1 Choice Theory

Rational preference relation (Choice 4; MWG 6–7) A binary relation $\succsim$ is a rational preference relation iff it satisfies

1. Completeness: $\forall x, y, x \succsim y \vee y \succsim x$ (NB: implies $x \succsim x$);
2. Transitivity: $\forall x, y, z, (x \succsim y \wedge y \succsim z) \implies x \succsim z$ (which rules out cycles, except where there's indifference).

If $\succsim$ is rational, then $\succ$ is both irreflexive and transitive; $\sim$ is reflexive, transitive, and symmetric; and $x \succ y \succsim z \implies x \succ z$.

Choice rule (Choice 6) Given a choice set B and preference relation $\succsim$, choice rule $C(B, \succsim) \equiv \{x \in B : \forall y \in B, x \succsim y\}$. This correspondence gives the set of "best" elements of B.

If $\succsim$ is complete and transitive and $|B|$ finite and non-empty, then $C(B, \succsim) \neq \varnothing$.

Revealed preference (Choice 6–8; MWG 11) We observe choices and deduce a preference relation. Consider revealed preferences $C : 2^B \to 2^B$ satisfying $\forall A$, $C(A) \subseteq A$. Assuming the revealed preference sets are always non-empty, there is a well-defined preference relation $\succsim$ (complete and transitive) satisfying $\forall A$, $C(A, \succsim) = C(A)$ iff C satisfies HARP (or WARP, and the set of budget sets contains all subsets of up to three elements).

Houthaker's Axiom of Revealed Preferences (Choice 6-88) A set of revealed preferences $C : 2^B \to 2^B$ satisfies HARP iff $\forall x, y \in U \cap V$ such that $x \in C(U)$ and $y \in C(V)$, it is also the case that $x \in C(V)$ and $y \in C(U)$. In other words, suppose two different choice sets both contain x and y; if x is preferred to all elements of one choice set, and y is preferred to all elements of the other, then x is also preferred to all elements of the second, and y is also preferred to all elements of the first.

Weak Axiom of Revealed Preference (MWG 10–11) Choice structure $(\mathcal{B}, C(\cdot))$—where $\mathcal{B}$ is the set of budget sets—satisfies WARP iff $B, B' \in \mathcal{B}$; $x, y \in B$; $x, y \in B'$; $x \in C(B)$; $y \in C(B')$ together imply that $x \in C(B')$. (Basically, HARP, but only for budget sets).

Generalized Axiom of Revealed Preference (Micro P.S. 1.3) A set of revealed preferences $C : \mathcal{A} \to \mathcal{B}$ satisfies GARP if for any sequences $A_1, \ldots, A_n$ and $x_1, \ldots, x_n$ where

1. $\forall i \in \{1, \ldots, n\}$, $x_i \in A_i$;
2. $\forall i \in \{1, \ldots, n-1\}$, $x_{i+1} \in C(A_i)$;
3. $x_1 \in C(A_n)$;

then $x_i \in C(A_i)$ for all i. (That is, there are no revealed preference cycles except for revealed indifference.)

Utility function (Choice 9-13; MWG 9) Utility function $u : X \to \mathbb{R}$ represents $\succsim$ on X iff $x \succsim y \iff u(x) \geq u(y)$.

1. This turns choice rule into a maximization problem: $C(B, \succsim) = \operatorname{argmax}_{y \in B} u(y)$.
2. A preference relation can be represented by a utility function only if it is rational (complete and transitive).
3. If $|X|$ is finite, then any rational preference relation $\succsim$ can be represented by a utility function; if $|X|$ is infinite, this is not necessarily the case.
4. If $X \subseteq \mathbb{R}^n$, then $\succsim$ (complete, transitive) can be represented by a continuous utility function iff $\succsim$ is continuous (i.e., $\lim_{n\to\infty}(x_n, y_n) = (x, y)$ and $\forall n$, $x_n \succsim y_n$ imply $x \succsim y$).
5. The property of representing $\succsim$ on X is ordinal (i.e., invariant to monotone increasing transformations).

Interpersonal comparison (Choice 13) It is difficult to weigh utility tradeoffs between people. Two possible systems are Rawls' "veil of ignorance" (which effectively makes all the choices one person's), and a system of "just noticeable differences" (which suffers transitivity issues).

Continuous preference (Choice 11; MWG 46–7, Micro P.S. 1-5) $\succsim$ on X is continuous if for any sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ with $\lim_{n\to\infty}(x_n, y_n) = (x, y)$ and $\forall n$, $x_n \succsim y_n$, we have $x \succsim y$. Equivalently, $\succsim$ is continuous iff for all x, the upper and lower contour sets of x are both closed sets. $\succsim$ is rational and continuous iff it can be represented by a continuous utility function.

Monotone preference (Choice 15–6; MWG 42–3) $\succsim$ is monotone iff $x \geq y \implies x \succsim y$ (i.e., more of something is better).

(MWG uses $x \gg y \implies x \succ y$; this is *not* equivalent.)

Strictly/strongly monotone iff $x > y \implies x \succ y$.

$\succsim$ is (notes) monotone iff $u(\cdot)$ is nondecreasing. $\succsim$ is strictly monotone iff $u(\cdot)$ monotone increasing. Strictly monotone $\implies$ (notes or MWG) monotone. MWG monotone $\implies$ locally non-satiated.

Locally non-satiated preference (Choice 15–6; MWG 42–3) $\succsim$ is locally non-satiated on X iff for any $y \in X$ and $\varepsilon > 0$, there exists $x \in X$ such that $\|x - y\| \leq \varepsilon$ and $x \succ y$ (i.e., there are no "thick" indifference curves). $\succsim$ is locally non-satiated iff $u(\cdot)$ has no local maxima in X. Strictly monotone $\implies$ MWG monotone $\implies$ locally non-satiated.

Convex preference (Choice 15–6; MWG 44–5) $\succsim$ is convex on X iff (*de facto*, X is a convex set, and) $x \succsim y$ and $x' \succsim y$ together imply that $\forall t \in (0, 1)$, $tx + (1-t)x' \succsim y$ (i.e., one never gets worse off by mixing goods). Equivalently, $\succsim$ is convex on X iff the upper contour set of any $y \in X$ (i.e., $\{x \in X : x \succsim y\}$) is a convex set. Can be interpreted as diminishing marginal rates of substitution.

$\succsim$ is strictly convex on X iff X is a convex set, and $x \succsim y$ and $x' \succsim y$ (with $x \neq x'$) together imply that $\forall t \in (0, 1)$, $tx + (1-t)x' \succ y$.

$\succsim$ is (strictly) convex iff $u(\cdot)$ is (strictly) quasi-concave.

Homothetic preference (MWG 45, Micro P.S. 1.6) $\succsim$ is homothetic iff for all $\lambda > 0$, $x \succsim y \iff \lambda x \succsim \lambda y$. (MWG uses $\forall \lambda \geq 0$, $x \sim y \implies \lambda x \sim \lambda y$.) A continuous preference relation is homothetic iff it can be represented by a utility function that is homogeneous of degree one (note it can also be represented by utility functions that aren't).

Separable preferences (Choice p.18–9) Suppose $\succsim$ on $X \times Y$ is represented by $u(x, y)$. Then preferences over x do not depend on y iff there exist functions $v : X \to \mathbb{R}$ and $U : \mathbb{R} \times Y \to \mathbb{R}$ such that U is increasing in its first argument and $\forall (x, y)$, $u(x, y) = U(v(x), y)$. Note that this property is asymmetric. Preferences over x given y will be represented by $v(x)$, regardless of y.

Quasi-linear preferences (Choice 20–1; MWG 45) Suppose $\succsim$ on $X = \mathbb{R} \times Y$ is complete and transitive, and that

1. The numeraire good ("good 1") is valuable: $(t, y) \succsim (t', y)$ iff $t \geq t'$;
2. Compensation is possible: For every $y, y' \in Y$, there exists some $t \in \mathbb{R}$ such that $(0, y) \sim (t, y')$;
3. No wealth effects: If $(t, y) \succsim (t', y')$, then for all $d \in \mathbb{R}$, $(t + d, y) \succsim (t' + d, y')$.

