Optimal Price Setting and Inflation Inertia in a Rational Expectations Model

TECHNICAL APPENDIX

(Not for Publication)

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1 UNIT ROOT SHOCKS AND SCALING OF VARIABLES

- Shocks to labor-augmenting technology S_t^y and to the inflation target π_t^* are a key feature of the model.
- To make the model stationary, this requires the following:
  - All real variables must be rescaled by the level of technology S_t^y.
  - All nominal variables must be rescaled by the target price level P_t^*.
- As we will do this rescaling throughout this Appendix rather than at the end, we present these two processes first.
- Technology Shocks:
  - In Levels:

$$S_t^y = S_{t-1}^y g_t$$
$$g_t = g_t^{gr} g_t^{lev}$$
$$\ln g_t^{gr} = (1-\rho_g)\ln \bar{g} + \rho_g \ln g_{t-1}^{gr} + \hat{\varepsilon}_t^{gr}$$
$$\ln g_t^{lev} = \hat{\varepsilon}_t^{lev}$$

  - Linearized:

$$\hat{g}_t = \hat{g}_t^{gr} + \hat{g}_t^{lev} \tag{1}$$

$$\hat{g}_t^{gr} = \rho_g \hat{g}_{t-1}^{gr} + \hat{\varepsilon}_t^{gr} \tag{2}$$

$$\hat{g}_t^{lev} = \hat{\varepsilon}_t^{lev} \tag{3}$$

- Inflation Target Shocks:
  - In Levels:

$$\pi_t^* = \pi_{t-1}^* \varepsilon_t^{\pi^*}$$

  - Linearized:

$$\hat{\pi}_t^* = \hat{\pi}_{t-1}^* + \hat{\varepsilon}_t^{\pi^*} \tag{4}$$

2 HOUSEHOLDS

2.1 Optimization Problem

- Indices for different heterogeneous agents in this paper:
  - i for households.
  - j for manufacturing firms.
  - z for financial intermediaries.
- Objective Function for Household i:

$$Max \quad E_0 \sum_{t=0}^{\infty} \beta^t \left\{ S_t^c (1 - \frac{v}{\bar{g}}) \log(H_t(i)) - S_t^L \psi \frac{L_t(i)^{1+\frac{1}{\gamma}}}{1+\frac{1}{\gamma}} + \frac{a}{1-\epsilon} \left(\frac{M_t(i)}{P_t} \right)^{1-\epsilon} \right\} \text{ , where}$$

$$H_t(i) = C_t(i) - \nu C_{t-1} \text{ (external habit)}$$

$$C_t(i) = \left(\int_0^1 c_t(i,j)^{\frac{\sigma_t - 1}{\sigma_t}} dj \right)^{\frac{\sigma_t}{\sigma_t - 1}} \quad , \quad P_t = \left(\int_0^1 P_t(j)^{1-\sigma_t} dj \right)^{\frac{1}{1-\sigma_t}}$$

- Budget Constraint (multiplier = $\lambda_t(i)/P_t$):

$$\begin{aligned} B_t(i) \quad = \quad & i_{t-1}B_{t-1}(i) + M_{t-1}(i) - M_t(i) \\ & + W_t(i)L_t(i) + R_t^k x_t K_t(i) - P_t a(x_t) K_t(i) \\ & + \int_0^1 \Pi_t(i,j) dj + \int_0^1 \Pi_t(i,z) dz - P_t \tau_t(i) \\ & - P_t C_t(i) - P_t I_t(i) S_t^{inv} \\ & - P_t \frac{\theta_k}{2} K_t(i) \left(\frac{I_t(i)}{K_t(i)} - \frac{\bar{I}}{\bar{K}} \right)^2 - P_t \frac{\theta_i}{2} K_t(i) \left(\frac{I_t(i)}{K_t(i)} - \frac{I_{t-1}}{K_{t-1}} \right)^2 \\ & - W_t \frac{\phi_w}{2} \frac{(L_t(i) - \ell_t)^2}{\ell_t} \end{aligned}$$

- Cost of deviating from “normal” labor supply of other households:
  - Quadratic in the % deviation from “normal” labor supply $\frac{\phi_w}{2} \left(L_t(i) - L_t \right)^2 / L_t^2$.
  - Proportional to the aggregate wage bill $W_t L_t$.
- Capital Accumulation (multiplier = $\lambda_t(i) q_t(i)$):

$$K_{t+1}(i) = (1 - \Delta) K_t(i) + I_t(i)$$

  - $K_t(i)$ is the (unscaled) capital stock.
  - $I_t(i)$ is (unscaled) investment.

- Technical Notes:
  (a) Linearization for any variable x around its steady state $\bar{x}$ uses the following notation: $\hat{x}_t = (x_t - \bar{x})/\bar{x}$. The exception is investment I_t, where we use $\hat{x}_t = (x_t - \bar{x})$.
  (b) Assume complete contingent claims markets for labor income and identical initial endowments of capital, bonds and money. Then all FOC are the same except for labor supply. Therefore drop subscript i except in the wage setting problem. The latter is dealt with much later, after firm and intermediary price setting.
  (c) All inflation and interest rates in our notation are gross rates.

2.2 First-Order Conditions

- FOC for B_t:

$$\lambda_t = \beta i_t E_t \left(\frac{\lambda_{t+1}}{\pi_{t+1}} \right)$$

  - Rescaled by technology ($\check{\lambda}_t = \lambda_t S_t^Y$) and by the inflation target ($\check{i}_t = i_t/\pi_t^*$, $\check{\pi}_{t+1} = \pi_{t+1}/\pi_{t+1}^*$):

$$\check{\lambda}_t = \beta \check{i}_t E_t \left(\frac{\check{\lambda}_{t+1}}{\check{\pi}_{t+1} g_{t+1} \varepsilon_{t+1}^{\pi^*}} \right)$$

  - Linearization (remember that $E_t \hat{\varepsilon}_{t+1}^{\pi^*} = 0$):

$$\hat{\lambda}_t = \hat{\imath}_t + E_t \left(\hat{\lambda}_{t+1} - \hat{\pi}_{t+1} - \hat{g}_{t+1} \right) \tag{5}$$

- FOC for C_t:

$$\frac{S_t^c (1 - \frac{v}{g})}{H_t} = \lambda_t$$

  - Rescaled by technology ($\check{H}_t = H_t / S_t^y$):

$$\frac{S_t^c (1 - \frac{v}{g})}{\check{H}_t} = \check{\lambda}_t$$

  - Steady state:

$$\frac{(1 - v)}{\bar{H}} = \bar{\lambda}$$

  - Linearization:

$$\hat{S}_t^c - \hat{H}_t = \hat{\lambda}_t \tag{6}$$

- Habit H_t:

$$H_t = C_t - \nu C_{t-1}$$

  - Rescaled by technology ($\check{C}_t = C_t / S_t^y$):

$$\check{H}_t = \check{C}_t - \nu \frac{\check{C}_{t-1}}{g_t}$$

  - Steady state:

$$\bar{H} = \bar{C}(1 - \frac{\nu}{\bar{g}})$$

  - Linearization:

$$\hat{H}_t = \frac{1}{1-\frac{\nu}{\bar{g}}}\hat{C}_t - \frac{\frac{\nu}{\bar{g}}}{1-\frac{\nu}{\bar{g}}}\left(\hat{C}_{t-1} - \hat{g}_t\right) \tag{7}$$

- Capital Accumulation K_t:

$$K_t = (1-\Delta)K_{t-1} + I_{t-1}$$

  - Rescaled by technology:

$$\check{K}_t g_t = (1-\Delta)\check{K}_{t-1} + \check{I}_{t-1}$$

  - Steady state:

$$\bar{I} = (\bar{g} + \Delta - 1)\,\bar{K}$$

  - Linearization:

$$\bar{K}\bar{g}\left(\hat{K}_t - \hat{g}_t\right) = (1-\Delta)\bar{K}\hat{K}_{t-1} + \hat{I}_{t-1} \tag{8}$$

- FOC for I_t:

$$q_t = S_t^{inv} + \theta_k\left(\frac{I_t}{K_t} - \frac{\bar{I}}{\bar{K}}\right) + \theta_i\left(\frac{I_t}{K_t} - \frac{I_{t-1}}{K_{t-1}}\right)$$

  - Rescaled by technology ($\check{K} = K_t/S_t^y$, $\check{I} = I_t/S_t^y$):

$$q_t = S_t^{inv} + \theta_k\left(\frac{\check{I}_t}{\check{K}_t} - \frac{\bar{I}}{\bar{K}}\right) + \theta_i\left(\frac{\check{I}_t}{\check{K}_t} - \frac{\check{I}_{t-1}}{\check{K}_{t-1}}\right)$$

  - Steady state:

$$\beta_g = \frac{\beta}{\bar{g}} \quad \text{(the inverse of the steady state gross real interest rate)}$$

$$\bar{q} = 1$$

  - Linearization:

$$\hat{q}_t = \frac{\theta_k}{\bar{K}}\hat{I}_t - \frac{\theta_k \bar{I}}{\bar{K}}\hat{K}_t + \frac{\theta_i}{\bar{K}}\left(\hat{I}_t - \hat{I}_{t-1}\right) - \frac{\theta_i \bar{I}}{\bar{K}}\left(\hat{K}_t - \hat{K}_{t-1}\right) + \hat{S}_t^{inv} \tag{9}$$

