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Math 2A
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Lecture 15b Examples

These examples correspond to Sections 3.7 and 3.9 in the text.

Example Given a curve that is described by the equation

$$(x^2 + y)^2 = 10(x^2 - y^3),$$

compute the equation of the tangent line to the curve at the point $(\sqrt{10}, 0)$.

Solution We begin by differentiating both sides of the equation that describes the curve with respect to x. Differentiating the left side, using the Power Rule and the Chain Rule, yields

$$\frac{d}{dx}[(x^2 + y)^2] = 2(x^2 + y)\frac{d}{dx}[x^2 + y] = 2(x^2 + y)\left(2x + \frac{dy}{dx}\right).$$

To differentiate the right side, we use the Constant Multiple Rule, the Sum Rule, the Power Rule and the Chain Rule to obtain

$$\frac{d}{dx}[10(x^2 - y^3)] = 10\frac{d}{dx}[x^2 - y^3] = 10\left(2x - 3y^2\frac{dy}{dx}\right).$$

Having differentiated both sides, we now have an equation that describes dy/dx,

$$2(x^2 + y)\left(2x + \frac{dy}{dx}\right) = 10\left(2x - 3y^2\frac{dy}{dx}\right).$$

We now need to solve this equation for dy/dx. Expanding both sides, we have

$$2(x^2)(2x) + 2y(2x) + 2x^2\frac{dy}{dx} + 2y\frac{dy}{dx} = 10(2x) - 10(3y^2)\frac{dy}{dx},$$

or

$$4x^3 + 4xy + 2x^2\frac{dy}{dx} + 2y\frac{dy}{dx} = 20x - 30y^2\frac{dy}{dx}.$$

We move all terms involving dy/dx to the left side, and all terms not involving dy/dx to the right side, to obtain

$$2x^2\frac{dy}{dx} + 2y\frac{dy}{dx} + 30y^2\frac{dy}{dx} = -4x^3 - 4xy + 20x$$

or

$$[2x^2 + 2y + 30y^2]\frac{dy}{dx} = -4x^3 - 4xy + 20x.$$

It follows that

$$\frac{dy}{dx} = \frac{-4x^3 - 4xy + 20x}{2x^2 + 2y + 30y^2}.$$

To obtain the slope of the tangent line at the point $(\sqrt{10}, 0)$, we substitute $\sqrt{10}$ for x and 0 for y and obtain

$$\frac{dy}{dx} = \frac{-4(\sqrt{10})^3 + 20(\sqrt{10})}{2(\sqrt{10})^2} = \frac{-4(10)\sqrt{10} + 20\sqrt{10}}{2(10)} = \frac{-40\sqrt{10} + 20\sqrt{10}}{20} = \frac{-20\sqrt{10}}{20} = -\sqrt{10}.$$

We can now obtain the equation of this tangent line using the point-slope form,

$$y - y_0 = m(x - x_0),$$

where m is the slope, $-\sqrt{10}$, and (x_0, y_0) is the point of tangency, with $x_0 = \sqrt{10}$ and $y_0 = 0$. Substituting the values of m, x_0 and y_0 yields the equation

$$y = -\sqrt{10}(x - \sqrt{10}) = -\sqrt{10}x + 10.$$

□

Example Suppose that the length of the side of a perfect cube is increasing at the rate of 5 cm/s. How fast is the volume of the cube changing when the side length is 10 cm?

Solution We follow the general approach to solving related rates problems:

1. *We identify all relevant quantities in the problem.* Let s denote the side length of the cube, V denote the volume of the cube, and t denote time. While time is not explicitly mentioned, it is implied by the mention of the rate of change of the side length, which is in cm/s.

2. *We identify what exactly needs to be computed.* The problem asks for the rate of change of the volume of the cube with respect to time, which is dV/dt, when the side length is 10 cm; that is, $s = 10$.

3. *We identify relationships among the quantities in the problem from the problem statement.* We are given that the side length, s, is increasing at the rate of 5 cm/s, which implies that $ds/dt = 5$. Furthermore, the side length s of the cube and the volume V of the cube are related by the formula for the volume of a cube; that is, $V = s^3$.