Then there exists a utility function representing $\succsim$ of the form $u(t, y) = t + v(y)$ for some $v : Y \to \mathbb{R}$. (Note it can also be represented by utility functions that aren't of this form.) Conversely, any preference relation $\succsim$ on $X = \mathbb{R} \times Y$ represented by a utility function of the form $u(t, y) = t + v(y)$ satisfies the above conditions. (MWG uses slightly different formulation.)

Lexicographic preferences (MWG 46) A preference relation $\succsim$ on $\mathbb{R}^2$ defined by $(x, y) \succsim (x', y')$ iff $x > x'$ or $x = x' \wedge y \geq y'$. Lexicographic preferences are complete, transitive, strongly monotone, and strictly convex; however, they is not continuous and can*not* be represented by any utility function.

2 Producer theory

Competitive producer behavior (Producer 1–2) Firms choose a production plan (technologically feasible set of inputs and outputs) to maximize profits. Assumptions include:

1. Firms are price takers (applies to both input and output markets);
2. Technology is exogenously given;
3. Firms maximize profits; should be true as long as
   - The firm is competitive;
   - There is no uncertainty about profits;
   - Managers are perfectly controlled by owners.

Production plan (Producer 4) A vector $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ where an output has $y_k > 0$ and an input has $y_k < 0$.

Production set (Producer 4) Set $Y \subseteq \mathbb{R}^n$ of feasible production plans; generally assumed to be non-empty and closed.

Free disposal (Producer 5) $y \in Y$ and $y' \le y$ imply $y' \in Y$.

Shutdown (Producer 5) $\mathbf{0} \in Y$.

Nonincreasing returns to scale (Producer 5) $y \in Y$ implies $\alpha y \in Y$ for all $\alpha \in [0, 1]$. Implies shutdown.

Nondecreasing returns to scale (Producer 5, Micro P.S. 2-1) $y \in Y$ implies $\alpha y \in Y$ for all $\alpha \ge 1$. Along with shutdown, implies that $\pi(p) = 0$ or $\pi(p) = +\infty$ for all p.

Constant returns to scale (Producer 5) $y \in Y$ implies $\alpha y \in Y$ for all $\alpha \ge 0$; i.e., nonincreasing *and* nondecreasing returns to scale.

Convexity (Producer 6) $y, y' \in Y$ imply $ty + (1-t)y' \in Y$ for all $t \in [0, 1]$. Vaguely "nonincreasing returns to specialization." If $\mathbf{0} \in Y$, then convexity implies nonincreasing returns to scale. Strictly convex iff for $t \in (0, 1)$, the convex combination is in the interior of Y.

Transformation function (Producer 6, 24) A function $T\colon \mathbb{R}^n \to \mathbb{R}$ with $T(y) \le 0 \iff y \in Y$. Can be interpreted as the amount of technical progress required to make y feasible. The set $\{y\colon T(y) = 0\}$ is the transformation frontier.

Kuhn-Tucker FOC gives necessary condition $\nabla T(y^*) = \lambda p$, which means the price vector is normal to the production possibility frontier at the optimal production plan.

Marginal rate of transformation (Producer 6–7) When the transformation function T is differentiable, MRT between goods k and l is $\text{MRT}_{k,l}(y) \equiv \frac{\partial T(y)}{\partial y_l} / \frac{\partial T(y)}{\partial y_k}$. Measures the extra amount of good k that can be obtained per unit reduction of good l. Equals the slope of the transformation frontier.

Production function (Producer 7) For a firm with only a single output q (and inputs $-z$), defined as $f(z) \equiv \max q$ such that $T(q, -z) \le 0$. Thus $Y = \{(q, -z)\colon q \le f(z)\}$, allowing for free disposal.

Marginal rate of technological substitution (Producer 7) When the production function f is differentiable, MRTS between goods k and l is $\text{MRTS}_{k,l}(z) \equiv \frac{\partial f(z)}{\partial z_l} / \frac{\partial f(z)}{\partial z_k}$. Measures how much of input k must be used in place of one unit of input l to maintain the same level of output. Equals the slope of the isoquant.

Profit maximization (Producer 7–8) The firm's optimal production decisions are given by correspondence $y\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$

$$y(p) \equiv \operatorname*{argmax}_{y \in Y} p \cdot y = \{y \in Y\colon p \cdot y = \pi(p)\}.$$

Resulting profits are given by

$$\pi(p) \equiv \sup_{y \in Y} p \cdot y.$$

Rationalization: profit maximization functions (Producer 9–11, 13)

1. Profit function $\pi(\cdot)$ is rationalized by production set Y iff $\forall p$, $\pi(p) = \sup_{y \in Y} p \cdot y$.
2. Supply correspondence $y(\cdot)$ is rationalized by production set Y iff $\forall p$, $y(p) \subseteq \operatorname{argmax}_{y \in Y} p \cdot y$.
3. $\pi(\cdot)$ or $y(\cdot)$ is rationalizable if it is rationalized by some production set.
4. $\pi(\cdot)$ and $y(\cdot)$ are jointly rationalizable if they are both rationalized by the same production set.

We seek a Y that rationalizes both $y(\cdot)$ and $\pi(\cdot)$. Consider an "inner bound": all production plans the firm chooses must be feasible ($Y^I \equiv \bigcup_{p \in P} y(p)$). Consider an "outer bound": Y can only include points that don't give higher profits than $\pi(p)$ ($Y^O \equiv \{y\colon p \cdot y \le \pi(p) \text{ for all } p \in P\}$).* A nonempty-valued supply correspondence $y(\cdot)$ and profit function $\pi(\cdot)$ on a price set are jointly rationalized by production set Y iff:

1. $p \cdot y = \pi(p)$ for all $y \in y(p)$ (adding-up);
2. $Y^I \subseteq Y \subseteq Y^O$; i.e., $p \cdot y' \le \pi(p)$ for all p, p', and all $y' \in y(p')$ (Weak Axiom of Profit Maximization).

If we observe a firm's choices for all positive price vectors on an open convex set P, then necessary conditions for rationalizability include:

1. $\pi(\cdot)$ must be a convex function;
2. $\pi(\cdot)$ must be homogeneous of degree one; i.e., $\pi(\lambda p) = \lambda \pi(p)$ for all $p \in P$ and $\lambda > 0$;
3. $y(\cdot)$ must be homogeneous of degree zero; i.e., $y(\lambda p) = y(p)$ for all $p \in P$ and $\lambda > 0$.

Loss function (Producer 12) $L(p, y) \equiv \pi(p) - p \cdot y$. This is the loss from choosing y rather than the profit-maximizing feasible production plan. The outer bound can be written $Y^0 = \{y\colon \inf_p L(p, y) \ge 0\}$.

Hotelling's Lemma (Producer 14) $\nabla \pi(p) = y(p)$, assuming differentiable $\pi(\cdot)$. Equivalently, $\nabla_p L(p, y)|_{p=p'} = \nabla \pi(p') - y = 0$ for all $y \in y(p')$. An example of the Envelope Theorem. Implies that if $\pi(\cdot)$ is differentiable at p, then $y(p)$ must be a singleton.

Substitution matrix (Producer 15–6) The Jacobian of the optimal supply function, $Dy(p) = [\partial y_i / \partial p_j]$. By Hotelling's Lemma, $Dy(p) = D^2 \pi(p)$ (the Hessian of the profit function), hence the substitution matrix is symmetric. Convexity of $\pi(\cdot)$ implies positive semidefiniteness.

Law of Supply (Producer 16) $(p' - p) \cdot (y(p') - y(p)) \ge 0$; i.e., supply curves are upward-sloping. Law of Supply is the finite-difference equivalent of PSD of substitution matrix. Follows from WAPM ($p \cdot y(p) \ge p \cdot y(p')$).