- FOC for K_{t+1}:

$$\begin{aligned}\lambda_t q_t &= \beta E_t \lambda_{t+1}\Big[q_{t+1}(1-\Delta) + r^k_{t+1}x_{t+1} - a(x_{t+1}) \\ &\quad +\theta_k\left(\frac{I_{t+1}}{K_{t+1}} - \frac{\bar{I}}{\bar{K}}\right)\frac{I_{t+1}}{K_{t+1}} + \theta_i\left(\frac{I_{t+1}}{K_{t+1}} - \frac{I_t}{K_t}\right)\frac{I_{t+1}}{K_{t+1}} \\ &\quad -\frac{\theta_k}{2}\left(\frac{I_{t+1}}{K_{t+1}} - \frac{\bar{I}}{\bar{K}}\right)^2 - \frac{\theta_i}{2}\left(\frac{I_{t+1}}{K_{t+1}} - \frac{I_t}{K_t}\right)^2\Big]\end{aligned}$$

  - Rescaled by technology:

$$\begin{aligned}\check{\lambda}_t q_t &= E_t \frac{\beta}{g_{t+1}}\check{\lambda}_{t+1}\Big[q_{t+1}(1-\Delta) + r^k_{t+1}x_{t+1} - a(x_{t+1}) \\ &\quad +\theta_k\left(\frac{\check{I}_{t+1}}{\check{K}_{t+1}} - \frac{\bar{I}}{\bar{K}}\right)\frac{\check{I}_{t+1}}{\check{K}_{t+1}} + \theta_i\left(\frac{I_{t+1}}{\check{K}_{t+1}} - \frac{\check{I}_t}{\check{K}_t}\right)\frac{\check{I}_{t+1}}{\check{K}_{t+1}} \\ &\quad -\frac{\theta_k}{2}\left(\frac{\check{I}_{t+1}}{\check{K}_{t+1}} - \frac{\bar{I}}{\bar{K}}\right)^2 - \frac{\theta_i}{2}\left(\frac{\check{I}_{t+1}}{\check{K}_{t+1}} - \frac{\check{I}_t}{\check{K}_t}\right)^2\Big]\end{aligned}$$

  - Steady state:

$$\bar{r}^k = (r+\Delta) = \overline{a'(x)}, \text{ where } r = \frac{1-\beta_g}{\beta_g}$$

  - Linearization ($\hat{x}_{t+1}$ terms cancel):

$$\begin{aligned}\hat{\lambda}_t + \hat{q}_t &= E_t\Big\{\hat{\lambda}_{t+1} - \hat{g}_{t+1} + \beta_g(1-\Delta)\hat{q}_{t+1} + \beta_g(r+\Delta)\hat{r}^k_{t+1} \\ &\quad +\frac{\beta_g\theta_k\bar{I}}{\bar{K}^2}\hat{I}_{t+1} - \frac{\beta_g\theta_k\bar{I}^2}{\bar{K}^2}\hat{K}_{t+1} \\ &\quad +\frac{\beta_g\theta_i\bar{I}}{\bar{K}^2}\left(\hat{I}_{t+1} - \hat{I}_t\right) - \frac{\beta_g\theta_i\bar{I}^2}{\bar{K}^2}\left(\hat{K}_{t+1} - \hat{K}_t\right)\Big\}\end{aligned} \tag{10}$$

- FOC for x_t:

$$r^k_t = a'(x_t)$$

  - Steady state:

$$\epsilon = \frac{\overline{a''(x)}}{\overline{a'(x)}}$$

  - Linearization:

$$\hat{r}^k_t = \epsilon\hat{x}_t \tag{11}$$

- Wage Setting: See below, after derivation of price setting.

3 FIRMS

3.1 Cost Minimization

- Production Functions:

$$y_t(j) = (S_t^y \ell_t(j))^{1-\alpha} k_t(j)^\alpha$$

- Real Marginal Cost:
  - In levels:

$$mc_t = \frac{A w_t^{1-\alpha} (u_t)^\alpha}{(S_t^y)^{1-\alpha}} \quad , \quad \text{where } A = \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}$$

  - Rescaled by technology ($\breve{w}_t = w_t / S_t^y$):

$$mc_t = A \breve{w}_t^{1-\alpha} (u_t)^\alpha$$

  - Linearized:

$$\widehat{mc}_t = (1-\alpha)\hat{w}_t + \alpha \hat{u}_t \tag{12}$$

- Input Demands for Aggregate Firm Sector (see next page for definitions of aggregate variables):
  - In levels:

$$\begin{aligned} \ell_t &= (1-\alpha)\frac{mc_t}{w_t}\tilde{Y}_t \\ k_t &= \alpha \frac{mc_t}{u_t}\tilde{Y}_t \end{aligned}$$

  - Rescaled by technology ($\breve{Y}_t = \tilde{Y}_t / S_t^y$):

$$\begin{aligned} \ell_t &= (1-\alpha)\frac{mc_t}{\breve{w}_t}\breve{Y}_t \\ \breve{k}_t &= \alpha \frac{mc_t}{u_t}\breve{Y}_t \end{aligned}$$

  - Linearized:

$$\hat{\ell}_t = \widehat{mc}_t - \hat{w}_t + \hat{Y}_t \tag{13}$$

$$\hat{k}_t = \widehat{mc}_t - \hat{u}_t + \hat{Y}_t \tag{14}$$

- Definitions used above:
  - Labor:
    - Aggregate:

$$\ell_t = \int_0^1 \ell_t(j)dj \quad , \quad \text{where } \ell_t(j) = \left(\int_0^1 L_t(j,i)^{\frac{\sigma_t^w - 1}{\sigma_t^w}} di \right)^{\frac{\sigma_t^w}{\sigma_t^w - 1}}$$

$$L_t(i) = \int_0^1 L_t(j,i)dj$$

    - Varieties $L_t(j,i)$ supplied by households i, see above and also Section 5 on household wage setting.
    - Cost minimizing varieties demands:

$$L_t(j,i) = \ell_t(j) \left(\frac{W_t(i)}{W_t} \right)^{-\sigma_t^w} \quad , \quad \text{therefore } L_t(i) = \ell_t \left(\frac{W_t(i)}{W_t} \right)^{-\sigma_t^w}$$

  - Capital:
    - Aggregate:

$$k_t = \int_0^1 k_t(j)dj \quad , \quad \text{where } k_t(j) = \left(\int_0^1 k_t(j,z)^{\frac{\sigma^k - 1}{\sigma^k}} dz \right)^{\frac{\sigma^k}{\sigma^k - 1}}$$

$$k_t(z) = \int_0^1 k_t(j,z)dj$$

    - Varieties $k_t(j,z)$ supplied by intermediaries z, see below.
    - Cost minimizing varieties demands:

$$k_t(j,z) = k_t(j) \left(\frac{u_t(z)}{u_t} \right)^{-\sigma^k} \quad , \quad \text{therefore } k_t(z) = k_t \left(\frac{u_t(z)}{u_t} \right)^{-\sigma^k}$$

    - Equality between household supplied capital K_t and aggregate firm demand for capital k_t:

$$\bar{K} = \bar{k} \quad , \quad \hat{K}_t + \hat{x}_t = \hat{k}_t$$

  - Output:
    - Aggregate:

$$\tilde{Y}_t = \int_0^1 y_t(j)dj \quad , \quad \text{while } Y_t = \left(\int_0^1 y_t(j)^{\frac{\sigma_t - 1}{\sigma_t}} dj \right)^{\frac{\sigma_t}{\sigma_t - 1}}$$

    - It is easy to show that:

$$\overline{\tilde{Y}} = \bar{Y} \quad , \quad \widehat{\tilde{Y}}_t = \hat{Y}_t$$

3.2 Profit Maximization

- Discounted real profits:
  - Real revenue $\frac{P_{t+k}(j)}{P_{t+k}} y_{t+k}(j)$.
  - Real marginal cost $\frac{MC_{t+k}}{P_{t+k}} y_{t+k}(j)$.
  - Real cost of deviating from "normal" output of other firms:
    * Quadratic in the % deviation from "normal" output $\frac{\phi}{2} \frac{(y_{t+k}(j)-Y_{t+k})^2}{Y_{t+k}^2}$.
    * Proportional to aggregate real output Y_{t+k}.
- Pricing policy of firm j that reoptimizes at t, choosing V_t and v_t (a gross inflation rate):

$$P_{t+k}(j) = V_t (v_t)^k$$

- Profit maximization:

$$\underset{V_t, v_t}{Max} \quad E_t \sum_{k=0}^{\infty} (\delta\beta)^k \lambda_{t+k} \left[\frac{V_t (v_t)^k}{P_{t+k}} y_{t+k}(j) - mc_{t+k} y_{t+k}(j) - \frac{\phi}{2} \frac{(y_{t+k}(j) - Y_{t+k})^2}{Y_{t+k}} \right] \quad , \ s.t.$$

$$y_{t+k}(j) = Y_{t+k} \left(\frac{V_t (v_t)^k}{P_{t+k}} \right)^{-\sigma_{t+k}}$$

- Substitute constraints:

$$\underset{V_t, v_t}{Max} E_t \sum_{k=0}^{\infty} (\delta\beta)^k \lambda_{t+k} \left[\left(\frac{V_t (v_t)^k}{P_{t+k}} \right)^{1-\sigma_{t+k}} Y_{t+k} - mc_{t+k} \left(\frac{V_t (v_t)^k}{P_{t+k}} \right)^{-\sigma_{t+k}} Y_{t+k} - \frac{\phi}{2} \frac{(y_{t+k}(j) - Y_{t+k})^2}{Y_{t+k}} \right]$$

