

Substitute Valuations with Divisible Goods*

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Abstract

In a companion paper, we showed that weak and strong notions of substitutes in economies with discrete goods have different implications for auction theory and equilibrium theory. In contrast, for the divisible goods case with concave valuations, natural extensions of these concepts coincide. Concave substitute valuations are characterized by submodularity of the dual profit function over nonlinear prices and are robust with respect to additive concave perturbations, which extends a related notion of robustness established for complements, as in Milgrom (1994). When all bidders have concave substitute valuations, the Vickrey outcome is in the core but the law of aggregate demand, which holds for strong substitutes in discrete economies, can fail.

Keywords: Substitute Valuation, Nonlinear Price, Submodular Dual, Law of Aggregate Demand, Vickrey Auction.

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1 Introduction

In neoclassical theory, inputs to production are substitutes if when the price of one type of input rises, the number of units demanded of the other types of inputs does not fall. This definition, however, depends on the price domain that is considered. A priori, restricting the substitutes condition to linear prices would seem to make it weaker than allowing for nonlinear prices. In Milgrom and Strulovici (2006), we showed that a “strong” notion of substitutes based on the idea that each unit of input is treated as a distinct, unique good is necessary and sufficient for such results as the robust existence of equilibrium, the robust inclusion of the Vickrey outcome in the core, and the *law of aggregate demand*, while the “weak” notion of substitutes, which relies on linear prices, is necessary and sufficient for robust monotonicity of certain auction/tâtonnement processes. We also showed that the strong notion can be characterized as the weak notion plus two additional properties: concavity and the law of aggregate demand.

This paper draws a similar distinction for economies with divisible goods, introducing two notions of substitutes that are defined using linear and nonlinear prices.¹ We find important contrasts between the discrete and divisible good cases. First, concave valuations that are linear substitute valuations are also nonlinear substitute valuations, so the two concepts coincide, subject only to concavity. Second, concave substitutes valuations fail to imply the law of aggregate demand and even fail to imply a weaker *generalized law of aggregate demand* that we introduce.

Both concave and non-concave substitutes valuations have dual characterizations that use different price spaces. Each is equivalent to submodularity of a suitable indirect profit function. Both sets of valuations are closed under aggregation. The set of concave substitutes valuations has the robustness property that it is closed under the addition of additively separable concave functions.

¹Using nonlinear prices is not identical to treating goods individually, because treating goods as distinct expands both the domain and the range of the demand correspondence, while only the domain is changed by the switch to nonlinear pricing.

2 Definitions

Consider an economy with K goods, in which good k is available in quantity X_k for $k \in \mathcal{K} = \{1, \dots, K\}$. Let

$$\mathcal{X} = \prod_{k \in \mathcal{K}} [0, X_k]$$

represent the space of possible bundles of the exchange economy. Throughout the paper, we assume that agents have quasi-linear utilities.

ASSUMPTION 1 (QUASI-LINEARITY) *The utility of an agent with multi-unit valuation v acquiring a bundle x at price p is*

$$u(x, p) = v(x) - p(x).$$

Given a valuation v and a price vector $p \in \mathcal{P}$, define the demand function of the agent at price p by

$$D(p) = \arg \max_{x \in \mathcal{X}} \{v(x) - px\}.$$

3 Substitute Valuations with Divisible Goods

With quasi-linear preferences, there is no distinction to be made between gross and net substitutes, so we drop the modifier and make the following definitions.

DEFINITION 1 *v is a linear-substitute valuation if whenever $p_j \leq p'_j$, $p_k = p'_k$ for all $k \neq j$, and $x \in D(p)$, there exists $x' \in D(p')$ such that $x'_k \geq x_k$ for all $k \neq j$.*

In the discrete case with individual item pricing, a rational consumer who buys k units of some type of good always buys the cheapest k units. Therefore, one way to describe individual item pricing is to specify that the cost of acquiring goods is a convex function of the number of goods acquired from each class and is additive across classes of goods. Higher prices mean that the marginal cost of acquiring additional units is higher. This characterization of the cost of acquiring goods and the corresponding representation of higher prices can be applied directly to the continuous case. That is the approach we adopt in this section.