Rationalization: $y(\cdot)$ and differentiable $\pi(\cdot)$ (Producer 15) $y\colon P \to \mathbb{R}^n$ (the correspondence ensured to be a function by Hotelling's lemma, given differentiable $\pi(\cdot)$) and differentiable $\pi\colon P \to \mathbb{R}$ on an open convex set $P \subseteq \mathbb{R}^n$ are jointly rationalizable iff

1. $p \cdot y(p) = \pi(p)$ (adding-up);
2. $\nabla \pi(p) = y(p)$ (Hotelling's Lemma);
3. $\pi(\cdot)$ is convex.

The latter two properties imply WAPM. The second describes the FOC of the maximization problem, the third term describes the second-order condition.

Rationalization: differentiable $y(\cdot)$ (Producer 16) Differentiable $y\colon P \to \mathbb{R}^n$ on an open convex set $P \subseteq \mathbb{R}^n$ is rationalizable iff

1. $y(\cdot)$ is homogeneous of degree zero;
2. The Jacobian $Dy(p)$ is symmetric and positive semidefinite.

We construct $\pi(\cdot)$ by adding-up, ensure Hotelling's Lemma by symmetry and homogeneity of degree zero, and ensure convexity of $\pi(\cdot)$ by Hotelling's lemma and PSD.

Rationalization: differentiable $\pi(\cdot)$ (Producer 17) Differentiable $\pi\colon P \to \mathbb{R}$ on a convex set $P \subseteq \mathbb{R}^n$ is rationalizable iff

1. $\pi(\cdot)$ is homogeneous of degree one;
2. $\pi(\cdot)$ is convex.

We construct $y(\cdot)$ by Hotelling's Lemma, and ensure adding-up by homogeneity of degree one; convexity of $\pi(\cdot)$ is given.

*If Y is convex and closed and has free disposal, and $P = \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$, then $Y = Y^O$.

Rationalization: general $y(\cdot)$ and $\pi(\cdot)$ (Producer 17–9) $y\colon P \rightrightarrows \mathbb{R}^n$ and $\pi\colon P \to \mathbb{R}$ on a convex set $P \subseteq \mathbb{R}^n$ are jointly rationalizable iff for any selection $\hat{y}(p) \in y(p)$,

1. $p \cdot \hat{y}(p) = \pi(p)$ (adding-up);
2. (Producer Surplus Formula) For any p, $p' \in P$,
$$\pi(p') = \pi(p) + \int_0^1 (p' - p) \cdot \hat{y}(p + \lambda(p' - p))\, d\lambda;$$
3. $(p' - p) \cdot (\hat{y}(p') - \hat{y}(p)) \geq 0$ (Law of Supply).

Producer Surplus Formula (Producer 17–20) $\pi(p') = \pi(p) + \int_0^1 (p' - p) \cdot \hat{y}(p + \lambda(p' - p))\, d\lambda$.

1. "Works in the opposite direction of Hotelling's Lemma: it recovers the firm's profits from its choices, rather than the other way around."
2. If $\pi(\cdot)$ is differentiable, integrating Hotelling's Lemma along the linear path from p to p' gives PSF; however PSF is more general (doesn't require differentiability of $\pi(\cdot)$).
3. As written the integral is along a linear path, but it is actually path-independent.
4. PSF allows calculation of change in profits when price of good i changes by knowing only the supply function for good i; we need not know the prices or supply functions for other goods: $\pi(p_{-i}, b) - \pi(p_{-i}, a) = \int_a^b \hat{y}_i(p_i)\, dp_i$.

Single-output case (Producer 22) For a single-output firm with free disposal, production set described as $\{(q, -z)\colon z \in \mathbb{R}^m_+,\ q \in [0, f(z)]\}$. With positive output price p, profit-maximization requires $q = f(z)$, so firms maximize $\max_{z \in \mathbb{R}^m_+} pf(z) - w \cdot z$, where $w \in \mathbb{R}^m_+$ input prices.

Cost minimization (Producer 22, Micro P.S. 2-4) For a fixed output level $q \geq 0$, firms minimize costs, choosing inputs according to a conditional factor demand correspondence:

$$\begin{aligned} c(q, w) &\equiv \inf_{z\colon f(z) \geq q} w \cdot z; \\ Z^*(q, w) &\equiv \operatorname*{argmin}_{z\colon f(z) \geq q} w \cdot z \\ &= \{z\colon f(z) \geq q, \text{ and } w \cdot z = c(q, w)\}. \end{aligned}$$

Once these problems are solved, firms solve $\max_{q \geq 0} pq - c(q, w)$.

By the envelope theorem, $\frac{\partial c}{\partial w}(w, q) = Z^*(q, w)$.

Rationalization: single-output cost function (Producer 23, Micro P.S. 2-2) Conditional factor demand function $z\colon \mathbb{R} \times W \rightrightarrows \mathbb{R}^n$ and differentiable cost function $c\colon \mathbb{R} \times W \to \mathbb{R}$ for a fixed output q on an open convex set $W \subseteq \mathbb{R}^m$ of input prices are jointly rationalizable iff

1. $c(q, w) = w \cdot z(q, w)$ (adding-up);
2. $\nabla_w c(q, w) = z(q, w)$ (Shephard's Lemma);
3. $c(q, \cdot)$ is concave.

Other necessary properties follow from corresponding properties of profit-maximization, e.g.,

1. $c(q, \cdot)$ is homogeneous of degree one in w;
2. $Z^*(q, \cdot)$ is homogeneous of degree zero in w;
3. If $Z^*(q, \cdot)$ is differentiable, then the matrix $D_w Z^*(q, w) = D_w^2 c(q, w)$ is symmetric and negative semidefinite;
4. Under free disposal, $c(\cdot, w)$ is nondecreasing in q;
5. If the production function has nondecreasing (nonincreasing) RTS, the average cost function $c(q, w)/q$ is nonincreasing (nondecreasing) in q;
6. If the production function is concave, the cost function $c(q, w)$ is convex in q.

Monopoly pricing (MWG 384–7) Suppose demand at price p is $x(p)$, continuous and strictly decreasing at all p for which $x(p) > 0$. Suppose the monopolist faces cost function $c(q)$. Monopolist solves $\max_p px(p) - c(x(p))$ for optimal price, or $\max_{q \geq 0} p(q)q - c(q)$ for optimal quantity (where $p(\cdot) = x^{-1}(\cdot)$ is the inverse demand function). Further assumptions:

1. $p(q)$, $c(q)$ continuous and twice differentiable at all $q \geq 0$;
2. $p(0) > c'(0)$ (ensures that supply and demand curves cross);
3. There exists a unique socially optimal output level $q^O \in (0, \infty)$ such that $p(q^o) = c'(q^o)$.

A solution $q^m \in [0, q^o]$ exists, and satisfies FOC $p'(q^m)q^m + p(q^m) = c'(q^m)$. If $p'(q) < 0$, then $p(q^m) > c'(q^m)$; i.e., monopoly price exceeds optimal price.

3 Comparative statics

Implicit function theorem (Producer 27–8) Consider $x(t) \equiv \operatorname{argmax}_{x \in X} F(x, t)$. Suppose:

1. F is twice continuously differentiable;
2. X is convex;
3. $F_{xx} < 0$ (strict concavity of F in x; together with convexity of X, this ensures a unique maximizer);
4. $\forall t$, $x(t)$ is in the interior of X.

Then the unique maximizer is given by $F_x(x(t), t) = 0$, and

$$x'(t) = -\frac{F_{xt}(x(t), t)}{F_{xx}(x(t), t)}.$$

Note by strict concavity, the denominator is negative, so $x'(t)$ and $F_{xt}(x(t), t)$ have the same sign.

Envelope theorem (Clayton Producer I 6–8) Given a constrained optimization $v(\theta) = \max_x f(x, \theta)$ such that $g_1(x, \theta) \leq b_1$; ...; $g_K(x, \theta) \leq b_K$, comparative statics on the value function are given by:

$$\frac{\partial v}{\partial \theta_i} = \left.\frac{\partial f}{\partial \theta_i}\right|_{x^*} - \sum_{k=1}^K \lambda_k \left.\frac{\partial g_k}{\partial \theta_i}\right|_{x^*} = \left.\frac{\partial \mathcal{L}}{\partial \theta_i}\right|_{x^*}$$

(for Lagrangian $\mathcal{L}$) for all θ such that the set of binding constraints does not change in an open neighborhood.