- Define terms:
  - Front-loading term:

$$p_t \equiv \frac{V_t}{P_t}$$

  - Inflation rescaled by the inflation target:

$$\check{\pi}_t = \pi_t / \pi_t^*$$

  - Cumulative aggregate rescaled inflation:

$$\check{\Pi}_{t,k} \equiv \prod_{j=1}^{k} \check{\pi}_{t+j} \text{ for } k \geq 1 \ \ (\equiv 1 \text{ for } k = 0) \tag{15}$$

  - Cumulative aggregate rescaled inflation deviation:

$$\hat{\Pi}_{t,k} \equiv \sum_{j=1}^{k} \hat{\pi}_{t+j} \text{ for } k \geq 1 \ \ (\equiv 0 \text{ for } k = 0) \tag{16}$$

  - Mark-up:

$$\mu_t = \frac{\sigma_t}{\sigma_t - 1}$$

3.3 First-Order Conditions

- Rescaled by technology, with $\check{y}_t(j) = y_t(j)/S_t^y$.
- Rescaled by the inflation target, with $\check{v}_t = v_t/\pi_t^*$.
- FOC for V_t:

$$p_t = \frac{E_t \sum_{k=0}^{\infty} (\delta\beta)^k \check{\lambda}_{t+k}\check{y}_{t+k}(j)\sigma_{t+k}\left(mc_{t+k} + \phi\left(\frac{\check{y}_{t+k}(j)-\check{Y}_{t+k}}{\check{Y}_{t+k}}\right)\right)}{E_t \sum_{k=0}^{\infty} (\delta\beta)^k \check{\lambda}_{t+k}\check{y}_{t+k}(j)\left(\sigma_{t+k}-1\right)\left(\frac{(\check{v}_t)^k}{\check{\Pi}_{t,k}}\right)} \quad (17)$$

- FOC w.r.t. v_t (rescaled by technology):

$$p_t = \frac{E_t \sum_{k=0}^{\infty} (\delta\beta)^k k\check{\lambda}_{t+k}\check{y}_{t+k}(j)\sigma_{t+k}\left(mc_{t+k} + \phi\left(\frac{\check{y}_{t+k}(j)-\check{Y}_{t+k}}{\check{Y}_{t+k}}\right)\right)}{E_t \sum_{k=0}^{\infty} (\delta\beta)^k k\check{\lambda}_{t+k}\check{y}_{t+k}(j)\left(\sigma_{t+k}-1\right)\left(\frac{(\check{v}_t)^k}{\check{\Pi}_{t,k}}\right)} \quad (18)$$

- Rescaling by the inflation target - comments:
  - This takes the form of dividing all price levels by the target price level P_t^*.
  - In the preceding formulas this only affects $E_t\frac{(v_t)^k}{\Pi_{t,k}}$, i.e. future firm-specific and aggregate cumulative inflation rates have to be deflated by future cumulative target inflation rates.
  - However, under unit roots all **expected** future quarterly target inflation rates are simply equal to the current target rate.
  - In all **forward-looking** conditions like (17) and (18) we can therefore simply linearize around the current inflation target π_t^*.
  - Example: Our final conditions will contain **expected** inflation terms for periods t and $t+1$. For actual future inflation the correct definition is $\hat{\pi}_{t+1} = \ln(\pi_{t+1}) - \ln(\pi_{t+1}^*)$. But it is ok to linearize around π_t^* instead because only the **expected** future inflation deviation enters the equations, and for this we have (up to an approximation error that disappears in linearization):

$$E_t\hat{\pi}_{t+1} = E_t\left(\ln(\pi_{t+1}) - \ln(\pi_{t+1}^*)\right) = E_t\left(\ln(\pi_{t+1}) - \ln(\pi_t^*)\right)$$

  - The same is not true for **backward-looking** conditions. See Section 4 on "The Price Index".

3.4 Linearization

- Geometric distribution formulas:

$$\sum_{k=0}^{\infty} (\delta\beta)^k k = \frac{\delta\beta}{(1-\delta\beta)^2} \tag{19}$$

$$\sum_{k=0}^{\infty} (\delta\beta)^k k^2 = \frac{\delta\beta(1+\delta\beta)}{(1-\delta\beta)^3} \tag{20}$$

- Linearization of goods demand:

$$\left(\hat{y}_{t+k}(j) - \hat{Y}_{t+k}\right) = -\bar{\sigma}\left(\hat{p}_t + k\hat{v}_t - \hat{\Pi}_{t,k}\right) \tag{21}$$

- Linearization of markup:

$$\hat{\mu}_t = \hat{\sigma}_t - \bar{\mu}\hat{\sigma}_t \tag{22}$$

3.4.1 Linearization for V_t

- Rewrite (17):

$$p_t\left(\sigma_{t+k}-1\right) E_t \sum_{k=0}^{\infty} (\delta\beta)^k \check{\lambda}_{t+k}\check{y}_{t+k}(j)\left(\frac{(\check{v}_t)^k}{\check{\Pi}_{t,k}}\right)$$

$$= E_t \sum_{k=0}^{\infty} (\delta\beta)^k \check{\lambda}_{t+k}\check{y}_{t+k}(j)\sigma_{t+k}\left(mc_{t+k} + \phi\left(\frac{\check{y}_{t+k}(j) - \check{Y}_{t+k}}{\check{Y}_{t+k}}\right)\right)$$

- Linearization of the common terms $\check{\lambda}_{t+k}\check{y}_{t+k}(j)$ cancels.
- Remaining terms are:

$$\frac{\hat{p}_t}{1-\delta\beta} + E_t\Sigma_{k=0}^{\infty}(\delta\beta)^k\left[k\hat{v}_t - \hat{\Pi}_{t,k} + \bar{\mu}\hat{\sigma}_{t+k}\right]$$

$$= E_t\Sigma_{k=0}^{\infty}(\delta\beta)^k\left[\widehat{mc}_{t+k} + \phi\bar{\mu}\left(\hat{y}_{t+k}(j) - \hat{Y}_{t+k}\right) + \hat{\sigma}_{t+k}\right]$$

- Using (19), (21) and (22):

$$\frac{\hat{p}_t}{1-\delta\beta}(1+\phi\bar{\mu}\bar{\sigma}) + \frac{\hat{v}_t\delta\beta}{(1-\delta\beta)^2}(1+\phi\bar{\mu}\bar{\sigma}) = E_t\Sigma_{k=0}^{\infty}(\delta\beta)^k\left[\widehat{mc}_{t+k} + \hat{\Pi}_{t,k}(1+\phi\bar{\mu}\bar{\sigma}) + \hat{\mu}_{t+k}\right]$$

- Equivalently:

$$\frac{\hat{p}_t}{1-\delta\beta} + \frac{\hat{v}_t\delta\beta}{(1-\delta\beta)^2} = E_t\Sigma_{k=0}^{\infty}(\delta_p\beta)^k\left[\frac{\widehat{mc}_{t+k} + \hat{\mu}_{t+k}}{(1+\phi\bar{\mu}\bar{\sigma})} + \hat{\Pi}_{t,k}\right] \tag{23}$$

3.4.2 Linearization for v_t

- Rewrite (18):

$$\begin{aligned} & p_t E_t \Sigma_{k=0}^{\infty} (\delta\beta)^k k \check{\lambda}_{t+k} \check{y}_{t+k}(j)(\sigma_{t+k}-1)\left(\frac{(\check{v}_t)^k}{\check{\Pi}_{t,k}}\right) \\ = \quad & E_t \Sigma_{k=0}^{\infty} (\delta\beta)^k k \check{\lambda}_{t+k} \check{y}_{t+k}(j)\sigma_{t+k}\left(mc_{t+k}+\phi\left(\frac{\check{y}_{t+k}(j)-\check{Y}_{t+k}}{\check{Y}_{t+k}}\right)\right) \end{aligned}$$

- Linearization of the common terms $\check{\lambda}_{t+k}\check{y}_{t+k}(j)$ cancels.
- Applying (19) to the $\hat{p}_t$-term, the remaining terms are:

$$\begin{aligned} & \frac{\hat{p}_t \delta\beta}{(1-\delta\beta)^2} + E_t \Sigma_{k=0}^{\infty} (\delta\beta)^k k \left[k\hat{v}_t - \hat{\Pi}_{t,k} + \bar{\mu}\hat{\sigma}_{t+k}\right] \\ & = E_t \Sigma_{k=0}^{\infty} (\delta\beta)^k k \left[\widehat{mc}_{t+k} + \phi\bar{\mu}\left(\hat{y}_{t+k}(j) - \hat{Y}_{t+k}\right) + \hat{\sigma}_{t+k}\right] \end{aligned}$$

- Applying (21) and (22), and simplifying, we get:

$$\begin{aligned} & \frac{\hat{p}_t \delta\beta}{(1-\delta\beta)^2}(1+\phi\bar{\mu}\bar{\sigma}) + E_t \Sigma_{k=0}^{\infty} (\delta\beta)^k k^2 \hat{v}_t (1+\phi\bar{\mu}\bar{\sigma}) \\ = \quad & E_t \Sigma_{k=0}^{\infty} (\delta\beta)^k k \left[\widehat{mc}_{t+k} + \hat{\mu}_{t+k} + \hat{\Pi}_{t,k}(1+\phi\bar{\mu}\bar{\sigma})\right] \end{aligned}$$

- Applying (20) to the $\hat{v}_t$-term and simplifying further, we get:

$$\frac{\hat{p}_t \delta\beta}{(1-\delta\beta)^2} + \frac{\hat{v}_t \delta\beta(1+\delta\beta)}{(1-\delta\beta)^3} = E_t \Sigma_{k=0}^{\infty} (\delta\beta)^k k \left[\frac{\widehat{mc}_{t+k} + \hat{\mu}_{t+k}}{(1+\phi\bar{\mu}\bar{\sigma})} + \hat{\Pi}_{t,k}\right] \tag{24}$$