4. *We differentiate relationships among quantities implicitly with respect to the independent variable of our desired rate of change.* Our goal is to compute dV/dt, and the independent variable in this rate of change is t. Therefore, we differentiate relationships among the quantities in our problem with respect to t. This will indicate how the various rates of change in our problem are actually related to one another, which, in turn, will enable us to compute the desired rate of change.

In this problem, we only have one such relationship to differentiate with respect to t, $V = s^3$. We have

$$\frac{dV}{dt} = \frac{d}{dt}[s^3] = 3s^2\frac{ds}{dt}.$$

5. *We use given information to compute the desired rate of change at the indicated point.* We can see that our desired rate of change, dV/dt, can be obtained from our knowledge of ds/dt and s. We are given that $ds/dt = 5$, and are asked to compute dV/dt when $s = 10$. We have

$$\frac{dV}{dt} = 3s^2\frac{ds}{dt} = 3(10^2)(5) = 1500 \text{ cm}^3/\text{s},$$

and our solution is complete.

□

Example The volume of the box is changing at a rate of 100 cm^3/s, while the length remains constant and the width is changing at a rate of 4 cm/s. How fast is the height of the box changing when the volume is 2000 cm^3, the width is 8 cm, and the height is 20 cm?

Solution As in the previous example, we follow the general approach to solving related rates problems:

1. *We identify all relevant quantities in the problem.* Let ℓ denote the length of the box, w denote its width, h denote its height, and V denote its volume. Again, we let t denote time. While time is not explicitly mentioned, it is implied by the mention of the rate of change of the width, which is in cm/s.

2. *We identify what exactly needs to be computed.* The problem asks for the rate of change of the height of the box with respect to time, which is dh/dt, when $V = 2000$, $w = 8$, and $h = 20$.

3. *We identify relationships among the quantities in the problem from the problem statement.* We are given that the volume, V, is increasing at the rate of 100 cm^3/s, which implies that $dV/dt = 100$. Furthermore, the length, ℓ, remains constant, so we have $d\ell/dt = 0$. In addition, the width, w is changing at the rate of 4 cm/s, so $dw/dt = 4$. Finally, the dimensions ℓ, w and h of the box and the volume V of the box are related by the formula for the volume of a box; that is, $V = \ell wh$.

4. *We differentiate relationships among quantities implicitly with respect to the independent variable of our desired rate of change.* Our goal is to compute dh/dt, and the independent variable in this rate of change is t. Therefore, we differentiate relationships among the quantities in our problem with respect to t. This will indicate how the various rates of change in our problem are actually related to one another, which, in turn, will enable us to compute the desired rate of change.

In this problem, we only have one such relationship to differentiate with respect to t, $V = \ell wh$. We have, by the Product Rule,

$$\begin{aligned}
\frac{dV}{dt} &= \frac{d}{dt}[\ell wh] \\
&= \frac{d}{dt}[\ell(wh)] \\
&= wh\frac{d\ell}{dt} + \ell\frac{d}{dt}[wh] \\
&= wh\frac{d\ell}{dt} + \ell\left[h\frac{dw}{dt} + w\frac{dh}{dt}\right] \\
&= wh\frac{d\ell}{dt} + \ell h\frac{dw}{dt} + \ell w\frac{dh}{dt}.
\end{aligned}$$

5. *We use given information to compute the desired rate of change at the indicated point.* We can see that our desired rate of change, dh/dt, can be obtained from our knowledge of dV/dt, dw/dt, $d\ell/dt$, V, w and h. We are given that $dV/dt = 100$, $d\ell/dt = 0$, and $dw/dt = 4$, and are asked to compute dh/dt when $V = 2000$, $w = 8$, and $h = 20$. From

$$\frac{dV}{dt} = wh\frac{d\ell}{dt} + \ell h\frac{dw}{dt} + \ell w\frac{dh}{dt}$$