Can be thought of as application first of chain rule, and then of FOCs.

Envelope theorem (integral form) (Clayton Producer II 9–10) [a.k.a. Holmstrom's Lemma] Given an optimization $v(q) = \max_x f(x, q)$, the envelope theorem gives us $v'(q) = f_q'(x(q), q)$. Integrating gives

$$v(q_2) = v(q_1) + \int_{q_1}^{q_2} \frac{\partial f}{\partial q}(x(q), q)\, dq.$$

Increasing differences (Producer 30–1, Micro P.S. 2-4') $F\colon X \times T \to \mathbb{R}$ (with X, $T \subseteq \mathbb{R}$) has ID (a.k.a. weakly increasing differences) iff for all $x' > x$ and $t' > t$, $F(x', t') + F(x, t) \geq F(x', t) + F(x, t')$. Strictly/strongly increasing differences (SID) iff $F(x', t') + F(x, t) > F(x', t) + F(x, t')$.

Assuming $F(\cdot, \cdot)$ is sufficiently smooth, all of the following are equivalent:

1. F has ID;
2. $F_x(x, t)$ is nondecreasing in t for all x;
3. $F_t(x, t)$ is nondecreasing in x for all t;
4. $F_{xt}(x, t) \geq 0$ for all (x, t);
5. $F(x, t)$ is supermodular.

Additional results:

1. If $F(\cdot, \cdot)$ and G both have ID, then for all α, $\beta \geq 0$, the function $\alpha F + \beta G$ also has ID.
2. If F has ID, and $g_1(\cdot)$ and $g_2(\cdot)$ are nondecreasing functions, then $F(g_1(\cdot), g_2(\cdot))$ has ID.
3. Suppose $h(\cdot)$ is twice differentiable. Then $h(x - t)$ has ID in x, t iff $h(\cdot)$ is concave.

Supermodularity (Producer 37) $F\colon X \to \mathbb{R}^n$ on a sublattice X is supermodular iff for all x, $y \in X$, we have $F(x \wedge y) + F(x \vee y) \geq F(x) + F(y)$.

If X is a product set, $F(\cdot)$ is supermodular iff it has ID in all pairs (x_i, x_j) with $i \neq j$ (holding other variables x_{-ij} fixed).

Submodularity (Producer 41) $F(\cdot)$ is submodular iff $-F(\cdot)$ is supermodular.

Topkis' Theorem (Producer 31–2, 8) If

1. $F\colon X \times T \to \mathbb{R}$ (with $X, T \subseteq \mathbb{R}$) has ID,
2. $t' > t$,
3. $x \in X^*(t) \equiv \operatorname{argmax}_{\xi \in X} F(\xi, t)$, and $x' \in X^*(t')$, then

$\min\{x, x'\} \in X^*(t)$ and $\max\{x, x'\} \in X^*(t')$. In other words, $X^*(t) \le X^*(t')$ in strong set order. This implies $\sup X^*(\cdot)$ and $\inf X^*(\cdot)$ are nondecreasing; if $X^*(\cdot)$ is single-valued, then $X^*(\cdot)$ is nondecreasing.

If $F\colon X \times T \to \mathbb{R}$ (with X a lattice and T fully ordered) is supermodular in x and has ID in (x, t); $t' > t$; and $x \in X^*(t)$ and $x' \in X^*(t')$, then $(x \wedge x') \in X^*(t)$ and $(x \vee x') \in X^*(t')$. In other words, $X^*(\cdot)$ is nondecreasing in t in the stronger set order.

Monotone Selection Theorem (Producer 32) Analogue of Topkis' Theorem for SID. If $F\colon X \times T \to \mathbb{R}$ with $X, T \in \mathbb{R}$ has SID, $t' > t$, $x \in X^*(t)$, and $x' \in X^*(t')$, then $x' \ge x$.

Milgrom-Shannon Monotonicity Theorem (Producer 34) $X^*(t) \equiv \operatorname{argmax}_{x \in X} F(x, t)$ is nondecreasing in t in SSO for all sets $X \in \mathbb{R}$ iff it has the single-crossing condition (which is non-symmetric): for all $x' > x$ and $t' > t$,

$$F(x', t) \ge F(x, t) \implies F(x', t') \ge F(x, t'), \text{ and}$$
$$F(x', t) > F(x, t) \implies F(x', t') > F(x, t').$$

MCS: robustness to objective function perturbation (Producer 34–5) [Milgrom-Shannon] $X^*(t) \equiv \operatorname{argmax}_{x \in X}[F(x, t) + G(x)]$ is nondecreasing in t in SSO for all functions $G\colon X \to \mathbb{R}$ iff $F(\cdot)$ has ID. Note Topkis gives *sufficiency* of ID.

Complement inputs (Producer 40–2) Restrict attention to price vectors $(p, w) \in \mathbb{R}^{m+1}_+$ at which input demand correspondence $z(p, w)$ is single-valued. If production function $f(z)$ is increasing and supermodular, then $z(p, w)$ is nondecreasing in p and nonincreasing in w. That is, supermodularity of the production function implies price-theoretic complementarity of inputs.

If profit function $\pi(p, w)$ is continuously differentiable, then $z_i(p, w)$ is nonincreasing in w_j for all $i \ne j$ iff $\pi(p, w)$ is supermodular in w.

Substitute inputs (Producer 41–2) *Suppose there are only two inputs.* Restrict attention to price vectors $(p, w) \in \mathbb{R}^3_+$ at which input demand correspondence $z(p, w)$ is single-valued. If production function $f(z)$ is increasing and submodular, then $z_1(p, w)$ is nondecreasing in w_2 and $z_2(p, w)$ is nondecreasing in w_1. That is, submodularity of the production function implies price-theoretic substitutability of inputs *in the two input case.*

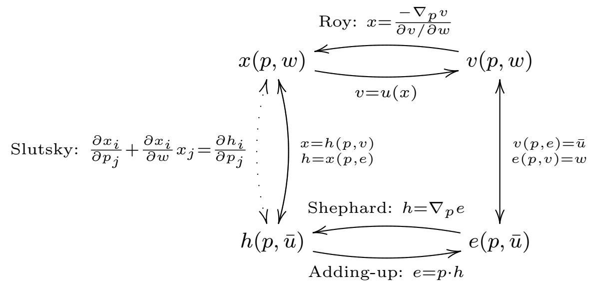
If there are ≥ 3 inputs, feedback between inputs with unchanging prices makes for unpredictable results.

If profit function $\pi(p, w)$ is continuously differentiable, then $z_i(p, w)$ is nondecreasing in w_j for all $i \ne j$ iff $\pi(p, w)$ is submodular in w.

LeChatelier principle (Producer 42–45) Argument (a.k.a Samuelson-LeChatelier principle) that firms react more to input price changes in the long-run than in the short-run, because it has more inputs that it can adjust. Does *not* consistently hold; only holds if each pair of inputs are substitutes everywhere or complements everywhere.

Suppose twice differentiable production function $f(k, l)$ satisfies either $f_{kl} \ge 0$ everywhere, or $f_{kl} \le 0$ everywhere. Then if wage w_l increases (decreases), the firm's labor demand will decrease (increase), and the decrease (increase) will be larger in the long-run than in the short-run.

4 Consumer theory



Budget set (Consumer 2) Given prices p and wealth w, $B(p, w) \equiv \{x \in \mathbb{R}^n_+ \colon p \cdot x \le w\}$.