Let \mathcal{C}_1 denote the space of continuously differentiable, convex functions from \mathbb{R}_+ to \mathbb{R} which vanish at 0, and $\mathcal{C} = \mathcal{C}_1^K$. An element of \mathcal{C} contains K convex cost functions, one

for each good. We endow \mathcal{C} with the following partial order: $C \preceq \hat{C}$ if for all k , $c_k \leq \hat{c}_k$, where c_k and \hat{c}_k are the derivatives of C_k and \hat{C}_k , respectively. With this order, \mathcal{C} is a lattice, where for any C and \hat{C} , the meet and the join satisfy, for all k and $x_k \geq 0$, $(C \vee \hat{C})'_k(x_k) = \max\{c_k(x_k), \hat{c}_k(x_k)\}$ and $(C \wedge \hat{C})'_k(x_k) = \min\{c_k(x_k), \hat{c}_k(x_k)\}$, respectively.² We extend the domain of any dual profit function π from linear prices to \mathcal{C} and denote $\bar{\pi}$ its extension:

$$\bar{\pi}(C) = \max_x \{v(x) - C(x)\},$$

where $C(x) = \sum_k C_k(x_k)$.

DEFINITION 2 *v is a nonlinear-substitute valuation if whenever $C_j \leq \hat{C}_j$, $C_k = \hat{C}_k$ for all $k \neq j$, and $x \in D(C)$, there exists $x' \in D(\hat{C})$ such that $x'_k \geq x_k$ for all $k \neq j$.*

For the discrete case, Milgrom and Strulovici (2006) have shown that there are several properties distinguishing weak substitutes and strong substitutes, so there is scope for judgment in creating the analogue of strong substitutes in the continuous case. For example, one could impose that the extended concept satisfy the law of aggregate demand. That would require that a dominant diagonal property hold for the matrix $[\partial x_i / \partial p_j]$ of partial derivatives of demand. The concept that we study below does not satisfy the law of aggregate demand.

In place of strong substitutes, we study the concept of *concave, nonlinear-substitute* valuations. This definition preserves properties distinguishing strong substitutes from weak substitutes in the discrete setting, including robustness of the substitutes property with respect to nonlinear price changes and existence of Walrasian equilibria. Moreover, we find below that these valuations are characterized by dual submodularity on the domain of nonlinear prices, which was also the characterization of strong substitutes in the discrete case. We find further that, given concavity, the linear-substitute and nonlinear-substitute properties are equivalent. Therefore, our divisible-good extensions of the two concepts coincide in the case of concave valuations. The next two theorems develop all of these relationships.

THEOREM 1 (DUAL SUBMODULARITY) *If v is a concave linear-substitute valuation, then $\bar{\pi}$ is submodular on \mathcal{C} .*

²As can be easily checked, the marginal costs of $(C \wedge \hat{C})'_k$ and $(C \vee \hat{C})'_k$ are continuous and nondecreasing for all k , and constructed cost functions both vanish at 0, so that $C \wedge \hat{C}$ and $C \vee \hat{C}$ belong to \mathcal{C} .

Proof. Convexity of π and C implies that the function $(p, x) \mapsto \pi(p) + px - C(x)$ is concave in x and convex in p . The minmax theorem (see e.g. Stoer and Witzgall (1970)) implies that

$$\max_x \min_p \{\pi(p) + px - C(x)\} = \min_p \max_x \{\pi(p) + px - C(x)\}.$$

Concavity of v implies that $v(x) = \min_p(\pi(p) + px)$ for all x . This, along with $\bar{\pi}(C)$'s definition, implies that

$$\bar{\pi}(C) = \max_x \{\min_p(\pi(p) + px) - C(x)\}.$$

This and the minmax theorem then imply that

$$\bar{\pi}(C) = \min_p \left\{ \pi(p) + \max_x \{px - C(x)\} \right\}.$$

The inner maximum equals

$$\sum_k \int_0^\infty (p_k - c_k(z_k))_+ dz_k.$$

Therefore,

$$\bar{\pi}(C) = \min_p \left\{ \pi(p) + \sum_k \int_0^\infty (p_k - c(z_k))_+ dz_k \right\}.$$

