FISCAL CRISIS RESOLUTION: TAXATION VERSUS INFLATION

Technical Appendix

Michael Kumhof, Stanford University

January 20, 2004

The following sections describe the computable general equilibrium methods used to obtain the solutions presented in Figures 4-8 of the paper. Section 1 deals with the steady state equilibria presented in Figure 4. Section 2 discusses the computations underlying Figures 5-7 for Ramsey plans with constant tax rates. Section 3 discusses the computations used to prepare Figure 8 using the primal approach to optimal taxation.

For simplicity it is assumed that the optimal quantity of money is attained at a near zero level of real money balances. Money is therefore disregarded in the following computations.

1 Steady States

This program repeatedly computes, for a given level of government spending $g = 0.1$, the steady states of the competitive equilibria associated with different tax rates τ. The Friedman rule is assumed to be followed at all times. Steady state values are denoted by a bar above the respective variable. To satisfy the nonnegativity part of the Kuhn-Tucker condition, two squared auxiliary terms are introduced, $\tilde{a}^2$ and $\tilde{\mu}^2$, with $\bar{\mu} = \tilde{\mu}^2$.

Steady state competitive equilibria are determined by the steady state versions of household optimality conditions, the Kuhn-Tucker conditions for the financial constraint, the government budget constraint, and the market clearing conditions for goods. The latter two imply by Walras' Law that the household budget constraint is satisfied.

The steady state versions of the household's first-order conditions for consumption, leisure and labor input can be combined with the market clearing conditions for goods as follows:

$$\frac{\kappa}{(1-\kappa)} \frac{\bar{l}^{\alpha}\bar{k}^{1-\alpha} - \delta\bar{k}}{(1-\bar{l})(1-\tau)} = \alpha \left(\frac{\bar{k}}{\bar{l}}\right)^{1-\alpha} . \tag{1}$$

The steady state first-order conditions for savings and capital combined with the first order condition for consumption, and again combined with goods market clearing, yield

$$(r+\delta) + \frac{\gamma}{1-\kappa}\tilde{\mu}^2(\bar{l}^{\alpha}\bar{k}^{1-\alpha} - \delta\bar{k}) = (1-\alpha)\left(\frac{\bar{l}}{\bar{k}}\right)^{\alpha} . \tag{2}$$

The first order condition for bonds is combined with that for consumption and with goods market clearing to get

$$\tilde{\mu}^2(1+\bar{r}^g)(\bar{l}^{\alpha}\bar{k}^{1-\alpha} - \delta\bar{k}) = (1-\kappa)(r-\bar{r}^g) . \tag{3}$$

The steady state version of the government's budget constraint is

$$\left(\frac{\bar{r}^g}{1+\bar{r}^g}\right)\bar{d} = \tau\alpha\bar{l}^{\alpha}\bar{k}^{1-\alpha} - g . \tag{4}$$

Finally, the Kuhn-Tucker condition for the financial constraint is

$$\tilde{\mu}^2\tilde{a}^2 = 0 , \tag{5}$$

with $\tilde{a}$ defined by

$$\tilde{a}^2 = (\bar{d} - \gamma \bar{k}) \; . \tag{6}$$

This is a system of six equations in six unknown variables, $\bar{l}$, $\bar{k}$, $\bar{d}$, $\bar{r}^g$, $\tilde{\mu}$ and $\tilde{a}$. To produce Figure 4, it is solved repeatedly for $\tau \in [0.15, 0.25]$ using a nonlinear equation solving algorithm.

2 Ramsey Problem with Constant Tax Rate

This program fixes an initial condition for debt and capital and then searches over possible competitive equilibria by varying the tax rate. Welfare is evaluated for each tax rate. The dynamic solution paths for the Ramsey equilibria shown in Figures 5 and 6 are produced using the welfare maximizing tax rate.

The requirement of government intertemporal solvency imposes a constraint on endogenous variables over the entire infinite horizon, making the problem of computing Ramsey equilibria non-recursive. The presence of capital and especially of capital stock adjustment costs means that today's optimal choices are affected by choices of endogenous variables many periods ahead, not just one period ahead as is common in economies without capital. Conventional solution methods can therefore not be easily applied.

The computable general equilibrium method used for this paper takes care of this problem for a nonstochastic economy. The idea is to simulteneously solve all equations characterizing a competitive equilibrium for a given tax rate (and given the Friedman rule) and over the infinite horizon. The equations include first order conditions, resource and budget constraints. The (infinite) dimensionality of this problem is limited by imposing and then verifying convergence of this system to a new steady state within a finite number of periods. The non-recursive nature of the problem is addressed by making the government's lifetime budget constraint one of the simultaneously solved for equations. This method solves for the solution paths of all variables, including their new steady state values, while ensuring fiscal solvency at all times.