3.5 Quasi-Differencing

3.5.1 Quasi-Differencing for V_t-Equation (23)

- Rewrite (23) as:

$$\hat{p}_t + \frac{\hat{v}_t \delta\beta}{(1-\delta\beta)} = (1-\delta\beta)\, E_t \sum_{k=0}^{\infty} (\delta\beta)^k \left[\frac{\widehat{mc}_{t+k} + \hat{\mu}_{t+k}}{(1+\phi\bar{\mu}\bar{\sigma})} + \hat{\Pi}_{t,k} \right] \tag{25}$$

- For future reference, lead this by one period and multiply by $(1-\delta\beta)$:

$$\begin{aligned} & (1-\delta\beta)\, E_t \hat{p}_{t+1} + \delta\beta E_t \hat{v}_{t+1} \\ = \quad & (1-\delta\beta)^2 E_t \left[\sum_{k=0}^{\infty} (\delta\beta)^k \frac{\widehat{mc}_{t+1+k} + \hat{\mu}_{t+1+k}}{(1+\phi\bar{\mu}\bar{\sigma})} + \sum_{k=1}^{\infty} (\delta\beta)^k \hat{\Pi}_{t+1,k} \right] \end{aligned} \tag{26}$$

- Note the following for $\hat{\Pi}_{t+1,k}$:[1]

$$\begin{aligned} \hat{\Pi}_{t+1,k} &= 0 \text{ for } k=0, \\ &= \hat{\pi}_{t+2} \text{ for k=1} \\ &= \hat{\pi}_{t+2} + ... + \hat{\pi}_{t+k+1} \text{ for k=2,3,4,...} \end{aligned} \tag{27}$$

- Write out terms in (25):

$$\hat{p}_t + \frac{\hat{v}_t \delta\beta}{(1-\delta\beta)} = (1-\delta\beta) *$$

$$E_t \left[\frac{\widehat{mc}_t + \hat{\mu}_t}{(1+\phi\bar{\mu}\bar{\sigma})} + (\delta\beta) \left(\frac{\widehat{mc}_{t+1} + \hat{\mu}_{t+1}}{(1+\phi\bar{\mu}\bar{\sigma})} + \hat{\pi}_{t+1} \right) + (\delta\beta)^2 \left(\frac{\widehat{mc}_{t+2} + \hat{\mu}_{t+2}}{(1+\phi\bar{\mu}\bar{\sigma})} + \hat{\pi}_{t+1} + \hat{\pi}_{t+2} \right) \right.$$
$$\left. + (\delta\beta)^3 \left(\frac{\widehat{mc}_{t+3} + \hat{\mu}_{t+3}}{(1+\phi\bar{\mu}\bar{\sigma})} + \hat{\pi}_{t+1} + \hat{\pi}_{t+2} + \hat{\pi}_{t+3} \right) + ... \right]$$

- Multiply the last equation by $\delta\beta$, and lead by one period:

$$\delta\beta E_t \hat{p}_{t+1} + \frac{(\delta\beta)^2}{(1-\delta\beta)} E_t \hat{v}_{t+1} = (1-\delta\beta) *$$

$$E_t \left[(\delta\beta) \frac{\widehat{mc}_{t+1} + \hat{\mu}_{t+1}}{(1+\phi\bar{\mu}\bar{\sigma})} + (\delta\beta)^2 \left(\frac{\widehat{mc}_{t+2} + \hat{\mu}_{t+2}}{(1+\phi\bar{\mu}\bar{\sigma})} + \hat{\pi}_{t+2} \right) \right.$$
$$\left. + (\delta\beta)^3 \left(\frac{\widehat{mc}_{t+3} + \hat{\mu}_{t+3}}{(1+\phi\bar{\mu}\bar{\sigma})} + \hat{\pi}_{t+2} + \hat{\pi}_{t+3} \right) + ... \right]$$

[1] The first line shows why for the last term in the previous equation we can let the subscript run from 1 instead of 0.

- Deduct the last equation from the preceding one:

$$[\hat{p}_t - \delta\beta E_t \hat{p}_{t+1}] + \frac{\delta\beta}{(1-\delta\beta)} [\hat{v}_t - \delta\beta E_t \hat{v}_{t+1}]$$
$$= (1-\delta\beta)\left[\frac{\widehat{mc}_t + \hat{\mu}_t}{(1+\phi\bar{\mu}\bar{\sigma})} + E_t \hat{\pi}_{t+1}\left(\delta\beta + (\delta\beta)^2 + + (\delta\beta)^3 + ...\right)\right]$$

- Equivalently:

$$[\hat{p}_t - E_t \hat{p}_{t+1} + (1-\delta\beta) E_t \hat{p}_{t+1}] + \frac{\delta\beta}{(1-\delta\beta)} [\hat{v}_t - E_t \hat{v}_{t+1} + (1-\delta\beta) E_t \hat{v}_{t+1}]$$
$$= (1-\delta\beta)\frac{\widehat{mc}_t + \hat{\mu}_t}{(1+\phi\bar{\mu}\bar{\sigma})} + \delta\beta E_t \hat{\pi}_{t+1}$$

- Equivalently:

$$[\hat{p}_t - E_t \hat{p}_{t+1}] + \frac{\delta\beta}{(1-\delta\beta)} [\hat{v}_t - E_t \hat{v}_{t+1}]$$
$$= \quad (1-\delta\beta)\left(\frac{\widehat{mc}_t + \hat{\mu}_t}{(1+\phi\bar{\mu}\bar{\sigma})} - E_t \hat{p}_{t+1}\right) + \delta\beta\left(E_t \hat{\pi}_{t+1} - E_t \hat{v}_{t+1}\right)$$

- Equivalently:

$$[E_t \hat{p}_{t+1} - \hat{p}_t] + \frac{\delta\beta}{(1-\delta\beta)} [E_t \hat{v}_{t+1} - \hat{v}_t]$$
$$= \quad -(1-\delta\beta)\left(\frac{\widehat{mc}_t + \hat{\mu}_t}{(1+\phi\bar{\mu}\bar{\sigma})} - E_t \hat{p}_{t+1}\right) + \delta\beta\left(E_t \hat{v}_{t+1} - E_t \hat{\pi}_{t+1}\right)$$

- Equivalently:

$$\delta\beta E_t \hat{p}_{t+1} - \hat{p}_t + \frac{\delta\beta}{(1-\delta\beta)} [E_t \hat{v}_{t+1} - \hat{v}_t] \tag{28}$$
$$= \quad -(1-\delta\beta)\left(\frac{\widehat{mc}_t + \hat{\mu}_t}{(1+\phi\bar{\mu}\bar{\sigma})}\right) + \delta\beta E_t \hat{v}_{t+1} - \delta\beta E_t \hat{\pi}_{t+1}$$

3.5.2 Quasi-Differencing for v_t-Equation (24)

- Rewrite (24) as:

$$\hat{p}_t + \frac{(1+\delta\beta)}{(1-\delta\beta)}\hat{v}_t = \frac{(1-\delta\beta)^2}{\delta\beta} E_t \sum_{k=0}^{\infty} (\delta\beta)^k \, k \left[\frac{\widehat{mc}_{t+k} + \hat{\mu}_{t+k}}{(1+\phi\bar{\mu}\bar{\sigma})} + \hat{\Pi}_{t,k} \right]$$

- Write out terms:

$$\hat{p}_t + \frac{(1+\delta\beta)}{(1-\delta\beta)}\hat{v}_t = \left(\frac{(1-\delta\beta)^2}{\delta\beta} \right) *$$
$$E_t \left[(\delta\beta) \left(\frac{\widehat{mc}_{t+1} + \hat{\mu}_{t+1}}{(1+\phi\bar{\mu}\bar{\sigma})} + \hat{\pi}_{t+1} \right) + 2\,(\delta\beta)^2 \left(\frac{\widehat{mc}_{t+2} + \hat{\mu}_{t+2}}{(1+\phi\bar{\mu}\bar{\sigma})} + \hat{\pi}_{t+1} + \hat{\pi}_{t+2} \right) \right.$$
$$\left. +3\,(\delta\beta)^3 \left(\frac{\widehat{mc}_{t+3} + \hat{\mu}_{t+3}}{(1+\phi\bar{\mu}\bar{\sigma})} + \hat{\pi}_{t+1} + \hat{\pi}_{t+2} + \hat{\pi}_{t+3} \right) + ... \right]$$

- Multiply the last equation by $\delta\beta$, and lead by one period:

$$\delta\beta E_t \hat{p}_{t+1} + \frac{\delta\beta(1+\delta\beta)}{(1-\delta\beta)} E_t \hat{v}_{t+1} = \left(\frac{(1-\delta\beta)^2}{\delta\beta} \right) *$$
$$E_t \left[(\delta\beta)^2 \left(\frac{\widehat{mc}_{t+2} + \hat{\mu}_{t+2}}{(1+\phi\bar{\mu}\bar{\sigma})} + \hat{\pi}_{t+2} \right) + 2\,(\delta\beta)^3 \left(\frac{\widehat{mc}_{t+3} + \hat{\mu}_{t+3}}{(1+\phi\bar{\mu}\bar{\sigma})} + \hat{\pi}_{t+2} + \hat{\pi}_{t+3} \right) + ... \right]$$