Utility maximization problem (Consumer 1–2, 6–8) $\max_{x \in \mathbb{R}^n_+} u(x)$ such that $p \cdot x \le w$, or equivalently $\max_{x \in B(p, w)} u(x)$. Assumes:

1. Perfect information,
2. Price taking,
3. Linear prices,
4. Divisible goods.

Construct Lagrangian $\mathcal{L} = u(x) + \lambda(w - p \cdot x) + \sum_i \mu_i x_i$. If u is concave and differentiable, Kuhn-Tucker conditions (FOCs, nonnegativity, complementary slackness) are necessary and sufficient. Thus $\partial u / \partial x_k \le \lambda p_k$ with equality if $x_k > 0$. For any two goods consumed in positive quantities, $p_j / p_k = \frac{\partial u / \partial x_j}{\partial u / \partial x_k} \equiv \mathrm{MRS}_{jk}$. The Lagrange multiplier on the budget constraint λ is the value in utils of an additional unit of wealth; i.e., the shadow price of wealth or marginal utility of wealth or income.

Indirect utility function (Consumer 3–4) $v(p, w) \equiv \sup_{x \in B(p, w)} u(x)$. Homogeneous of degree zero.

Marshallian demand correspondence (Consumer 3–4) [a.k.a. Walrasian or uncompensated demand] $x\colon \mathbb{R}^n_+ \times \mathbb{R}_+ \rightrightarrows \mathbb{R}^n_+$ with $x(p, w) \equiv \{x \in B(p, w) \colon u(x) = v(p, x)\} = \operatorname{argmax}_{x \in B(p, w)} u(x)$.

1. Given continuous preferences, $x(p, w) \ne \varnothing$ for $p \gg 0$ and $w \ge 0$.
2. Given convex preferences, $x(p, w)$ is convex-valued.
3. Given strictly convex preferences, $x(p, w)$ is single-valued.
4. $x(p, w)$ is homogeneous of degree zero.

Walras' Law (Consumer 4) Given locally non-satiated preferences:

1. For $x \in x(p, w)$, we have $p \cdot x = w$ (i.e., Marshallian demand is on budget line, and we can replace inequality constraint with equality in consumer problem);
2. For $z \in z(p)$ (where $z(\cdot)$ is excess demand $z(p) \equiv x(p, p \cdot e) - e$), we have $p \cdot z = 0$;
3. $v(p, w)$ is nonincreasing in p and strictly increasing in w.

Expenditure minimization problem (Consumer 9) $\min_{x \in \mathbb{R}^n_+} p \cdot x$ such that $u(x) \ge \bar{u}$; where $\bar{u} > u(0)$ and $p \gg 0$. Finds the cheapest bundle that yields utility at least $\bar{u}$. Equivalent to cost minimization for a single-output firm with production function u.

If $p \gg 0$, $u(\cdot)$ is continuous, and $\exists \hat{x}$ such that $u(\hat{x}) \ge \bar{u}$, then EMP has a solution.

Expenditure function (Consumer 9) $e(p, \bar{u}) \equiv \min_{x \in \mathbb{R}^n_+} p \cdot x$ such that $u(x) \ge \bar{u}$.

Hicksian demand correspondence (Consumer 9) [a.k.a. compensated demand] $h\colon \mathbb{R}^n_+ \times \mathbb{R}_+ \rightrightarrows \mathbb{R}^n_+$ with $h(p, \bar{u}) \equiv \{x \in \mathbb{R}^n_+ \colon u(x) \ge \bar{u}\} = \operatorname{argmin}_{x \in \mathbb{R}^n_+} p \cdot x$ such that $u(x) \ge \bar{u}$.

Relating Marshallian and Hicksian demand (Consumer 10) Suppose preferences are continuous and locally non-satiated, and $p \gg 0$, $w \ge 0$, $\bar{u} \ge u(0)$. Then:

1. $x(p, w) = h(p, v(p, w))$,
2. $e(p, v(p, w) = w$,
3. $h(p, \bar{u}) = x(p, e(p, \bar{u}))$,
4. $v(p, e(p, \bar{u}) = \bar{u}$.

Rationalization: h and differentiable e (Consumer 11) Hicksian demand function $h\colon P \times \mathbb{R}_+ \to \mathbb{R}^n_+$ and differentiable expenditure function $e\colon P \times \mathbb{R} \to \mathbb{R}$ on an open convex set $P \subseteq \mathbb{R}^n$ are jointly rationalizable by expenditure-minimization for a given utility level $\bar{u}$ of a monotone utility function iff:

1. $e(p, \bar{u}) = p \cdot h(p, \bar{u})$ (adding-up—together with Shephard's Lemma, ensures $e(\cdot, \bar{u})$ is homogeneous of degree one in prices);
2. $\nabla_p e(p, \bar{u}) = h(p, \bar{u})$ (Shephard's Lemma—equivalent to envelope condition applied to $e(p, \bar{u}) = \min_h p \cdot h$);
3. $e(\cdot, \bar{u})$ is concave in prices.

Rationalization: differentiable h (Consumer 12) A continuously differentiable Hicksian demand function $h\colon P \times \mathbb{R}_+ \to \mathbb{R}^n_+$ on an open convex set $P \subseteq \mathbb{R}^n$ is rationalizable by expenditure-minimization with a monotone utility function iff

1. Hicksian demand is increasing in $\bar{u}$; and
2. The Slutsky matrix
$$D_p h(p,\bar{u}) = \begin{bmatrix} \frac{\partial h_1(p,\bar{u})}{\partial p_1} & \cdots & \frac{\partial h_n(p,\bar{u})}{\partial p_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_1(p,\bar{u})}{\partial p_n} & \cdots & \frac{\partial h_n(p,\bar{u})}{\partial p_n} \end{bmatrix}$$
is symmetric,
3. Slutsky matrix is negative semidefinite (since $e(\cdot,\bar{u})$ is concave in prices),
4. Slutsky matrix satisfies $D_p h(p,\bar{u})p = 0$ (i.e., $h(\cdot,\bar{u}$ is homogeneous of degree zero in prices).

Rationalization: differentiable x (?) Slutsky matrix can be generated using Slutsky equation. Rationalizability requires Marshallian demand to be homogeneous of degree 0, and the Slutsky matrix to be symmetric and negative semidefinite. [Potentially also positive everywhere and/or increasing in w?]

Rationalization: differentiable e (?) Rationalizability requires e to be:

1. Homogeneous of degree one in prices;
2. Concave in prices;
3. Increasing in $\bar{u}$;
4. Positive everywhere, or equivalently nondecreasing in all prices.

Slutsky equation (Consumer 13–4) Relates Hicksian and Marshallian demand. Suppose preferences are continuous and locally non-satiated, $p \gg 0$, and demand functions $h(p,\bar{u})$ and $x(p,w)$ are single-valued and differentiable. Then for all i, j,

$$\underbrace{\frac{\partial x_i(p,w)}{\partial p_j}}_{\text{Total}} = \underbrace{\frac{\partial h_i(p,u(x(p,w)))}{\partial p_j}}_{\text{Substitution}} - \underbrace{\frac{\partial x_i(p,w)}{\partial w} x_j(p,w)}_{\text{Wealth}},$$

or more concisely, $\frac{\partial x_i}{\partial p_j} = \frac{\partial h_i}{\partial p_j} - \frac{\partial x_i}{\partial w} x_j$.

Derived by differentiating $h_i(p,\bar{u}) = x_i(p, e(p,\bar{u}))$ with respect to p_j and applying Shephard's lemma.

Normal good (Consumer 15) Marshallian demand $x_i(p,w)$ increasing in w. By Slutsky equation, normal goods must be regular.

Inferior good (Consumer 15) Marshallian demand $x_i(p,w)$ decreasing in w.

Regular good (Consumer 15) Marshallian demand $x_i(p,w)$ decreasing in p_i.

Giffen good (Consumer 15) Marshallian demand $x_i(p,w)$ increasing in p_i. By Slutsky equation, Giffen goods must be inferior.

Substitute goods (Consumer 15–6) Goods i and j substitutes iff Hicksian demand $h_i(p,\bar{u})$ is increasing in p_j. Symmetric relationship. In a two-good world, the goods must be substitutes.

