maximize $H_N(\sigma) \; : \; \sigma \in \{\pm 1\}^N$ generalized p-spin

$$H_N(\sigma) := \underline{\tfrac{1}{2}\langle \sigma, W\sigma\rangle}, \quad W \sim GOE(n) \; : \; \underset{\text{model}}{SK}$$

<u>AMP</u> $z^t \in \mathbb{R}^N, \quad t \in \{0, \delta, 2\delta, \dots, 1-\delta, 1\}$

$$\underline{z^{t+\delta}} = W f_t(z^{\le t}) - \sum_{s=\delta}^{t} \underline{d_{t,s}\, f_{s-\delta}(z^{\le s-\delta})} \qquad \underline{z^0 \sim N(0, \delta I_N)}$$

<u>Lem</u> $\hat{E}_N(\cdot) := \frac{1}{N}\sum_{i=1}^{N}(\cdot)$

$$\operatorname*{p\text{-}lim}_{N\to\infty} \underline{\hat{E}_N \psi(z_i^0, \dots, z_i^1)} = \underline{E\, \psi(Z_0^\delta, Z_\delta^\delta, \dots Z_1^\delta)}$$

$$(Z_0^\delta \dots Z_1^\delta) \sim N(0, Q) \qquad Z_0^\delta \perp (Z_\delta^\delta \dots Z_1^\delta)$$

$$Q_{s+\delta, t+\delta} = E\{ f_s(Z_{\le s}^\delta)\, f_t(Z_{\le t}^\delta) \}$$

$m^t := f_t(z^{\le t})$, Want

$$\frac{1}{N}\langle m^{t+\delta} - m^t, m^s \rangle \xrightarrow{N \to 0} 0 \qquad s \le t$$

$$\mathcal{F}^{t,N} = \sigma(\{z^0, \dots z^t\})$$

$$\boxed{m^t := m^0 + \sum_{s=0}^{t-\delta} u^s (z^{s+\delta} - z^s)} \qquad m^0 = \sqrt{\delta}\, \mathbf{1}$$

$u^s \in m\mathcal{F}^{s,N}$

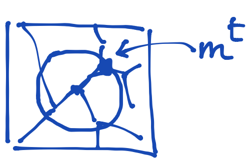
<u>$N < \infty$</u> <u>$N = \infty$</u>

$z^t, \; u^t, \; m^t$ $\qquad Z_t^\delta, \; U_t^\delta, \; M_t^\delta$

State evolution

$$\left[\begin{array}{l} M_t^\delta = \sqrt{\delta} + \sum_{s=0}^{t-\delta} U_s^\delta (Z_{s+\delta}^\delta - Z_s^\delta) \; ; \quad \underline{U_s^\delta \in m\mathcal{F}^\circ} \\ E(Z_{t+\delta}^\delta Z_{s+\delta}^\delta) = E(M_t^\delta M_s^\delta) \end{array}\right.$$

$\square$



Claim $(M^\delta_t)_{t\le 1}$ $(Z^\delta_t)_{t\le 1}$ are MG □

Proof
- Sufficient to prove for Z
- Suff to check $\mathbb{E}(Z^\delta_t Z^\delta_s) = q^\delta_{(t\wedge s)-\delta}$

By induction over t. Assume true up to t

WTS $\mathbb{E}(Z^\delta_{t+\delta} Z^\delta_{s+\delta}) \overset{?}{=} q^\delta_s \quad \forall s\le t$

$$= \delta + \sum_{\substack{t'\le t-\delta \\ s'\le s-\delta}} \mathbb{E}\{U_{s'}(Z_{s'+\delta}-Z_{s'})U_{t'}(Z_{t'+\delta}-Z_{t'})\}$$

$$q^\delta_s = \delta \sum_{s'\le s-\delta} \mathbb{E}[(U_{s'})^2](q^\delta_{t'+\delta}-q^\delta_{t'}) \quad \square$$

How to choose U^δ_s ?

$$U^\delta_s = \frac{\bar{U}^\delta_s}{\mathbb{E}[(\bar{U}^\delta_s)^2]^{1/2}} \qquad \begin{cases} q^\delta_s = \delta + \sum_{t\le s-\delta}(q^\delta_{t+\delta}-q^\delta_t) \\ q^\delta_0 = 0 \end{cases}$$

$$\boxed{q^\delta_t = t}$$

$$\bar{U}^\delta_t = u(X^\delta_t;t)$$

$$\boxed{X^\delta_{t+\delta} = X^\delta_t + v(X^\delta_t;t)\cdot\delta + (Z^\delta_{t+\delta}-Z^\delta_t)}$$

$v,u: \mathbb{R}\times[0,1]\to\mathbb{R}$.