We now show that the function $h : (p, C) \rightarrow h(p, C) = \int_0^\infty (p - c(z))_+ dz$ is submodular on $\mathbb{R}_+ \times \mathcal{C}_1$. For $q < r$, $h(r, C) - h(q, C)$ is the area of the region $\{(z, p) : p \in [q, r] \text{ and } C(z) \leq p\}$, which is also equal to $\int_q^r z(p, C) dp$, where $z(p, C) = \sup\{z : c(z) \leq p\}$. Since $z(p, C)$ is nonincreasing in C for all p , so is $h(r, C) - h(q, C)$, which proves submodularity of h in (p, C) . Moreover, for any numbers p, c and d , $(p - c)_+ + (p - d)_+ = (p - \max(c, d))_+ + (p - \min(c, d))_+$, which implies that h is modular in C . Linear substitutes implies that π is submodular in p . Therefore, $\bar{\pi}$ is the minimum over $p \in \mathcal{P} = \mathbb{R}_+^K$ of an objective function that is submodular on $\mathcal{P} \times \mathcal{C}$. This implies (Topkis (1998)) that π is submodular on \mathcal{C} . ■

Theorem 1 allows us to prove the equivalence of three candidate definitions for the divisible-good extension of strong substitutes.

THEOREM 2 *Suppose that v is concave. Then the three following statements are equivalent.*

- (i) v is a linear-substitute valuation.
- (ii) v is a nonlinear-substitute valuation.

(iii) π is submodular on \mathcal{C} .

Proof. Clearly, (ii) implies (i). From Theorem 1, (i) implies (iii). To conclude the proof, we show that (iii) implies (ii). We adapt the proof of Ausubel and Milgrom (2002), Theorem 10. We fix a direction of price increase for some good, and show that along this direction, the demand for any other good is nondecreasing. Fix goods $j \neq k$ and a direction of increase δ (i.e. δ is nondecreasing, vanishes at 0, and is such that $C + \delta$ is convex) for good j . Consider the restriction

$$\tilde{\pi}(\lambda, \mu) = \max_x \{v(x) - C(x) - \lambda x_k - \mu \delta(x_j)\}$$

of π , defined³ on $\mathbb{R}_+ \times [0, 1]$. Since π is submodular, so is $\tilde{\pi}$. $\tilde{\pi}$ is convex as the pointwise maximum of a family of functions that are affine in (λ, μ) . In particular, $\partial \tilde{\pi} / \partial \lambda$ exists almost everywhere. By an envelope theorem⁴ $\partial \tilde{\pi} / \partial \lambda$ exists everywhere that $x_k(\lambda, \mu)$ is a singleton and at those prices, $\partial \tilde{\pi} / \partial \lambda = -x_k(\lambda, \mu)$. Submodularity of $\tilde{\pi}$ implies that $\partial \tilde{\pi} / \partial \lambda(\lambda, \mu)$ is nonincreasing in μ or, equivalently, that x_k is nondecreasing in μ . ■

4 Robustness

Theorems 1 and 2 have an important consequence: concave nonlinear-substitute valuations are stable under perturbation by any concave modular function. Thus comparative statics results are robust with respect to such perturbations, as stated in the following theorem.

THEOREM 3 *If v is a concave nonlinear-substitute valuation, then $v + f$ is a concave nonlinear-substitute valuation for all f modular and concave.*

Proof. Suppose that v is a concave nonlinear-substitute valuation. Then, $v + f$ is concave whenever f is concave. By Theorem 2, it remains to show that $v + f$ is a linear-substitute valuation. Let

$$x^f(p) = \arg \max_x \{v(x) + f(x) - px\}.$$

Without loss of generality, we can assume that $f_i(0) = 0$ for all terms of f . Let $C(x, p) = px - f(x)$. Since f is modular and concave, C is modular and for each i , C_i is convex and

³The function $C(x) + \lambda x_k + \mu \delta(x_j)$ is convex for $\mu \in [0, 1]$, as can be easily checked.

⁴Milgrom and Segal (2002), Corollary 4.

vanishes at 0. Therefore, C belongs to \mathcal{C} . Moreover, increasing p_k implies increasing C_k . Since v is a nonlinear-substitute valuation and

$$x^f(p) = \arg \max_x \{v(x) - C(x, p)\},$$

$x_j^f(p)$ is nondecreasing in C_k , thus in p_k . ■

We now turn to the consequences of the substitutes properties in settings with multiple firms.