Complement goods (Consumer 15–6) Goods i and j complements iff Hicksian demand $h_i(p,\bar{u})$ is decreasing in p_j. Symmetric relationship.

Gross substitute (Consumer 15–7) Good i is a gross substitute for good j iff Marshallian demand $x_i(p,w)$ is increasing in p_j. *Not* necessarily a symmetric relationship.

Gross complement (Consumer 15–7) Good i is a gross complement for good j iff Marshallian demand $x_i(p,w)$ is decreasing in p_j. *Not* necessarily a symmetric relationship.

Engle curve (Consumer 15–6) [a.k.a. income expansion curve] For a given price p, the locus of Marshallian demands at various wealth levels.

Offer curve (Consumer 16–7) [a.k.a. price expansion path] For a given wealth w and prices (for goods other than good i) p_{-i}, the locus of Marshallian demands at various prices p_i.

Roy's identity (Consumer 17–8) Gives Marshallian demand from indirect utility:

$$x_i(p,w) = -\frac{\partial v(p,w)/\partial p_i}{\partial v(p,w)/\partial w},$$

Derived by differentiating $v(p, e(p,\bar{u})) = \bar{u}$ with respect to p and applying Shephard's lemma. Alternately, by applying envelope theorem to utility maximization problem $v(p,w) = \max_{x\colon p\cdot x \leq w} u(x)$ (giving $\frac{\partial v}{\partial w} = \frac{\partial \mathcal{L}}{\partial w} = \lambda$ and $\frac{\partial v}{\partial p} = \frac{\partial \mathcal{L}}{\partial p} = -\lambda x$.)

Consumer welfare: price changes (Consumer 20-2, 4) Welfare change in utils when prices change from p to p' is $v(p',w) - v(p,w)$, but because utility is ordinal, this is meaningless. More useful to have dollar-denominated measure. So measure amount of additional wealth required to reach some reference utility, generally either previous utility (CV) or new utility (EV).

If preferences are quasi-linear, then CV = EV.

On any range where the good in question is either normal or inferior, $\min\{\text{CV}, \text{EV}\} \leq \text{CS} \leq \max\{\text{CV}, \text{EV}\}$.

Compensating variation (Consumer 21, 3) How much less wealth consumer needs to achieve same utility at prices p' as she had at p (*compensating* for price change—consumer faces both new prices *and* new wealth).

$$\begin{aligned} \text{CV} &\equiv e(p,\bar{u}) - e(p',\bar{u}) \\ &= w - e(p',\bar{u}) \end{aligned}$$

which gives—when only price i is changing—the area to the left of the Hicksian demand curve corresponding to the old utility u by the consumer surplus formula and Shephard's Lemma:

$$= \int_{p'_i}^{p_i} \frac{\partial e(p,\bar{u})}{\partial p_i}\, dp_i = \int_{p'_i}^{p_i} h_i(p,\bar{u})\, dp_i.$$

Equivalent variation (Consumer 21, 3) How much additional expenditure is required at old prices p to achieve same utility as consumption at p' (*equivalent* to price change—consumer faces either new prices *or* revised wealth).

$$\begin{aligned} \text{EV} &\equiv e(p,\bar{u}') - e(p',\bar{u}') \\ &= e(p,\bar{u}') - w \end{aligned}$$

which gives—when only price i is changing—the area to the left of the Hicksian demand curve corresponding to the new utility u' by the consumer surplus formula and Shephard's Lemma:

$$= \int_{p'_i}^{p_i} \frac{\partial e(p,\bar{u}')}{\partial p_i}\, dp_i = \int_{p'_i}^{p_i} h_i(p,\bar{u}')\, dp_i.$$

Marshallian consumer surplus (Consumer 23–4) Area to the left of Marshallian demand curve: $\text{CS} \equiv \int_{p'_i}^{p_i} x_i(p,w)\, dp_i$.

Price index (Consumer 25–6) Given a basket of goods consumed at quantity x given price p, and quantity x' given price p',

1. Laspeyres index: $\frac{p'\cdot x}{p\cdot x} = \frac{p'\cdot x}{e(p,\bar{u})}$ (basket is old purchases). Overestimates welfare effects of inflation due to substitution bias.
2. Paasche index: $\frac{p'\cdot x'}{p\cdot x'} = \frac{e(p',\bar{u}')}{p\cdot x'}$ (basket is new purchases). Underestimates welfare effects of inflation.
3. Ideal index: $\frac{e(p',\bar{u})}{e(p,\bar{u})}$ for some fixed utility level $\bar{u}$, generally either $u(x)$ or $u(x')$; the percentage compensation in the wealth of a consumer with utility $\bar{u}$ needed to make him as well off at the new prices as he was at the old ones.

Paasche $\leq$ Ideal $\leq$ Laspeyres. Substitution biases result from using the same basket of goods at new and old prices. Include

1. New good bias,
2. Outlet bias.

Aggregating consumer demand (Consumer 29–32)

1. Can we predict aggregate demand knowing only aggregate wealth (not distribution)? True iff indirect utility functions take Gorman form: $v_i(p, w_i) = a_i(p) + b(p)w_i$ with the same function $b(\cdot)$ for all consumers.
2. Can aggregate demand be explained as though there were a single "positive representative consumer"?
3. (If 2 holds), can the welfare of the representative consumer be used as a proxy for some welfare aggregate of individual consumers? (i.e., Do we have a "normative representative consumer"?)

5 Choice under uncertainty

Lottery (Uncertainty 2–4) A vector of probabilities adding to 1 assigned to each possible outcome (prize). The set of lotteries for a given prize space is convex.

Preference axioms under uncertainty (Uncertainty 5–6) In addition to (usual) completeness and transitivity, assume preferences are:

1. Continuous: For any p, p', $p'' \in \mathcal{P}$ with $p \succsim p' \succsim p''$, there exists $\alpha \in [0,1]$ such that $\alpha p + (1-\alpha)p'' \sim p'$.
2. Independent: [a.k.a. substitution axiom] For any p, p', $p'' \in \mathcal{P}$ and $\alpha \in [0,1]$, we have $p \succsim p' \iff \alpha p + (1-\alpha)p'' \succsim \alpha p' + (1-\alpha)p''$.

Expected utility function (Uncertainty 6–10) Utility function $u\colon \mathcal{P} \to \mathbb{R}$ has expected utility form iff there are numbers $(u_1, \dots, u_n)$ for each of the n (certain) outcomes such that for every $p \in \mathcal{P}$, $u(p) = \sum_i p_i \cdot u_i$.

Equivalently, for any p, $p' \in \mathcal{P}$, $\alpha \in [0,1]$, we have $u(\alpha p + (1-\alpha)p') = \alpha U(p) + (1-\alpha)U(p')$.

Unlike a more general utility function, an expected utility functions is not merely ordinal—it is *not* invariant to *any* increasing transformation, only to affine transformations. If $u(\cdot)$ is an expected utility representation of $\succsim$, then $v(\cdot)$ is also an expected utility representation of $\succsim$ iff $\exists a \in \mathbb{R}$, $\exists b > 0$ such that $v(p) = a + bu(p)$ for all $p \in \mathcal{P}$.

Preferences can be represented by an expected utility function iff they are complete, transitive, *and* satisfy continuity and independence (assuming $|\mathcal{P}| < \infty$; otherwise we also need the "sure thing principle"). Obtains since both require indifference curves to be parallel straight lines.

Bernoulli utility function (Uncertainty 12) Assuming prize space $\mathcal{X}$ is an interval on the real line, Bernoulli utility function $u\colon \mathcal{X} \to \mathbb{R}$ assumed increasing and continuous.

von Neumann-Morganstern utility function (Uncertainty 12) An expected utility representation of preferences over lotteries characterized by a cdf over prizes $\mathcal{X}$ (an interval on the real line). If $F(x)$ is the probability of receiving less than or equal to x, and $u(\cdot)$ is the Bernoulli utility function, then vN-M utility function $U(F) \equiv \int_{\mathbb{R}} u(x)\,dF(x)$.