It is found necessary to assume that, following a shock, the economy will converge to its new steady state within 80 periods. It is not possible to pin down a priori the new steady state values of any variables except for the shadow price of capital, which of course is $q_{80} = 1$. The government budget constraint implies that the final steady state real interest rate in period 80 is r_{81}^g. As in the previous section, there are two auxiliary variables, with $\tilde{\mu}_t^2 = \mu_t$ and $\tilde{a}_t^2 = d_t - \gamma k_t$. The variables to be determined are therefore $\left\{c_t, l_t, k_t, I_t, d_t, r_t^k, \tilde{\mu}_t, \tilde{a}_t\right\}_{t=1}^{80}$, $\{r_t^g\}_{t=2}^{81}$, and $\{q_t\}_{t=1}^{79}$. There are therefore a total of 799 variables. The processes for government transfer payments and the tax rate, $\{g_t\}_{t=1}^{80}$ and τ, are taken as exogenous. The initial conditions for the debt and capital stocks are given by d_0 and k_0. The following equations (7) - (17) show the system of equations used to determine the 799 endogenous variables.

The first equation combines the household's first-order conditions for consumption, leisure and labor input:

$$\frac{\kappa}{1-\kappa}\frac{c_t}{(1-l_t)} - \alpha\left(\frac{k_t}{l_t}\right)^{1-\alpha}(1-\tau_t) = 0 \quad \text{(80 equations)} . \tag{7}$$

There are four equations relating to capital

$$q_t - 1 - \xi\left(\frac{I_t - \delta k_t}{k_t^2}\right) = 0 \quad \text{(80 equations)} , \tag{8}$$

$$k_{t+1} - I_t - (1-\delta)k_t = 0 \text{ and } k_1 = k_0 \quad \text{(80 equations)} , \tag{9}$$

$$q_t(1+r)\frac{c_{t+1}}{c_t} - q_{t+1}(1-\delta) - (q_{t+1}-1)\frac{I_{t+1}}{k_{t+1}} - r_{t+1}^k = 0 \quad \text{(79 equations)} , \tag{10}$$

$$r_t^k + \frac{\gamma}{1-\kappa}c_t\tilde{\mu}_t^2 - (1-\alpha)\left(\frac{l_t}{k_t}\right)^{\alpha} = 0 \quad \text{(80 equations)} . \tag{11}$$

The two equations relating to debt are

$$\frac{d_{t+1}}{(1+r_{t+1}^g)} - d_t - g_t + \alpha\tau_t l_t^{\alpha}k_t^{1-\alpha} = 0 \text{ and } d_1 = d_0 \quad \text{(80 equations)} , \tag{12}$$

$$(1+r)\frac{c_{t+1}}{c_t} - (1+r^g_{t+1})\left(1+\tilde{\mu}^2_{t+1}\frac{c_{t+1}}{1-\kappa}\right) = 0 \quad \text{(79 equations)}. \tag{13}$$

The Kuhn-Tucker conditions are given by

$$\tilde{a}^2_t - d_t + \gamma k_t = 0 \quad \text{(80 equations)} \ , \tag{14}$$

$$\tilde{\mu}^2_t \tilde{a}^2_t = 0 \quad \text{(80 equations)}. \tag{15}$$

The market clearing condition for goods is

$$l^{\alpha}_t k^{1-\alpha}_t - c_t - I_t - \frac{\xi}{2}\left(\frac{I_t}{k_t} - \delta\right)^2 = 0 \quad \text{(80 equations)}. \tag{16}$$

This is a total of 798 equations. The system is incomplete because the entire infinite horizon budget constraint of the government must be taken into account. The first 80 periods of that constraint are contained in equations (12). The constraint for the terminal period that needs to be added is

$$\left(\frac{1+r^g_{81}}{r^g_{81}}\right)\left(\alpha\tau_{80} l^{\alpha}_{80} k^{1-\alpha}_{80} - g_{80}\right) - d_{80} = 0 \text{ (1 equation)}. \tag{17}$$

This system is solved with a nonlinear equation solving algorithm. Given the very large number of equations, excellent starting values are critical for this exercise. These are generated by first solving the system at the original steady state, with the starting values taken from the solutions in Section 1 above but substituting the new path of government spending $\{g_t\}^{\infty}_{t=0}$. It is found that this system always converges, and the procedure thereafter is to iterate tax rates away from the original value. Starting values are updated by using the solutions of the previous iteration.