- Deduct the last equation from the preceding one:

$$[\hat{p}_t - \delta\beta E_t \hat{p}_{t+1}] + \frac{(1+\delta\beta)}{(1-\delta\beta)} [\hat{v}_t - \delta\beta E_t \hat{v}_{t+1}] = \left(\frac{(1-\delta\beta)^2}{\delta\beta} \right) *$$
$$E_t \left[(\delta\beta) \frac{\widehat{mc}_{t+1} + \hat{\mu}_{t+1}}{(1+\phi\bar{\mu}\bar{\sigma})} + (\delta\beta)^2 \left(\frac{\widehat{mc}_{t+2} + \hat{\mu}_{t+2}}{(1+\phi\bar{\mu}\bar{\sigma})} + \hat{\pi}_{t+2} \right) + (\delta\beta)^3 \left(\frac{\widehat{mc}_{t+3} + \hat{\mu}_{t+3}}{(1+\phi\bar{\mu}\bar{\sigma})} + \hat{\pi}_{t+2} + \hat{\pi}_{t+3} \right) + ... \right.$$
$$\left. + \hat{\pi}_{t+1} \left(\delta\beta + 2\,(\delta\beta)^2 + 3\,(\delta\beta)^3 + ... \right) \right]$$

- Use (19) for the final term:

$$E_t \hat{\pi}_{t+1} \left(\frac{(1-\delta\beta)^2}{\delta\beta} \right) \left(\delta\beta + 2\,(\delta\beta)^2 + 3\,(\delta\beta)^3 + ... \right) = E_t \hat{\pi}_{t+1}$$

- Simplify:

$$[\hat{p}_t - \delta\beta E_t \hat{p}_{t+1}] + \frac{(1+\delta\beta)}{(1-\delta\beta)} [\hat{v}_t - \delta\beta E_t \hat{v}_{t+1}] = E_t \hat{\pi}_{t+1} + \left((1-\delta\beta)^2 \right) *$$
$$E_t \left[\frac{\widehat{mc}_{t+1} + \hat{\mu}_{t+1}}{(1+\phi\bar{\mu}\bar{\sigma})} + (\delta\beta) \left(\frac{\widehat{mc}_{t+2} + \hat{\mu}_{t+2}}{(1+\phi\bar{\mu}\bar{\sigma})} + \hat{\pi}_{t+2} \right) + (\delta\beta)^2 \left(\frac{\widehat{mc}_{t+3} + \hat{\mu}_{t+3}}{(1+\phi\bar{\mu}\bar{\sigma})} + \hat{\pi}_{t+2} + \hat{\pi}_{t+3} \right) + ... \right]$$

- Rewrite:

$$[\hat{p}_t - \delta\beta E_t\hat{p}_{t+1}] + \frac{(1+\delta\beta)}{(1-\delta\beta)}[\hat{v}_t - \delta\beta E_t\hat{v}_{t+1}] = E_t\hat{\pi}_{t+1} + \left((1-\delta\beta)^2\right) *$$

$$\left[E_t\sum_{k=0}^{\infty}(\delta\beta)^k\frac{\widehat{mc}_{t+1+k}+\hat{\mu}_{t+1+k}}{(1+\phi\bar{\mu}\bar{\sigma})} + E_t\left[(\delta\beta)\hat{\pi}_{t+2} + (\delta\beta)^2(\hat{\pi}_{t+2}+\hat{\pi}_{t+3}) + ...\right]\right]$$

- The final term can be rewritten using (27):

$$E_t\left[(\delta\beta)\hat{\pi}_{t+2} + (\delta\beta)^2(\hat{\pi}_{t+2}+\hat{\pi}_{t+3}) + ...\right] = \sum_{k=1}^{\infty}(\delta\beta)^k\hat{\Pi}_{t+1,k}$$

- Then we have:

$$[\hat{p}_t - \delta\beta E_t\hat{p}_{t+1}] + \frac{(1+\delta\beta)}{(1-\delta\beta)}[\hat{v}_t - \delta\beta E_t\hat{v}_{t+1}] =$$

$$E_t\hat{\pi}_{t+1} + \left((1-\delta\beta)^2\right)E_t\left[\sum_{k=0}^{\infty}(\delta\beta)^k\frac{\widehat{mc}_{t+1+k}+\hat{\mu}_{t+1+k}}{(1+\phi\bar{\mu}\bar{\sigma})} + \sum_{k=1}^{\infty}(\delta\beta)^k\hat{\Pi}_{t+1,k}\right]$$

- Now we can replace the right-hand side using (26):

$$\begin{aligned}&[\hat{p}_t - \delta\beta E_t\hat{p}_{t+1}] + \frac{(1+\delta\beta)}{(1-\delta\beta)}[\hat{v}_t - \delta\beta E_t\hat{v}_{t+1}]\\ = \;& E_t\hat{\pi}_{t+1} + (1-\delta\beta)E_t\hat{p}_{t+1} + \delta\beta E_t\hat{v}_{t+1}\end{aligned}$$

- Further simplification:

$$[\hat{p}_t - E_t\hat{p}_{t+1} + (1-\delta\beta)E_t\hat{p}_{t+1}] + \frac{(1+\delta\beta)}{(1-\delta\beta)}[\hat{v}_t - E_t\hat{v}_{t+1} + (1-\delta\beta)E_t\hat{v}_{t+1}]$$

$$= E_t\hat{\pi}_{t+1} + (1-\delta\beta)E_t\hat{p}_{t+1} + \delta\beta E_t\hat{v}_{t+1}$$

- Cancel terms:

$$\begin{aligned}&[\hat{p}_t - E_t\hat{p}_{t+1}] + \frac{(1+\delta\beta)}{(1-\delta\beta)}[\hat{v}_t - E_t\hat{v}_{t+1}]\\ = \;& E_t\hat{\pi}_{t+1} - E_t\hat{v}_{t+1}\end{aligned}$$

- Equivalently:

$$\begin{aligned}&[E_t\hat{p}_{t+1} - \hat{p}_t] + \frac{(1+\delta\beta)}{(1-\delta\beta)}[E_t\hat{v}_{t+1} - \hat{v}_t]\\ = \;& E_t\hat{v}_{t+1} - E_t\hat{\pi}_{t+1}\end{aligned}$$

- Equivalently:

$$E_t\hat{p}_{t+1} = \hat{p}_t + E_t\hat{v}_{t+1} - E_t\hat{\pi}_{t+1} - \frac{(1+\delta\beta)}{(1-\delta\beta)}[E_t\hat{v}_{t+1} - \hat{v}_t] \tag{29}$$

- Equivalently:

$$E_t\hat{p}_{t+1} - \hat{p}_t = \frac{-2\delta\beta}{(1-\delta\beta)} E_t\hat{v}_{t+1} - E_t\hat{\pi}_{t+1} + \frac{(1+\delta\beta)}{(1-\delta\beta)}\hat{v}_t \tag{30}$$

3.5.3 Combine (28) and (29)

- (28) for ease of reference:

$$\begin{aligned} &\delta\beta E_t\hat{p}_{t+1} - \hat{p}_t + \frac{\delta\beta}{(1-\delta\beta)}\left[E_t\hat{v}_{t+1} - \hat{v}_t\right] \\ = &\ -(1-\delta\beta)\left(\frac{\widehat{mc}_t + \hat{\mu}_t}{(1+\phi\bar{\mu}\bar{\sigma})}\right) + \delta\beta E_t\hat{v}_{t+1} - \delta\beta E_t\hat{\pi}_{t+1} \end{aligned}$$

- Plug in (29):

$$\begin{aligned} &\delta\beta\left[\hat{p}_t + E_t\hat{v}_{t+1} - E_t\hat{\pi}_{t+1} - \frac{(1+\delta\beta)}{(1-\delta\beta)}\left[E_t\hat{v}_{t+1} - \hat{v}_t\right]\right] \\ &-\hat{p}_t + \frac{\delta\beta}{(1-\delta\beta)}\left[E_t\hat{v}_{t+1} - \hat{v}_t\right] \\ &= -(1-\delta\beta)\left(\frac{\widehat{mc}_t + \hat{\mu}_t}{(1+\phi\bar{\mu}\bar{\sigma})}\right) + \delta\beta E_t\hat{v}_{t+1} - \delta\beta E_t\hat{\pi}_{t+1} \end{aligned}$$

- Equivalently:

$$\begin{aligned} &(\delta\beta - 1)\hat{p}_t + \delta\beta E_t\hat{v}_{t+1} - \delta\beta E_t\hat{\pi}_{t+1} + \frac{\delta\beta}{(1-\delta\beta)}(1-1-\delta\beta)\left(E_t\hat{v}_{t+1} - \hat{v}_t\right) \\ &= -(1-\delta\beta)\left(\frac{\widehat{mc}_t + \hat{\mu}_t}{(1+\phi\bar{\mu}\bar{\sigma})}\right) + \delta\beta E_t\hat{v}_{t+1} - \delta\beta E_t\hat{\pi}_{t+1} \end{aligned}$$

- Cancel terms and multiply by -1:

$$\begin{aligned} &(1-\delta\beta)\hat{p}_t + \frac{(\delta\beta)^2}{(1-\delta\beta)}\left(E_t\hat{v}_{t+1} - \hat{v}_t\right) \\ &= (1-\delta\beta)\left(\frac{\widehat{mc}_t + \hat{\mu}_t}{(1+\phi\bar{\mu}\bar{\sigma})}\right) \end{aligned}$$

