STATS 300A Theory of Statistics
Stanford University, Fall 2015

Problem Set 5

Due: Thursday, November 5, 2015

Instructions:

- You may appeal to any result proved in class or proved in the course textbooks.
- Any request to "find" requires proof that all requested properties are satisfied.

Problem 1. Let $X = (X_1, \dots, X_n)$ be an i.i.d. sample from $\mathrm{Unif}[\theta - \frac{1}{2}, \theta + \frac{1}{2}]$ where $\theta \in \mathbb{R}$ is unknown. Show that $\delta(X) = \frac{X_{(1)} + X_{(n)}}{2}$ is a minimax estimator under squared error loss. [Hint: You may find the proof of TPE 5.3.5 relevant.]

Problem 2. Suppose that X is a random variable with probability mass function

$$p(x; \theta) = (1 - \theta)\theta^x$$

for $x \in \{0, 1, 2, \dots\}$ and unknown $\theta \in [0, 1)$. Consider the loss function

$$L(\theta, d) = (\theta - d)^2/(1 - \theta),$$

and the decision space $\mathcal{D} = [0, 1]$.

(a) Write the risk function $R(\theta, \delta)$ as a power series in θ.

(b) Show that the only (non-randomized) estimator with constant risk $R(\theta, \delta)$ is given by $\delta(0) = \frac{1}{2}$ and $\delta(i) = 1$ for $i \geq 1$.

(c) Suppose that Λ is a prior distribution that does not place all mass on 0. Show that an estimator is Bayes with respect to Λ if and only if $\delta(0) = \mu_1$ and $\delta(i) = \mu_{i+1}/\mu_i$ for $i \geq 1$, where $\mu_i = \mathbb{E}_\Lambda[\Theta^i]$ is the i-th moment of Θ under Λ.

(d) Use your work to find a minimax estimator of θ.

Problem 3 (TPE 5.1.15)**.** Note that $p + q = 1$ in this problem. The hint for part (b) should read, "The risk of δ from part (a) at p = 0, 1/2, 1 is, respectively, $P(U \geq 1/4), 1/2[P(U \leq 1/4) + P(V \geq 3/4)]$, and $P(V \leq 3/4)$."

Problem 4 (TPE 5.1.24)**.** If you choose, you may solve this problem assuming $F(1) = G(1) = 1$ and $F(-\epsilon) = G(-\epsilon) = 0$ for all $\epsilon > 0$ instead of the more restrictive constraint $F(1) - F(0) = G(1) - G(0) = 1$.

Problem 5 (TPE 5.4.5)**.** In part (c), the undefined symbol π_n is g_n from part (b).