we have

$$100 = 8(20)(0) + \ell(20)(4) + \ell 8\frac{dh}{dt},$$

which simplifies to

$$100 = 80\ell + 8\ell\frac{dh}{dt}.$$

We are not given ℓ, but we can obtain ℓ from the formula $V = \ell wh$ and the fact that $V = 2000$, $w = 8$ and $h = 20$. This yields

$$\ell = \frac{V}{wh} = \frac{2000}{8(20)} = \frac{100}{8} = 12.5 \text{ cm}.$$

It follows that

$$100 = 80(12.5) + 8(12.5)\frac{dh}{dt},$$

which becomes

$$100 = 1000 + 100\frac{dh}{dt}$$

and therefore

$$\frac{dh}{dt} = \frac{100 - 1000}{100} = \frac{-900}{100} = -9 \text{ cm/s},$$

and our solution is complete.

□

Example Suppose that $y = \cos x$ and $z = (x^2+1)^2$. Compute dy/dz when $x = 3$.

Solution As in the previous example, we follow the general approach to solving related rates problems:

1. *We identify all relevant quantities in the problem.* In this problem, the quantities are already identified; they are x, y and z.

2. *We identify what exactly needs to be computed.* This is already explicitly prescribed in the problem statement; we are to compute dy/dz when $x = 3$.

3. *We identify relationships among the quantities in the problem from the problem statement.* These relationships are already given in the problem statement; x and y are related by the equation $y = \cos x$, and x and z are related by the equation $z = (x^2+1)^2$.

4. *We differentiate relationships among quantities implicitly with respect to the independent variable of our desired rate of change.* Our goal is to compute dy/dz, and the independent variable in this rate of change is z. Therefore, we differentiate relationships among the quantities in our problem with respect to z. This will indicate how the various rates of change in our problem are actually related to one another, which, in turn, will enable us to compute the desired rate of change.

   In this problem, we have two such relationship to differentiate with respect to z, $y = \cos x$ and $z = (x^2+1)^2$. Differentiating $y = \cos x$ implicitly with respect to z yields, by the Chain Rule,

   $$\frac{dy}{dz} = \frac{d}{dz}[\cos x] = -\sin x \frac{dx}{dz}.$$

   The left side of this new equation has our desired rate of change, dy/dz, but we have no information about dx/dz. We can obtain information about this rate of change by differentiating our other known relationship, $z = (x^2+1)^2$, with respect to z. Using the Power Rule and the Chain Rule, we obtain

   $$\frac{dz}{dz} = \frac{d}{dz}[(x^2+1)^2] = 2(x^2+1)\frac{d}{dz}[x^2+1] = 2(x^2+1)(2x)\frac{dx}{dz}.$$

   Simplifying yields the equation

   $$1 = 4x(x^2+1)\frac{dx}{dz},$$

   or

   $$\frac{dx}{dz} = \frac{1}{4x(x^2+1)}.$$

5. *We use given information to compute the desired rate of change at the indicated point.* We can see that our desired rate of change, dy/dz, can be obtained from our knowledge of dx/dz and x. We are asked to compute dy/dz when $x = 3$. From

$$\frac{dy}{dz} = -\sin x \frac{dx}{dz}$$

and

$$\frac{dx}{dz} = \frac{1}{4x(x^2+1)}$$

we have

$$\frac{dy}{dz} = -\sin x \frac{1}{4x(x^2+1)}.$$

Substituting $x = 3$ yields

$$\frac{dy}{dz} = -\frac{\sin 3}{4(3)(3^2+1)} = -\frac{\sin 3}{120} \approx -0.001176,$$

and our solution is complete.

It is worthwhile to note that we can differentiate the relationships given in the problem statement with respect to x and obtain

$$\frac{dy}{dx} = -\sin x, \quad \frac{dz}{dx} = 4x(x^2+1).$$

Earlier, we found that

$$\frac{dy}{dz} = \frac{-\sin x}{4x(x^2+1)}$$

which implies

$$\frac{dy}{dz} = \frac{dy/dx}{dz/dx},$$

which can also be obtained directly from the Chain Rule, which states that

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}.$$

□