- Simplify further:

$$\frac{(\delta\beta)^2}{(1-\delta\beta)}\left(E_t\hat{v}_{t+1} - \hat{v}_t\right) = (1-\delta\beta)\left(\frac{\widehat{mc}_t + \hat{\mu}_t}{(1+\phi\bar{\mu}\bar{\sigma})} - \hat{p}_t\right)$$

- Preliminary difference equation for $\hat{v}_t$:

$$\left(E_t\hat{v}_{t+1} - \hat{v}_t\right) = \frac{(1-\delta\beta)^2}{(\delta\beta)^2}\left(\frac{\widehat{mc}_t + \hat{\mu}_t}{(1+\phi\bar{\mu}\bar{\sigma})} - \hat{p}_t\right) \tag{31}$$

4 THE PRICE INDEX

4.1 Formula for the Index

- Price Level:

$$P_t = \left[(1-\delta) \sum_{s=0}^{\infty} \delta^s \left[V_{t-s} (v_{t-s})^s \right]^{1-\sigma_t} \right]^{\frac{1}{1-\sigma_t}}$$

- Deflate by current target price level P_t^* and write out terms ($\check{P}_t = P_t/P_t^*$, $\check{V}_t = V_t/P_t^*$):

$$\left(\check{P}_t\right)^{1-\sigma_t} = (1-\delta)\check{V}_t^{1-\sigma_t} + (1-\delta)\delta\check{V}_{t-1}^{1-\sigma_t} \left(\frac{v_{t-1}}{\pi_t^*}\right)^{1-\sigma_t}$$

$$+(1-\delta)\delta^2\check{V}_{t-2}^{1-\sigma_t} \left(\frac{v_{t-2}^2}{\pi_t^*\pi_{t-1}^*}\right)^{(1-\sigma_t)} + (1-\delta)\delta^3\check{V}_{t-3}^{1-\sigma_t} \left(\frac{v_{t-3}^3}{\pi_t^*\pi_{t-1}^*\pi_{t-2}^*}\right)^{(1-\sigma_t)} + ...$$

- Divide by $\check{P}_{t-1}$:

$$\begin{aligned}
\left(\frac{\check{P}_t}{\check{P}_{t-1}}\right)^{1-\sigma_t} &= (1-\delta)\left(\frac{\check{V}_t}{\check{P}_t}\right)^{1-\sigma_t} \left(\frac{\check{P}_t}{\check{P}_{t-1}}\right)^{1-\sigma_t} \\
&+(1-\delta)\delta\left(\frac{\check{V}_{t-1}}{\check{P}_{t-1}}\right)^{1-\sigma_t} \left(\frac{v_{t-1}}{\pi_t^*}\right)^{1-\sigma_t} \\
&+(1-\delta)\delta^2\left(\frac{\check{V}_{t-2}}{\check{P}_{t-2}}\right)^{1-\sigma_t} \left(\frac{\check{P}_{t-2}}{\check{P}_{t-1}}\right)^{1-\sigma_t} \left(\frac{v_{t-2}^2}{\pi_t^*\pi_{t-1}^*}\right)^{(1-\sigma_t)} \\
&+(1-\delta)\delta^3\left(\frac{\check{V}_{t-3}}{\check{P}_{t-3}}\right)^{1-\sigma_t} \left(\frac{\check{P}_{t-3}}{\check{P}_{t-2}}\right)^{1-\sigma_t} \left(\frac{\check{P}_{t-2}}{\check{P}_{t-1}}\right)^{1-\sigma_t} \left(\frac{v_{t-3}^3}{\pi_t^*\pi_{t-1}^*\pi_{t-2}^*}\right)^{(1-\sigma_t)} + ...
\end{aligned}$$

- Use $\check{P}_t/\check{P}_{t-1} = \check{\pi}_t = \frac{\pi_t}{\pi_t^*}$ (this is the deviation of gross inflation from its target):

$$\begin{aligned}
(\check{\pi}_t)^{1-\sigma_t} &= (1-\delta)\left(p_t\right)^{1-\sigma_t}(\check{\pi}_t)^{1-\sigma_t} \\
&+(1-\delta)\delta\left(p_{t-1}\right)^{1-\sigma_t}\left(\frac{v_{t-1}}{\pi_t^*}\right)^{1-\sigma_t} \\
&+(1-\delta)\delta^2\left(p_{t-2}\right)^{1-\sigma_t}\left(\frac{1}{\check{\pi}_{t-1}}\right)^{1-\sigma_t}\left(\frac{v_{t-2}^2}{\pi_t^*\pi_{t-1}^*}\right)^{(1-\sigma_t)} \\
&+(1-\delta)\delta^3\left(p_{t-3}\right)^{1-\sigma_t}\left(\frac{1}{\check{\pi}_{t-2}}\right)^{1-\sigma_t}\left(\frac{1}{\check{\pi}_{t-1}}\right)^{1-\sigma_t}\left(\frac{v_{t-3}^3}{\pi_t^*\pi_{t-1}^*\pi_{t-2}^*}\right)^{(1-\sigma_t)} + ...
\end{aligned}$$

- Divide through by $(\breve{\pi}_t)^{1-\sigma_t}$:

$$
\begin{aligned}
1 \;=\;& (1-\delta)p_t^{1-\sigma_t} \\
&+(1-\delta)\delta p_{t-1}^{1-\sigma_t}\left(\frac{\upsilon_{t-1}}{\breve{\pi}_t\pi_t^*}\right)^{1-\sigma_t} \\
&+(1-\delta)\delta^2 p_{t-2}^{1-\sigma_t}\left(\frac{(\upsilon_{t-2})^2}{\breve{\pi}_{t-1}\breve{\pi}_t\pi_{t-1}^*\pi_t^*}\right)^{1-\sigma_t} \\
&+(1-\delta)\delta^3 p_{t-3}^{1-\sigma_t}\left(\frac{(\upsilon_{t-3})^3}{\breve{\pi}_{t-2}\breve{\pi}_{t-1}\breve{\pi}_t\pi_{t-2}^*\pi_{t-1}^*\pi_t^*}\right)^{1-\sigma_t} + \ldots
\end{aligned}
$$

- Let $\breve{\upsilon}_{t-1} = \upsilon_{t-1}/\pi_{t-1}^*$, the deviation of firm-specific gross inflation from the inflation target, and use $\pi_t^*/\pi_{t-1}^* = \varepsilon_t^{\pi^*}$:

$$
\begin{aligned}
1 \;=\;& (1-\delta)p_t^{1-\sigma_t} \\
&+(1-\delta)\delta p_{t-1}^{1-\sigma_t}\left(\frac{\breve{\upsilon}_{t-1}}{\breve{\pi}_t\varepsilon_t^{\pi^*}}\right)^{1-\sigma_t} \\
&+(1-\delta)\delta^2 p_{t-2}^{1-\sigma_t}\left(\frac{(\breve{\upsilon}_{t-2})^2}{\breve{\pi}_{t-1}\breve{\pi}_t\left(\varepsilon_{t-1}^{\pi^*}\right)^2\varepsilon_t^{\pi^*}}\right)^{1-\sigma_t} \\
&+(1-\delta)\delta^3 p_{t-3}^{1-\sigma_t}\left(\frac{(\breve{\upsilon}_{t-3})^3}{\breve{\pi}_{t-2}\breve{\pi}_{t-1}\breve{\pi}_t\left(\varepsilon_{t-2}^{\pi^*}\right)^3\left(\varepsilon_{t-1}^{\pi^*}\right)^2\varepsilon_t^{\pi^*}}\right)^{1-\sigma_t} + \ldots
\end{aligned}
\tag{32}
$$

4.2 Linearize (32)

- Linearize:

$$
\begin{aligned}
0 \;=\;& (1-\delta)(1-\sigma_t)\hat{p}_t \\
&+(1-\delta)(1-\sigma_t)\delta\left(\hat{p}_{t-1}+\hat{\upsilon}_{t-1}-\hat{\pi}_t-\hat{\varepsilon}_t^{\pi^*}\right) \\
&+(1-\delta)(1-\sigma_t)\delta^2\left(\hat{p}_{t-2}+2\hat{\upsilon}_{t-2}-\hat{\pi}_{t-1}-\hat{\pi}_t-2\hat{\varepsilon}_{t-1}^{\pi^*}-\hat{\varepsilon}_t^{\pi^*}\right) \\
&+(1-\delta)(1-\sigma_t)\delta^3\left(\hat{p}_{t-3}+3\hat{\upsilon}_{t-3}-\hat{\pi}_{t-2}-\hat{\pi}_{t-1}-\hat{\pi}_t-3\hat{\varepsilon}_{t-2}^{\pi^*}-2\hat{\varepsilon}_{t-1}^{\pi^*}-\hat{\varepsilon}_t^{\pi^*}\right)+\ldots
\end{aligned}
$$

- Cancel terms and bring $\hat{\pi}_t$ onto left-hand side:

$$
\hat{\pi}_t\left(\delta+\delta^2+\delta^3+\ldots\right)=\hat{p}_t+\delta\left(\hat{p}_{t-1}+\hat{\upsilon}_{t-1}-\hat{\varepsilon}_t^{\pi^*}\right)
$$

$$
+\delta^2\left(\hat{p}_{t-2}+2\hat{\upsilon}_{t-2}-\hat{\pi}_{t-1}-2\hat{\varepsilon}_{t-1}^{\pi^*}-\hat{\varepsilon}_t^{\pi^*}\right)+\delta^3\left(\hat{p}_{t-3}+3\hat{\upsilon}_{t-3}-\hat{\pi}_{t-2}-\hat{\pi}_{t-1}-3\hat{\varepsilon}_{t-2}^{\pi^*}-2\hat{\varepsilon}_{t-1}^{\pi^*}-\hat{\varepsilon}_t^{\pi^*}\right)+\ldots
$$