5 Multiple firms: Aggregation and Core

We first show that both linear and concave nonlinear substitute valuations are closed under aggregation. We recall the following result, which is a simple adaptation of Ausubel and Milgrom (2002), Theorem 10.

THEOREM 4 (AUSUBEL AND MILGROM (2002)) *v is a linear-substitute valuation if and only if π is submodular over \mathcal{P} .*

Theorem 4 will be used in conjunction with the following result. We define the market-valuation v of the economy by

$$v(x) = \max \left\{ \sum v_i(x_i) : \sum x_i = x, x_i \geq 0 \right\}.$$

and the market dual profit function of the economy by $\pi(p) = \max_{x \in \mathbb{R}^K} \{v(x) - px\}$. The function π is convex, as can be checked easily.

THEOREM 5 *For all $p \in \mathcal{P}$, $\pi(p) = \sum_{i=1}^n \pi_i(p)$.*

Proof.

$$\begin{aligned} \pi(p) &= \max_x \{ \max \{ \sum_i v_i(x_i) : \sum_i x_i = x \} - px \} \\ &= \max_{x_1, \dots, x_n} \sum_i \{ v_i(x_i) - px_i \} \\ &= \sum_i \pi_i(x_i), \end{aligned}$$

which concludes the proof. ■

THEOREM 6 *The class of linear-substitute valuations is closed under aggregation.*⁵

⁵This proof is an adaptation of Milgrom and Strulovici (2006).

Proof. If individual firms have linear substitute valuations, Theorem 4 implies that individual profit functions are submodular. By Theorem 5, the market dual profit function is therefore a sum of submodular functions, and so itself submodular. Applying Theorem 4 once again, we conclude that v is a linear substitute valuation. ■

With divisible goods, concavity is also closed under aggregation: the maximization

$$v(x) = \max_x \sum_i v_i(x_i)$$

subject to $\sum x_i \leq x$ has a concave objective function and a convex constraint function, so v is concave⁶ in the constraint bound x . This shows the following result.

THEOREM 7 *Concave nonlinear-substitute valuations are closed under aggregation.*

Proof. The above discussion shows that concave linear-substitute valuations are closed under aggregation. This, along with Theorem 2, implies that the same is true of concave nonlinear-substitute valuations. ■

With divisible goods, concavity is a sufficient condition for the existence of a Walrasian equilibrium. We show that if, in addition, firms have nonlinear-substitute valuations, then the Vickrey outcome is in the core. The setting considered in this section is the same as Ausubel and Milgrom (2002). We first recall the definitions of coalitional value functions, the core, and Vickrey payoffs.

Suppose that, in addition to bidders, there exists a single owner (labeled “0”) of all units of all goods, who has zero utility for her endowment.

DEFINITION 3 *The coalitional value function of a set S of bidders is*

$$w(S) = \max \left\{ \sum_{i \in S} v_i(x_i) : \sum x_i \in \mathcal{X} \right\}$$

if $0 \in S$, and $w(S) = 0$ otherwise.

Denote L the set consisting of all bidders and the owner of the good.

DEFINITION 4 *The core of the economy is the set*

$$\text{Core}(L, w) = \left\{ \pi : w(L) = \sum_{l \in L} \pi_l, w(S) \leq \sum_{l \in S} \pi_l \text{ for all } S \subset L \right\}.$$

⁶See for example Luenberger (1969, p.216).

DEFINITION 5 *The Vickrey payoff vector (the payoff at the dominant-strategy equilibrium of the generalized Vickrey auction) is*

$$\bar{\pi}_l = w(L) - w(L \setminus l)$$

for $l \in L \setminus 0$, and

$$\bar{\pi}_0 = w(L) - \sum_{l \in L \setminus 0} \bar{\pi}_l.$$

DEFINITION 6 *The coalitional value function w is bidder-submodular if for all $l \in L \setminus 0$ and sets S and S' such that $0 \in S \subset S'$,*

$$w(S) - w(S \setminus l) \geq w(S') - w(S' \setminus l).$$

THEOREM 8 *If all bidders have concave nonlinear-substitute valuations, the Vickrey outcome is in the core.*

