STATS 218 Homework 1 Solutions.

Question 1.

The times between consecutive hits against the container wall, given the constant container size and particle speed, are uniquely determined by the random angle of deflection. These random angles are independent, and therefore, the times between hits are independent, with identical distributions. Therefore the process $N(t)$ that counts the number of hits up to time t is a renewal process.

Question 2.

The interarrival times, as mentioned above, are determined by the angle of deflection off the wall on the last hit. For the angle θ with the normal to the surface, the distance the particle must travel is $2R\cos\theta$, because the point of departure, its reflection through the center, and the point of arrival form a right triangle, as shown below in the case $R = 1$.

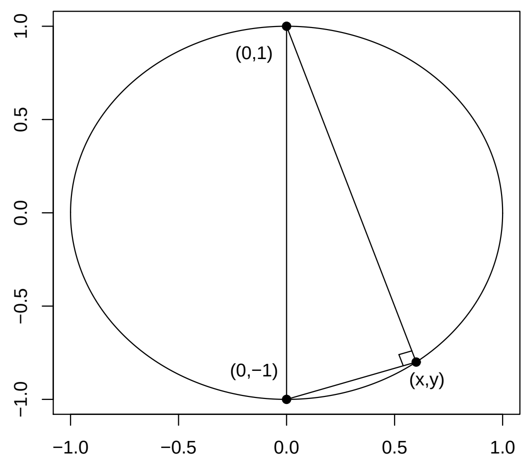

Fig. 1: x^2+y^2=1 yields Pythagoras.

Therefore the interarrival time for angle θ is $2v^{-1}R\cos\theta$, and mean of the interarrival times is given by

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2R}{v\pi} \cos\theta \; d\theta = \frac{4R}{v\pi}.$$

So by Proposition 3.3.1 in Ross, $N(t)/t \to v\pi/4R$ a.s. The force $F(t) = \epsilon N(t)$, and the container size $L = 2\pi R$, so we use this to derive the desired limit a.s.:

$$\lim_{t\to\infty} \frac{F(t)}{Lt} = \frac{v\epsilon}{8R^2}.$$

Question 3.

The force exerted on each hit is now nonconstant with mean force given by

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\epsilon}{\pi} \cos\theta \; d\theta = \frac{2\epsilon}{\pi}.$$

Note that given an angle θ, the time between hits and the resulting force have a constant ratio ($\epsilon \cos\theta$ vs. $2v^{-1}R\cos\theta$), so with probability 1, $F(S_k)/S_k = v\epsilon/2R$ for all k (S_k the sum of the first k interarrival times.) As $F(t) = \max\{F(S_k) : S_k \leq t\}$, $F(t)/t \leq v\epsilon/2R$ for all t.

We know that the time between hits is bounded above by $2R/v$, and so $\max\{S_k : S_k \leq t\} \geq t - 2R/v$. AS $F(S_k) = v\epsilon S_k/2R$ for all k, we get the inequality

$$\frac{F(t)}{t} \geq \frac{\frac{v\epsilon}{2R} \cdot (t - 2R/v)}{t} = \frac{v\epsilon}{2R} - \frac{\epsilon}{t}.$$

We have bounded $F(t)/t$ above and below by sequences with a common limit. Applying the squeeze principle and including the constant factor L, we get the result:

$$\lim_{t\to\infty} \frac{F(t)}{Lt} = \frac{v\epsilon}{4\pi R^2}.$$

Observe that the same result is obtainable via the framework of renewal-reward process (Ross 3.6). Theorem 3.6.1 yields the same limit.