\delta x^2} p_t(x|y). \tag{14}$$