$\mathbb{E}(Z^\delta_t Z^\delta_s) = q^\delta_{t\wedge s} = t\wedge s \qquad Z^\delta_t = B_t$ BM

$$\boxed{M^\delta_t = \sqrt{\delta} + \sum_{s=0}^{t-\delta} \frac{u(X^\delta_s;s)}{\mathbb{E}[u(X^\delta_s;s)^2]^{1/2}}(B_{s+\delta}-B_s)}$$

$\delta\to 0 \quad (X^\delta_t)\to(X_t)$ ..

$$dX_t = v(X_t,t)\,dt + dB_t$$

$$M_t = \int^t \frac{u(X_s,s)}{\mathbb{E}[u(X_s,s)^2]^{1/2}}\, dB_s$$

$_0 \ \mathbb{E}[u(X_s,s)]$

$$\mathbb{E}u(X_s,s)^2=1 \quad \forall s \iff \mathbb{E}(M_t^2)=t$$

$$\begin{cases} M_t = \int_0^t \sqrt{\xi''(s)}\, u(s,X_s)\, dB_s \\ dX_t = v(t,X_t)\,dt + \sqrt{\xi''(t)}\, dB_t \end{cases}$$

$m^t \in \mathbb{R}^N \qquad m^1 \to \text{Thresh} \to \text{Round} \to \sigma^{alg}$

<u>Lem</u> $\quad \frac{1}{N} H_N(\sigma^{alg}) \geq \frac{1}{N} H_N(m^1) + o_N(1).$

<u>Lem</u> $\quad \frac{1}{N} H_N(m^1) = \mathcal{E}(u,v) + o_N(1)$

$$\mathcal{E}(u,v) := \int_0^1 \xi''(t)\, \mathbb{E}u(t,X_t)\, dt \qquad \square$$

Alg design

$$\begin{cases} \text{maximize} & \mathcal{E}(u,v) \\ \text{subj to} & \mathbb{E}(M_t^2)=t \ \forall t, \quad M_1 \in (-1,1) \text{ a.s.} \end{cases}$$

1) Relax

2) UB via duality

3) Dual $\to$ Primal

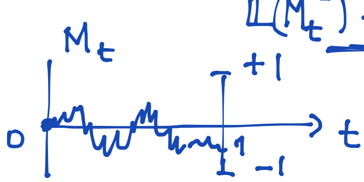
$\mathbb{E}(M_1^2)=1 \quad (\|m^1\|^2=N) \qquad \xi(x)=x^2/2 \qquad \text{VAL}=1$

<u>REL</u> $\quad u_t \in m\mathcal{F}_t \qquad \mathcal{F}_t = \sigma(\{B_s\}_{s\leq t})$

$$\max \int_0^1 \xi''(t)\, \mathbb{E}u_t\, dt \quad *$$

$$\text{s.t.} \quad M_t = \int_0^t \sqrt{\xi''(s)}\, u_s\, dB_s \qquad M_1 \in (-1,1)$$

$$\underline{\mathbb{E}(M_t^2)=t} \quad \forall t$$



$\gamma : [0,1] \to \mathbb{R}_{\geq 0}$, $\nu(t) := \int_t^1 \xi''(s)\gamma(s)\, ds$

$$J_\gamma := \sup_u \mathbb{E}\left\{ \left[\int_0^1 \xi''(t) u_t + \frac{1}{2}\nu(t)\left(u_t^2\, \xi''(t) - 1 \right) \right] dt \right\}$$

s.t. $\int_0^1 \sqrt{\xi''(t)}\; u_t \, dB_t \in (-1,1)$

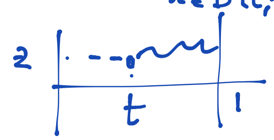
REL $\leq J_\gamma$

$$\mathbb{E}\int_0^1 \nu(t)\left(\xi''(t)u_t^2 - 1\right) dt = \mathbb{E}\int_0^1 \int_0^1 \mathbb{1}_{s \geq t}\, \xi''(s)\gamma(s)\left(\xi''(t)u_t^2 - 1\right) ds\, dt$$

$$= \int_0^1 \xi''(s)\gamma(s)\left(\mathbb{E}M_s^2 - s\right) ds \quad = 0$$

Dynamic programming

$$J_\gamma(t,z) := \sup_{u \in D(t,1)} \mathbb{E}\left\{ \int_t^1 \left[\xi''(s)u_s + \frac{1}{2}\nu(s)\left(\xi''(s)\, u_s^2 - 1\right) \right] ds \right\}$$



s.t. $z + \int_t^1 \sqrt{\xi''(s)}\; u_s \, dB_s \in (-1,1)$

$J_\gamma(t,z) = \max_{Y \cdot u \in D(t,\theta)} \ldots \quad J_\gamma(\theta, Y) \qquad J_\gamma = J_\gamma(0,0)$

$\theta > t$

$\theta \to t$: H.J.B equation

PDE for J_γ

REL $\leq J_\gamma(0,0)$

Legendre Fenchel dual of J_γ

$$\Phi_\gamma(t,x) = \min_z \left[J_\gamma(t,z) - zx + \ldots \right]$$

$\Rightarrow$ Φ satisfies Parisi's PDE

REL $\leq \boxed{\inf\, \mathcal{P}(\gamma)}$ $\quad \underline{\gamma \in \mathcal{L}}$.

if inf is achieved at γ_*

$v(t,x) = V_\gamma(t,x)$ $\quad u_\gamma(t,x)$ $\cancel{\approx}$:

$\inf_{\gamma \in \mathcal{L}} P(\gamma)$, $\inf_{\gamma \in \mathcal{U}} P(\gamma)$

Q1: Complexity of solving these within ϵ error.

Q2: $d(\phi_{\gamma_*}, \phi^\epsilon)$ $\quad \cancel{\phi}^\epsilon \leq \epsilon$

β $\quad m^1$ $\quad TAP(m^1) \approx 0$

$\frac{1}{N} D(\mu_\beta \| \hat{\mu}) = o(1) \leftarrow$

? $\| \mu_\beta - \hat{\mu} \|_{TV} = o(1) \leftarrow$