Let the (799x1) solution vector be z. To test the conjecture of convergence within 80 periods, the procedure is repeated for a higher number of periods, specifically for 85. The last 5 observations are dropped from the new solution vectors for each variable, and the vectors are arranged into an overall (799x1) solution vector z' whose elements correspond to those of z. Convergence is tested by checking whether the element-by-element maximum absolute percentage difference between z and z' is less than 0.01% For the choice of parameter values presented in the paper this condition is found to be satisfied. The conclusion is that convergence is achieved in at most 80 periods.

3 Ramsey Problem with Arbitrary Tax Rates

There are two differences to the constant tax case discussed in the previous section. First, the tax rate is now allowed to be different in each period, so that the solution of this Ramsey problem is a sequence of taxes $\left\{\tau_t^{opt}\right\}_{t=0}^{\infty}$. Second, the primal approach to optimal taxation (Chari and Kehoe (1999)) is used to obtain solutions. The financial constraint is assumed to hold with equality, and solutions are checked to verify that this is indeed satisfied in equilibrium. In the primal approach the government chooses optimal allocations, after all prices and taxes are substituted out of the household budget constraint using the optimality conditions of the competitive equilibrium. The modified household budget constraint can then be shown to have the following form:

$$\begin{aligned} & \frac{1}{1+r}\frac{1}{c_{t+1}}\left[(1-\alpha)l_{t+1}^{\alpha}k_{t+1}^{1-\alpha}+k_{t+1}(1-\delta+\gamma)+\xi\frac{k_{t+2}^2}{k_{t+1}^2}-\xi\frac{k_{t+2}}{k_{t+1}}\right] \qquad (18) \\ = \; & \frac{1}{c_t}\left[(1-\alpha)l_t^{\alpha}k_t^{1-\alpha}+k_t(1-\delta+\gamma)+\frac{\xi}{2}\frac{k_{t+1}^2}{k_t^2}-\frac{\xi}{2}\right]+\frac{g_t}{c_t}+\frac{\kappa}{1-\kappa}\frac{l_t}{1-l_t}-1 \; . \end{aligned}$$

The multiplier of this constraint in the Ramsey problem is denoted by χ_t. The goods market clearing condition is given by

$$l_t^{\alpha}k_t^{1-\alpha}=c_t+(k_{t+1}-(1-\delta)k_t)+\frac{\xi}{2}\left(\frac{k_{t+1}-k_t}{k_t}\right)^2 \quad . \qquad (19)$$

The system of first-order conditions then consists of (18), (19) and of the following Ramsey optimality conditions for consumption (20), labor (21) and capital (22):

$$\begin{aligned} & \frac{1-\kappa}{c_t}+\chi_t\frac{g_t}{c_t^2}+(\chi_t-\chi_{t-1})\left[\frac{(1-\alpha)l_t^{\alpha}k_t^{1-\alpha}+k_t(1-\delta+\gamma)}{c_t^2}\right] \\ & -\theta_t+\frac{\xi}{c_t^2k_t^2}\left[\frac{1}{2}\chi_t(k_{t+1}^2-k_t^2)-\chi_{t-1}(k_{t+1}^2-k_tk_{t+1})\right]=0 \; . \qquad (20) \end{aligned}$$

$$\frac{\kappa}{1-l_t}+\chi_t\frac{\kappa}{1-\kappa}\frac{1}{(1-l_t)^2}+(\chi_t-\chi_{t-1})\left[\frac{(1-\alpha)\alpha\left(\frac{l_t}{k_t}\right)^{\alpha-1}}{c_t}\right]-\theta_t\alpha\left(\frac{l_t}{k_t}\right)^{\alpha-1}=0\ . \quad (21)$$

$$(\chi_t-\chi_{t-1})\left[\frac{(1-\alpha)^2\left(\frac{l_{t+1}}{k_{t+1}}\right)^{\alpha}+(1-\delta+\gamma)}{(1+r)c_{t+1}}\right]-\theta_t+\frac{1}{1+r}\theta_{t+1}\left[(1-\alpha)\left(\frac{l_{t+1}}{k_{t+1}}\right)^{\alpha}+1-\delta\right]$$
$$+\frac{\xi}{c_tk_t^2}\left[\chi_{t-1}(2k_{t+1}-k_t)-\chi_tk_{t+1}-\theta_tc_t(k_{t+1}-k_t)\right]$$
$$+\frac{\xi}{(1+r)c_{t+1}k_{t+1}^3}\left[\chi_t(k_{t+2}k_{t+1}-2k_{t+2}^2)+\chi_{t+1}k_{t+2}^2+\theta_{t+1}c_{t+1}(k_{t+2}^2-k_{t+1}k_{t+2})\right]=0\ . \quad (22)$$

This system of equations is solved in an analogous fashion to that outlined in the previous section. In this case a system of 400 simultaneous equations is solved using a computable general equilibrium method.