- Equivalently:

$$\hat{\pi}_t = \frac{1-\delta}{\delta}\hat{p}_t + (1-\delta)\left(\hat{p}_{t-1} + \hat{v}_{t-1} - \hat{\varepsilon}_t^{\pi^*}\right) + (1-\delta)\,\delta\left(\hat{p}_{t-2} + 2\hat{v}_{t-2} - \hat{\pi}_{t-1} - 2\hat{\varepsilon}_{t-1}^{\pi^*} - \hat{\varepsilon}_t^{\pi^*}\right)$$
$$+ (1-\delta)\,\delta^2\left(\hat{p}_{t-3} + 3\hat{v}_{t-3} - \hat{\pi}_{t-2} - \hat{\pi}_{t-1} - 3\hat{\varepsilon}_{t-2}^{\pi^*} - 2\hat{\varepsilon}_{t-1}^{\pi^*} - \hat{\varepsilon}_t^{\pi^*}\right) + \ldots$$

4.3 Quasi-Differencing

- Lag the last equation and multiply by δ:

$$\begin{aligned} \delta\hat{\pi}_{t-1} &= (1-\delta)\,\hat{p}_{t-1} + (1-\delta)\,\delta\left(\hat{p}_{t-2} + \hat{v}_{t-2} - \hat{\varepsilon}_{t-1}^{\pi^*}\right) \\ &\quad + (1-\delta)\,\delta^2\left(\hat{p}_{t-3} + 2\hat{v}_{t-3} - \hat{\pi}_{t-2} - 2\hat{\varepsilon}_{t-2}^{\pi^*} - \hat{\varepsilon}_{t-1}^{\pi^*}\right) + \ldots \end{aligned}$$

- Deduct the last equation from the preceding one:

$$\begin{aligned} \hat{\pi}_t - \delta\hat{\pi}_{t-1} &= \frac{1-\delta}{\delta}\hat{p}_t + (1-\delta)\left(\hat{v}_{t-1} - \hat{\varepsilon}_t^{\pi^*}\right) + (1-\delta)\,\delta\left(\hat{v}_{t-2} - \hat{\pi}_{t-1} - \hat{\varepsilon}_{t-1}^{\pi^*} - \hat{\varepsilon}_t^{\pi^*}\right) \\ &\quad + (1-\delta)\,\delta^2\left(\hat{v}_{t-3} - \hat{\pi}_{t-1} - \hat{\varepsilon}_{t-2}^{\pi^*} - \hat{\varepsilon}_{t-1}^{\pi^*} - \hat{\varepsilon}_t^{\pi^*}\right) + \ldots \end{aligned}$$

- Equivalently:

$$\begin{aligned} \hat{\pi}_t - \delta\hat{\pi}_{t-1} &= \frac{1-\delta}{\delta}\hat{p}_t - \delta\hat{\pi}_{t-1} + (1-\delta)\left(\hat{v}_{t-1} - \hat{\varepsilon}_t^{\pi^*}\right) + (1-\delta)\,\delta\left(\hat{v}_{t-2} - \hat{\varepsilon}_{t-1}^{\pi^*} - \hat{\varepsilon}_t^{\pi^*}\right) \\ &\quad + (1-\delta)\,\delta^2\left(\hat{v}_{t-3} - \hat{\varepsilon}_{t-2}^{\pi^*} - \hat{\varepsilon}_{t-1}^{\pi^*} - \hat{\varepsilon}_t^{\pi^*}\right) + \ldots \end{aligned}$$

- Equivalently:

$$\begin{aligned} \hat{\pi}_t &= \frac{1-\delta}{\delta}\hat{p}_t + (1-\delta)\left(\hat{v}_{t-1} - \hat{\varepsilon}_t^{\pi^*}\right) + (1-\delta)\,\delta\left(\hat{v}_{t-2} - \hat{\varepsilon}_{t-1}^{\pi^*} - \hat{\varepsilon}_t^{\pi^*}\right) \\ &\quad + (1-\delta)\,\delta^2\left(\hat{v}_{t-3} - \hat{\varepsilon}_{t-2}^{\pi^*} - \hat{\varepsilon}_{t-1}^{\pi^*} - \hat{\varepsilon}_t^{\pi^*}\right) + \ldots \end{aligned}$$

- We finally obtain the key expression discussed at some length in the paper:

$$\hat{\pi}_t = \frac{1-\delta}{\delta}\hat{p}_t + \hat{\psi}_t \tag{33}$$

$$\hat{\psi}_t = (1-\delta)\left(\hat{v}_{t-1} - \hat{\varepsilon}_t^{\pi^*}\right) + (1-\delta)\,\delta\left(\hat{v}_{t-2} - \hat{\varepsilon}_{t-1}^{\pi^*} - \hat{\varepsilon}_t^{\pi^*}\right) + (1-\delta)\,\delta^2\left(\hat{v}_{t-3} - \hat{\varepsilon}_{t-2}^{\pi^*} - \hat{\varepsilon}_{t-1}^{\pi^*} - \hat{\varepsilon}_t^{\pi^*}\right) + \ldots$$

4.4 The Auxiliary Variable $\hat{\psi}_t$

- Lag the last equation and multiply by δ:

$$\delta\hat{\psi}_{t-1} = (1-\delta)\delta\left(\hat{v}_{t-2} - \hat{\varepsilon}^{\pi^*}_{t-1}\right) + (1-\delta)\delta^2\left(\hat{v}_{t-3} - \hat{\varepsilon}^{\pi^*}_{t-2} - \hat{\varepsilon}^{\pi^*}_{t-1}\right) + ...$$

- Deduct the last equation from the preceding one:

$$\hat{\psi}_t = \delta\hat{\psi}_{t-1} + (1-\delta)\hat{v}_{t-1} - \hat{\varepsilon}^{\pi^*}_t \tag{34}$$

- Also, for future reference:

$$\left(E_t\hat{\psi}_{t+1} - \hat{\psi}_t\right) = (1-\delta)\hat{v}_t - (1-\delta)\,\hat{\psi}_t \tag{35}$$

- Finally we will also be able to use (33) to substitute $\hat{p}_t$ out of equation (31):

$$\hat{p}_t = \frac{\delta}{1-\delta}\left(\hat{\pi}_t - \hat{\psi}_t\right) \tag{36}$$

4.5 More Intuition for (34)

- Assume that we incorrectly linearize (32) around the period t inflation target π^*_t regardless of the time subscript of the variables. We get the following, after changing notation to distinguish this linearization from (34):

$$\widehat{\psi_t} = \delta\widehat{\psi_{t-1}} + (1-\delta)\widehat{v_{t-1}} \tag{37}$$

- Note that for $\widehat{v_{t-1}}$ we have (the same holds for $\widehat{\psi_{t-1}}$)

$$\begin{aligned}\widehat{v_{t-1}} &= \ln(v_{t-1}) - \ln(\pi^*_t) \quad \text{, while} \\ \hat{v}_{t-1} &= \ln(v_{t-1}) - \ln(\pi^*_{t-1})\end{aligned}$$

- This implies that (again similarly for $\widehat{\psi_{t-1}}$)

$$\widehat{v_{t-1}} = \hat{v}_{t-1} - \hat{\varepsilon}^{\pi^*}_t$$

- Also

$$\widehat{\psi_t} = \hat{\psi}_t$$

- Substituting the foregoing into (37) we obtain (34):

$$\hat{\psi}_t = \delta\hat{\psi}_{t-1} + (1-\delta)\hat{v}_{t-1} - \hat{\varepsilon}^{\pi^*}_t$$

5 FINAL INFLATION DYNAMICS

5.1 Final Equation for $\hat{v}_t$

- Equation (31) reproduced for ease of reference:

$$(E_t\hat{v}_{t+1} - \hat{v}_t) = \frac{(1-\delta\beta)^2}{(\delta\beta)^2}\left(\frac{\widehat{mc}_t + \hat{\mu}_t}{(1+\phi\bar{\mu}\bar{\sigma})} - \hat{p}_t\right)$$

- Plug (36) into this:

$$E_t\hat{v}_{t+1} = \hat{v}_t + \frac{(1-\delta\beta)^2}{(\delta\beta)^2}\frac{\delta}{1-\delta}\hat{\psi}_t - \frac{(1-\delta\beta)^2}{(\delta\beta)^2}\frac{\delta}{1-\delta}\hat{\pi}_t + \frac{(1-\delta\beta)^2}{(\delta\beta)^2}\frac{\widehat{mc}_t + \hat{\mu}_t}{(1+\phi\bar{\mu}\bar{\sigma})} \tag{38}$$

5.2 Final Equation for $\hat{\pi}_t$

- Deduct (33) from its once led version to get:

$$(E_t\hat{\pi}_{t+1} - \hat{\pi}_t) = \frac{1-\delta}{\delta}(E_t\hat{p}_{t+1} - \hat{p}_t) + \left(E_t\hat{\psi}_{t+1} - \hat{\psi}_t\right)$$

- To substitute for the first term on the right-hand side, use (30), which we reproduce here:

$$E_t\hat{p}_{t+1} - \hat{p}_t = \frac{-2\delta\beta}{(1-\delta\beta)}E_t\hat{v}_{t+1} - E_t\hat{\pi}_{t+1} + \frac{(1+\delta\beta)}{(1-\delta\beta)}\hat{v}_t$$