Risk aversion (Uncertainty 12–4) A decision maker is (strictly) risk-averse iff for any non-degenerate lottery $F(\cdot)$ with expected value $\mathrm{E}_F = \int_{\mathbb{R}} x\,dF(x)$, the lottery δ_{E_F} which pays E_F for certain is (strictly) preferred to F.

Stated mathematically, $\int u(x)\,dF(x) \le u(\int_{\mathbb{R}} x\,dF(x))$ for all $F(\cdot)$, which by Jensen's inequality obtains iff $u(\cdot)$ is concave.

The following notions of $u(\cdot)$ being "more risk-averse" then $v(\cdot)$ are equivalent:

1. $F \succsim_u \delta_x \implies F \succsim_v \delta_x$ for all F and x.*
2. Certain equivalent $c(F,u) \le c(F,v)$ for all F.
3. $u(\cdot)$ is "more concave" than $v(\cdot)$; i.e., there exists an increasing concave function $g(\cdot)$ such that $u = g \circ v$.
4. Arrow-Pratt coefficient $A(x,u) \ge A(x,v)$ for all x.

Certain equivalent (Uncertainty 13–4) $c(F,u)$ is the certain payout such that $\delta_{c(F,u)} \sim_u F$, or equivalently $u(c(F,u)) = \int_{\mathbb{R}} u(x)\,dF(x)$. Given risk aversion (i.e., concave u), $c(F,u) \le \mathrm{E}_F$.

Absolute risk aversion (Uncertainty 14–6) For a twice differentiable Bernoulli utility function $u(\cdot)$, the Arrow-Pratt coefficient of absolute risk aversion is $A(x) \equiv -u''(x)/u'(x)$.

$u(\cdot)$ has decreasing (constant, increasing) absolute risk aversion iff $A(x)$ is decreasing (...) in x. Under DARA, if I will gamble \$10 when poor, I will gamble \$10 when rich.

Since $R(x) = xA(x)$, we have IARA $\implies$ IRRA.

Certain equivalent rate of return (Uncertainty 16) A proportionate gamble pays tx where t is a non-negative random variable with cdf F. The certain equivalent rate of return is $\mathrm{cr}(F,x,u) \equiv \hat{t}$ where $u(\hat{t}x) = \int u(tx)\,dF(t)$.

Relative risk aversion (Uncertainty 16) For a twice differentiable Bernoulli utility function $u(\cdot)$, the coefficient of relative risk aversion is $R(x) \equiv -xu''(x)/u'(x) = xA(x)$.

$u(\cdot)$ has decreasing (constant, increasing) relative risk aversion iff $R(x)$ is decreasing (...) in x. An agent exhibits DRRA iff certain equivalent rate of return $\mathrm{cr}(F,x)$ is increasing in x. Under DRRA, if I will invest 10% of my wealth in a risky asset when poor, I will invest 10% when rich.

Since $R(x) = xA(x)$, we have DRRA $\implies$ DARA.

First-order stochastic dominance (Uncertainty 17–8) cdf G first-order stochastically dominates cdf F iff $G(x) \le F(x)$ for all x.

Equivalently, for every nondecreasing function $u\colon \mathbb{R} \to \mathbb{R}$, $\int u(x)\,dG(x) \ge \int u(x)\,dF(x)$.

Equivalently, we can construct G as a compound lottery starting with F and followed by (weakly) upward shifts.

Second-order stochastic dominance (Uncertainty 18–21) cdf G second-order stochastically dominates cdf F (where F and G have the same mean†) iff for every x, $\int_{-\infty}^{x} G(y)\,dy \le \int_{-\infty}^{x} F(y)\,dy$.

Equivalently, for every (nondecreasing?) concave function $u\colon \mathbb{R} \to \mathbb{R}$, $\int u(x)\,dG(x) \ge \int u(x)\,dF(x)$.

Equivalently, we can construct F as a compound lottery starting with G and followed by mean-preserving spreads.

Demand for insurance (Uncertainty 21–3) A risk-averse agent with wealth w faces a probability of p of incurring a loss L. She can insure against this loss by buying a policy that will pay out a in the event the loss occurs, at cost qa.

If insurance is actuarially fair ($q = p$), the agent fully insures ($a^* = L$) for all wealth levels. If $p < q$, the agent's insurance coverage a^* will decrease (increase) with wealth if the agent has decreasing (increasing) absolute risk aversion.

Portfolio problem (Uncertainty 23–5) A risk-averse agent with wealth w must choose to allocate investment between a "safe" asset that returns r and a risky asset that pays return z with cdf F.

If risk-neutral, the agent will invest all in the asset with higher expected return (r or $\mathrm{E}\,z$). If (strictly) risk-averse, she will invest at least some in the risky asset as long as its real expected return is positive. (To see why, consider marginal utility to investing in the risky asset at investment $a = 0$.)

If u is more risk-averse than v, then u will invest less in the risky asset than v for any initial level of wealth. An agent with decreasing (constant, increasing) absolute risk aversion will invest more (same, less) in the risky asset at higher levels of wealth.

Subjective probabilities (Uncertainty 26–8) We relax the assumption that there are objectively correct probabilities for various states of the world to be realized. If preferences over acts (bets) satisfy a set of properties "similar in spirit" to the vN-M axioms (completeness, transitivity, something like continuity, the sure thing principle, and two axioms that have the flavor of substitution), then decision makers' choices are consistent with some utility function and some prior probability distribution (Savage 1954).

*Note this does *not* mean $F \succsim_u G \implies F \succsim_v G$ where G is also a risky prospect—this would be a stronger version of "more risk averse."

†If $\mathrm{E}\,F > \mathrm{E}\,G$, there is always a concave utility function that will prefer F to G.

Savage requires an exhaustive list of possible states of the world. No reason to assume different decision makers are using the same implied probability distribution over states, although we often make a "common prior" assumption, which implies that "differences in opinion are due to differences in information."

6 General equilibrium

Walrasian model (G.E. 3–4, 5) An economy $\mathcal{E} \equiv ((u^i, e^i)_{i\in\mathcal{I}})$ comprises:

1. L commodities (indexed $l \in \mathcal{L} \equiv \{1, \dots, L\}$);
2. I agents (indexed $i \in \mathcal{I} \equiv \{1, \dots, I\}$), each with
   - Endowment $e^i \in \mathbb{R}^L_+$, and
   - Utility function $u^i : \mathbb{R}^L_+ \to \mathbb{R}$.

Given market prices $p \in \mathbb{R}^L_+$, each agent chooses consumption to maximize utility given a budget constraint: $\max_{x\in\mathbb{R}^L_+} u^i(x)$ such that $p \cdot x \le p \cdot e^i$, or equivalently $p \in \mathcal{B}^i(p) \equiv \{x : p \cdot x \le p \cdot e^i\}$.

We often assume (some or all of):

1. $\forall i$, $u^i(\cdot)$ is continuous;
2. $\forall i$, $u^i(\cdot)$ is increasing; i.e., $u^i(x') > u^i(x)$ whenever $x' \gg x$;
3. $\forall i$, $u^i(\cdot)$ is concave;
4. $\forall i$, $e^i \gg 0$;

Walrasian equilibrium (G.E. 4, 17, 24–9) A WE for economy $\mathcal{E}$ is a vector of prices and allocations $(p, (x^i)_{i\in\mathcal{I}})$ such that:

1. Agents maximize their utilities: $\max_{x\in\mathcal{B}^i(p)} u^i(x)$ for all $i \in \mathcal{I}$;
2. Markets clear: $\sum_{i\in\mathcal{I}} x^i_l = \sum_{i\in\mathcal{I}} e^i_l$ for all $l \in \mathcal{L}$, or equivalently $\sum_{i\in\mathcal{I}} x^i = \sum_{i\in\mathcal{I}} e^i$.

Under assumptions 1–4 above, a WE exists (proof using fixed point theorem). WE are not generally unique, but are locally unique (and there are an odd number of them). Price adjustment process ("tatonnement") may not converge to an equilibrium.