Proof. From Theorem 7 of Ausubel and Milgrom (2002), it is enough to show that the coalitional value function is bidder submodular. Therefore, we need to show that $w(S \cup \{l\}) - w(S)$ is nonincreasing in S . Let x denote the quantity of goods available. Then,

$$w(S \cup \{l\}) - w(S) = \max_{y \leq x} \{v_l(x - y) + v_S(y) - v_S(x)\},$$

where $v_T(z)$ denote the optimal value of bundle z for coalition T . Therefore, it is enough to show that $v_S(y) - v_S(x)$ is non increasing in S or simply that $v_S(z)$ is submodular in $(-z, S)$. Concavity⁷ of v_S implies that

$$v_S(z) = \min_p \{\pi_S(p) + pz\}, \tag{1}$$

where π_S is the dual profit function of v_S and is equal to $\sum_{i \in S} \pi_i(p)$. Since π is submodular in p , the objective in (1) is submodular in $(-z, S, p)$. Hence, $v_S(z)$ is submodular in $(-z, S)$, as required. ■

6 Law of Aggregate Demand

We have focused so far on monotone comparative statics of the demand function. In the discrete case, Milgrom and Strulovici (2006) proved that nonlinear substitutes not only

⁷See discussion preceding Theorem 7.

implied that x_j is nondecreasing in the price of other goods, but also that $\sum_j x_j(\tilde{p})$ was nonincreasing in \tilde{p} , which is the discrete law of aggregate demand. In this section, we show that this property no longer holds, and thus is not necessary for Theorems 1 and 8 to hold.

What is the analogue of the law of aggregate demand for divisible substitute goods? One problem is to determine the units in which such a law might be expressed. For example, suppose that one unit of good i represents a 10-ride train pass between two cities, while one unit of good j is a one-way bus ticket between the same cities. One expects that, starting from prices where a consumer chooses the train pass, a large price increase in the train pass results in the consumer buying several bus tickets to replace the train pass, implying that the sum $x_i + x_j$ *increases* as p_i increases, which violates the law of aggregate demand. One way to pose the problem without units is to ask whether there exist constants a_i such that $\sum_i a_i x_i$ be nondecreasing in prices? In the previous example, a natural choice would be $a_i = 1$ and $a_j = 10$, given the relative similarity of a train trip and a bus trip. More generally, we say that a valuation v satisfies the *generalized law of aggregate demand* (GLoAD) if there exist increasing functions f_i such that

$$\sum_i f_i(x_i(C))$$

is nonincreasing in C . It satisfies the *law of aggregate demand* if one can take $f_i(x_i) = x_i$ for all i . The GLoAD seems so much more flexible than the law of aggregate demand that one is led to wonder whether it is satisfied by linear-substitute valuations, or at least concave nonlinear-substitute valuations. However, the following theorem and its corollary show that the GLoAD is equivalent to the law of aggregate demand up to a mere convex re-scaling of goods. For the remaining of this section we assume that the cost functions are nondecreasing.⁸ To simplify the exposition, let $(f \circ g)(x) = f(g_1(x_1), \dots, g_K(x_K))$, for any function f and modular function g . Clearly, $f \circ g$ is modular if f is also modular. Restricted to the class of increasing modular functions, let f^{-1} denote the sum of component-wise inverse functions: $f^{-1}(x) = \sum_k f_k^{-1}(x_k)$. For functions of one variable, these definitions coincide with the usual ones. For the next two theorems, we assume that valuations are nondecreasing.

THEOREM 9 *Let v be a nondecreasing concave nonlinear-substitute valuation satisfying the generalized law of aggregate demand for some function f , and g be an increasing,*

⁸This assumption is used in the proof of Theorem 9. We did not make this assumption earlier in order to prove Theorem 3, where we consider $C(x) = px - f(x)$ and f may be increasing.

concave, modular function. Then $\tilde{v} = v \circ g$ is a nondecreasing, concave nonlinear-substitute valuation satisfying the generalized law of aggregate demand for the modular function $\tilde{f} = f \circ g$.