Feasible allocation (G.E. 4) An allocation $(x^i)_{i\in\mathcal{I}} \in \mathbb{R}^{IL}_+$ is feasible iff $\sum_{i\in\mathcal{I}} x^i \le \sum_{i\in\mathcal{I}} e^i$.

Pareto optimality (G.E. 5) A feasible allocation $x \equiv (x^i)_{i\in\mathcal{I}}$ for economy $\mathcal{E}$ is Pareto optimal iff there is no other feasible allocation $\hat{x}$ such that $u^i(\hat{x}^i) \ge u^i(x^i)$ for all $i \in \mathcal{I}$ with strict inequality for some $i \in \mathcal{I}$.

Edgeworth box (G.E. 6–10) Graphical representation of the two-good, two-person exchange economy. Bottom left corner is origin for one consumer; upper right corner is origin for other consumer (with direction of axes reversed). Budget line has slope $-p_1/p_2$, and passes through endowment e.

1. Locus of Marshallian demands for each consumer as relative prices shift is her *offer curve*. WE are intersections of the two consumers' offer curves.
2. Set of PO allocations is locus of points of tangency between the two consumers' indifference curves, generally a path from the upper right to lower left of the Edgeworth box.
3. Portion of Pareto set that lies between the indifference curves that pass through e is the *contract curve*: PO outcomes preferred by both consumers to their endowments.

First Welfare Theorem (G.E. 11) If $\forall i$, $u^i(\cdot)$ is increasing (i.e., $u^i(x') > u^i(x)$ whenever $x' \gg x$) and $(p, (x^i)_{i\in\mathcal{I}})$ is a WE, then the allocation $(x^i)_{i\in\mathcal{I}}$ is PO. Note implicit assumptions such as

1. All agents face same prices;
2. All agents are price takers;
3. Markes exist for all goods, and individuals can freely participate;
4. Prices are somehow arrived at.

Proof by adding results of Walras' Law across consumer at potentially Pareto-improving allocation. Result shows that allocation cannot be feasible.

Second Welfare Theorem (G.E. 11-3) If allocation $(e^i)_{i\in\mathcal{I}}$ is PO and

1. $\forall i$, $u^i(\cdot)$ is continuous;
2. $\forall i$, $u^i(\cdot)$ is increasing; i.e., $u^i(x') > u^i(x)$ whenever $x' \gg x$;
3. $\forall i$, $u^i(\cdot)$ is concave;
4. $\forall i$, $e^i \gg 0$;

then there exists a price vector $p \in \mathbb{R}^L_+$ such that $(p, (e^i)_{i\in\mathcal{I}})$ is a WE.

Note this does *not* say that starting from a given endowment, every PO allocation is a WE. Thus decentralizing a PO allocation is not simply a matter of identifying the correct prices—large-scale redistribution may be required as well.

Proof by separating hyperplane theorem. Consider the set of changes to total endowment that strictly improve every consumer's utility; by concavity of $u^i(\cdot)$, this set is convex. Separate this set from 0, and show that the resulting prices are nonnegative, and that at these prices, e^i maximizes each consumer's utility.

Excess demand (G.E. 18) $z^i(p) \equiv x^i(p, p\cdot e^i) - e^i$, where x^i is the agent's Marshallian demand. Walras' Law gives $p \cdot z^i(p) = 0$.

Aggregate excess demand is $z(p) \equiv \sum_{i\in\mathcal{I}} z^i(p)$. If $z(p) = 0$, then $(p, (x^i(p, p\cdot e^i))_{i\in\mathcal{I}})$ is a WE.

Sonnenschein-Mantel-Debreu Theorem (G.E. 30) Let $B \subseteq \mathbb{R}^L_{++}$ be open and bounded, and $f : B \to \mathbb{R}^L$ be continuous, homogeneous of degree zero, and satisfy $p \cdot z(p) = 0$ for all p. Then there exist an economy $\mathcal{E}$ with aggregate excess demand function $z(p)$ satisfying $z(p) = f(p)$ on B.

Interpretation is that without special assumptions, pretty much any any comparative statics result could be obtained in a GE model. However, Brown and Matzkin show that if we can observe endowments as well as prices, GE *may* be testable.

Gross substitutes property (G.E. 32–5) A Marshallian demand function $x(p)$ satisfies the gross substitutes property if for all k, whenever $p'_k > p_k$ and $p'_{-k} = p_{-k}$, then $x_{-k}(p') > x_{-k}(p)$; i.e., all pairs of goods are (strict) gross substitutes. Implies that excess demand function satisfies gross substitutes. If every individual satisfies gross substitutes, then so does aggregate excess demand.

If aggregate excess demand satisfies gross substitutes,

1. The economy has at most one (price-normalized) WE.
2. If $z(p^*) = 0$ (i.e., p^* are WE prices), then for any p not colinear with p^*, we have $p^* \cdot z(p) > 0$.
3. The tatonnement process $\frac{dp}{dy} = \alpha z(p(t))$ with $\alpha > 0$ converges to WE prices for any initial condition $p(0)$.
4. Any change that raises the excess demand for good k will increase the equilibrium price of good k.

7 Mathematics

Elasticity (Wikipedia) Elasticity of $f(x)$ is

$$\left|\frac{\partial \log f(x)}{\partial \log x}\right| = \left|\frac{\partial f}{\partial x} \cdot \frac{x}{f(x)}\right| = \left|\frac{\partial f/f}{\partial x/x}\right| = \left|\frac{x f'(x)}{f(x)}\right|.$$

Euler's law (Producer 14) If f is differentiable, it is homogeneous of degree k iff $p \cdot \nabla f(p) = k f(p)$.

One direction proved by differentiating $f(\lambda p) = \lambda^k f(p)$ with respect to λ, and then setting $\lambda = 1$. Implies that if f is homogeneous of degree one, then ∇f is homogeneous of degree zero.

Strong set order (Producer 32) $A \le B$ in the strong set order iff for all $a \in A$ and $b \in B$ with $a \ge b$, then $a \in B$ and $b \in A$. Equivalently, every element in $A \setminus B$ is $\le$ every element in $A \cap B$, which is $\le$ every element in $B \setminus A$.

Meet (Producer 36) For $x, y \in \mathbb{R}^n$, the meet is $x \wedge y \equiv (\min\{x_1, y_1\}, \dots, \min\{x_n, y_n\})$. More generally, on a partially ordered set, $x \wedge y$ is the greatest lower bound of x and y.

Join (Producer 36) For x, $y \in \mathbb{R}^n$, the meet is $x \vee y \equiv (\max\{x_1, y_1\}, \dots, \max\{x_n, y_n\})$. More generally, on a partially ordered set, $x \vee y$ is the least upper bound of x and y.

Sublattice (Producer 37) A set X is a sublattice iff $\forall x, y \in X$, we have $x \wedge y \in X$ and $x \vee y \in X$. Any sublattice in $\mathbb{R}^n$ can be described as an intersection of sets of the forms

1. A product set $X_1 \times \cdots \times X_n$; or
2. A set $\{(x_1, \dots, x_n) \colon x_i \leq g(x_j)\}$, where $g(\cdot)$ is an increasing function.

Orthants of Euclidean space (?)

1. $\mathbb{R}^n_+ \equiv \{\mathbf{x} \colon \mathbf{x} \geq \mathbf{0}\} \equiv \{\mathbf{x} \colon x_i \geq 0 \, \forall i\}$, which includes the axes and $\mathbf{0}$.
2. $\{\mathbf{x} \colon \mathbf{x} > \mathbf{0}\} \equiv \{\mathbf{x} \colon x_i \geq 0 \, \forall i\} \setminus \mathbf{0}$, which includes the axes, but not $\mathbf{0}$.
3. $\mathbb{R}^n_{++} \equiv \{\mathbf{x} \colon \mathbf{x} \gg \mathbf{0}\} \equiv \{\mathbf{x} \colon x_i > 0 \, \forall i\}$, which includes neither the axes nor $\mathbf{0}$.

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