Proof. Since v and g are nondecreasing concave, so is \tilde{v} . Let \tilde{C} be a convex price schedule, and $y(\tilde{C}) = \arg \max \tilde{v}(y) - \tilde{C}(y)$. We wish to show that y_j is nondecreasing in \tilde{C}_k for $j \neq k$, and that there exists an increasing modular function \tilde{f} such that $\tilde{f}(y(\tilde{C}))$ is nonincreasing in \tilde{C} . The function $\gamma = g^{-1}$ is increasing, convex, and modular. By assumption, there exists a modular function f such that $f(x(C))$ is nondecreasing in C , where $x(C)$ is the demand of v at the convex price schedule C . Let $C = \tilde{C} \circ \gamma$. Since all components of γ and \tilde{C} are nondecreasing convex, so are the components of C . Increasing \tilde{C}_k to \tilde{C}'_k is equivalent to increasing C_k to $C'_k = \tilde{C}'_k \circ \gamma_k$. Therefore, if $j \neq k$, $y_j(\tilde{C}) = \gamma_j(x_j(C))$ is nondecreasing when \tilde{C}_k increases. Moreover, letting $\tilde{f} = f \circ g$, we have $\tilde{f}(y(\tilde{C})) = f(x(\tilde{C} \circ \gamma))$, which is nonincreasing in \tilde{C} . ■

COROLLARY 1 *Suppose that v is a nondecreasing, concave nonlinear-substitute valuation satisfying the generalized law of aggregate demand for some convex function f . Then, $\tilde{v} = v \circ f^{-1}$ satisfies the law of aggregate demand.*

Thus, the generalized law of aggregate demand corresponds to a quantitative rather than a qualitative relaxation of the law of aggregate demand. In fact, it is possible to construct a concave nonlinear-substitute valuation that does not satisfy *any* generalized law of aggregate demand. We provide a sketch of this counter-example below, which establishes a fundamental difference between the cases of discrete and divisible goods.

COUNTER-EXAMPLE 1 *There exist concave nonlinear-substitute valuations that do not satisfy the generalized law of aggregate demand.*

Proof. [Sketch] We consider the case of two goods. Let $x < x'$ and $y < y'$ be positive numbers, and consider the bundles $A = (x, y)$, $B = (x', y)$, $C = (x, y')$, and $D = (x', y')$ and let (p_i, q_i) be the supporting prices for $i = A, B, C, D$ (such prices exist by concavity). If GLoAD held, there would exist some increasing functions f and g with $f(x(p, q)) + g(y(p, q))$ nonincreasing in (p, q) , where (p, q) is the price vector of the two goods and $(x(p, q), y(p, q))$ is the demand at that price. Suppose that at (p_B, q_B) and (p_C, q_C) , a small increase in price p reduces $x(p, q)$ by a very small amount and increases

$y(p, q)$ by a very large amount (as in the ticket/pass example above). GLoAD can⁹ only hold if $f'(x')$ is much larger than $g'(y)$ (looking at B), and if $f'(x)$ is much larger than $g'(y')$ (looking at C). Now suppose that at (p_A, q_A) and (p_D, q_D) , a small increase in price q reduces $y(q)$ by a very small amount and increases $x(q)$ by a very large amount. GLoAD can only hold if $g'(y)$ is much larger than $f'(x)$ (looking at A) and if $g'(y')$ much larger than $f'(x')$ (looking at D). These two sets of conditions are incompatible proving that GLoAD cannot hold. To conclude the counter-example, it remains to show that there exist concave nonlinear-substitute valuations satisfying the demand behavior described at points A, B, C , and D . Demand variations are determined by the Hessian of the valuation at these points. As is easily checked, one can choose for each bundle in $\{A, B, C, D\}$ a Hessian matrix that is negative definite with negative cross derivatives and that satisfies the demand behavior specified at that point. It is also possible to extend these Hessian matrices over the entire consumption space, while keeping negative definiteness and negative cross derivatives. This construction is achieved by superposition of four concave submodular functions, one for each bundle, whose Hessian coincides with the specified Hessian at that bundle and vanishes around the three remaining bundles. Such superposition defines a valuation (up to an affine term) that is submodular and concave. In two dimensions, submodularity implies the linear-substitute property. By Theorem 2, the constructed valuation is therefore a concave nonlinear-substitute valuation. ■

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⁹The proof is easily adapted if f and g are not differentiable at these points.

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