- To substitute for the second term on the right-hand side, use (35), which we reproduce here:

$$\left(E_t\hat{\psi}_{t+1} - \hat{\psi}_t\right) = (1-\delta)\hat{v}_t - (1-\delta)\,\hat{\psi}_t$$

- Substitute:

$$(E_t\hat{\pi}_{t+1} - \hat{\pi}_t) = (1-\delta)\hat{v}_t - (1-\delta)\,\hat{\psi}_t$$
$$+\frac{1-\delta}{\delta}\left[\frac{-2\delta\beta}{(1-\delta\beta)}E_t\hat{v}_{t+1} - E_t\hat{\pi}_{t+1} + \frac{(1+\delta\beta)}{(1-\delta\beta)}\hat{v}_t\right]$$

- Equivalently:

$$E_t\hat{\pi}_{t+1}\left(1+\frac{1-\delta}{\delta}\right) = \hat{\pi}_t - \frac{1-\delta}{\delta}\frac{2\delta\beta}{(1-\delta\beta)}E_t\hat{v}_{t+1} + \left(\frac{1-\delta}{\delta}\frac{(1+\delta\beta)}{(1-\delta\beta)} + (1-\delta)\right)\hat{v}_t - (1-\delta)\,\hat{\psi}_t$$

- Simplify and substitute from (38) for $E_t \hat{v}_{t+1}$:

$$E_t \hat{\pi}_{t+1} \frac{1}{\delta} = \hat{\pi}_t + \left(\frac{1-\delta}{\delta} \frac{(1+\delta\beta)}{(1-\delta\beta)} + (1-\delta) \right) \hat{v}_t - (1-\delta)\hat{\psi}_t$$

$$-\frac{1-\delta}{\delta} \frac{2\delta\beta}{(1-\delta\beta)} \left[\hat{v}_t + \frac{(1-\delta\beta)^2}{(\delta\beta)^2} \frac{\delta}{1-\delta} \hat{\psi}_t - \frac{(1-\delta\beta)^2}{(\delta\beta)^2} \frac{\delta}{1-\delta} \hat{\pi}_t + \frac{(1-\delta\beta)^2}{(\delta\beta)^2} \frac{\widehat{mc}_t + \hat{\mu}_t}{(1+\phi\bar{\mu}\bar{\sigma})} \right]$$

- Equivalently:

$$E_t \hat{\pi}_{t+1} \frac{1}{\delta} = \hat{\pi}_t \left(1 + \frac{2(1-\delta\beta)}{(\delta\beta)} \right) + \hat{v}_t \left(\frac{1-\delta}{\delta} \frac{(1+\delta\beta)}{(1-\delta\beta)} + (1-\delta) - \frac{1-\delta}{\delta} \frac{2\delta\beta}{(1-\delta\beta)} \right)$$

$$-\hat{\psi}_t \left(\frac{2(1-\delta\beta)}{(\delta\beta)} + (1-\delta) \right) - \frac{1-\delta}{\delta} \frac{2(1-\delta\beta)}{(\delta\beta)} \frac{\widehat{mc}_t + \hat{\mu}_t}{(1+\phi\bar{\mu}\bar{\sigma})}$$

- Equivalently:

$$E_t \hat{\pi}_{t+1} \frac{1}{\delta} = \hat{\pi}_t \left(\frac{\delta\beta + 2 - 2\delta\beta}{\delta\beta} \right) + \hat{v}_t \left(\frac{1-\delta}{\delta} \frac{(1+\delta\beta-2\delta\beta)}{(1-\delta\beta)} + (1-\delta) \right)$$

$$-\hat{\psi}_t \left(\frac{2 - 2\delta\beta + \delta\beta - \delta^2\beta}{\delta\beta} \right) - \frac{1-\delta}{\delta} \frac{2(1-\delta\beta)}{(\delta\beta)} \frac{\widehat{mc}_t + \hat{\mu}_t}{(1+\phi\bar{\mu}\bar{\sigma})}$$

- Equivalently:

$$E_t \hat{\pi}_{t+1} \frac{1}{\delta} = \hat{\pi}_t \left(\frac{2-\delta\beta}{\delta\beta} \right) + \hat{v}_t \left(\frac{(1-\delta)(1+\delta)}{\delta} \right)$$

$$-\hat{\psi}_t \left(\frac{2-\delta\beta-\delta^2\beta}{\delta\beta} \right) - \frac{1-\delta}{\delta} \frac{2(1-\delta\beta)}{(\delta\beta)} \frac{\widehat{mc}_t + \hat{\mu}_t}{(1+\phi\bar{\mu}\bar{\sigma})}$$

- Equivalently:

$$E_t \hat{\pi}_{t+1} = \hat{\pi}_t \left(\frac{2}{\beta} - \delta \right) + \hat{v}_t \left((1-\delta)(1+\delta) \right) \tag{39}$$

$$+\hat{\psi}_t \left(\delta(1+\delta) - \frac{2}{\beta} \right) - \frac{2(1-\delta)(1-\delta\beta)}{(\delta\beta)(1+\phi\bar{\mu}\bar{\sigma})} \left(\widehat{mc}_t + \hat{\mu}_t \right)$$

6 THE CALVO-YUN CASE

6.1 Profit Maximization

- Pricing policy of firm j that reoptimizes at t, choosing V_t and indexing to π_t^* (the current inflation target):

$$P_{t+k}(j) = V_t \Pi_{t,k}^*$$

- Define additional terms:
  - Cumulative inflation targets:

$$\Pi_{t,k}^* \equiv \prod_{j=1}^{k} \pi_{t+j}^* \text{ for } k \geq 1 \ \ (\equiv 1 \text{ for } k = 0)$$

  - Cumulative lagged aggregate inflation deviation:

$$\hat{\Pi}_{t,k}^* \equiv \sum_{j=1}^{k} \hat{\pi}_{t+j}^* \text{ for } k \geq 1 \ \ (\equiv 0 \text{ for } k = 0)$$

- Profit maximization:

$$\underset{V_t}{Max} \quad E_t \sum_{k=0}^{\infty} (\delta\beta)^k \lambda_{t+k} \left[\frac{V_t \Pi_{t,k}^*}{P_t \Pi_{t,k}} y_{t+k}(j) - mc_{t+k} y_{t+k}(j) - \frac{\phi}{2} \frac{(y_{t+k}(j) - Y_{t+k})^2}{Y_{t+k}} \right] \quad , \ s.t.$$

$$y_{t+k}(j) = Y_{t+k} \left(\frac{V_t \Pi_{t,k}^*}{P_t \Pi_{t,k}} \right)^{-\sigma_{t+k}}$$

- Substitute constraints:

$$\underset{V_t, v_t}{Max} E_t \sum_{k=0}^{\infty} (\delta\beta)^k \lambda_{t+k} \left[\left(\frac{V_t \Pi_{t,k}^*}{P_t \Pi_{t,k}} \right)^{1-\sigma_{t+k}} Y_{t+k} - mc_{t+k} \left(\frac{V_t \Pi_{t,k}^*}{P_t \Pi_{t,k}} \right)^{-\sigma_{t+k}} Y_{t+k} - \frac{\phi}{2} \frac{(y_{t+k}(j) - Y_{t+k})^2}{Y_{t+k}} \right]$$

- FOC for V_t, normalized by inflation target:

$$p_t = \frac{E_t \sum_{k=0}^{\infty} (\delta\beta)^k \check{\lambda}_{t+k} \check{y}_{t+k}(j) \sigma_{t+k} \left(mc_{t+k} + \phi \left(\frac{\check{y}_{t+k}(j) - \check{Y}_{t+k}}{\check{Y}_{t+k}} \right) \right)}{E_t \sum_{k=0}^{\infty} (\delta\beta)^k \check{\lambda}_{t+k} \check{y}_{t+k}(j) (\sigma_{t+k} - 1) \left(\frac{1}{\check{\Pi}_{t,k}} \right)} \tag{40}$$

- Linearization of goods demand:

$$\left(\hat{y}_{t+k}(j) - \hat{Y}_{t+k} \right) = -\bar{\sigma} \left(\hat{p}_t - \hat{\Pi}_{t,k} \right)$$

- Rewrite (40):

$$p_t E_t \sum_{k=0}^{\infty} (\delta\beta)^k \check{\lambda}_{t+k} \check{y}_{t+k}(j)(\sigma_{t+k}-1)\left(\frac{1}{\check{\Pi}_{t,k}}\right)$$
$$= \quad E_t \sum_{k=0}^{\infty} (\delta\beta)^k \check{\lambda}_{t+k} \check{y}_{t+k}(j)\sigma_{t+k}\left(mc_{t+k}+\phi\left(\frac{\check{y}_{t+k}(j)-\check{Y}_{t+k}}{\check{Y}_{t+k}}\right)\right)$$

- Linearization of (40):

$$\frac{\hat{p}_t}{1-\delta\beta} - E_t \sum_{k=0}^{\infty} (\delta\beta)^k \hat{\Pi}_{t,k}$$
$$= E_t \sum_{k=0}^{\infty} (\delta\beta)^k \left[\widehat{mc}_{t+k} + \hat{\mu}_{t+k} + \phi\bar{\mu}\left(\hat{y}_{t+k}(j) - \hat{Y}_{t+k}\right)\right]$$

- Equivalently:

$$\frac{\hat{p}_t}{1-\delta\beta} - E_t \sum_{k=0}^{\infty} (\delta\beta)^k \hat{\Pi}_{t,k} = E_t \sum_{k=0}^{\infty} (\delta\beta)^k \frac{\left(\widehat{mc}_{t+k} + \hat{\mu}_{t+k}\right)}{(1+\phi\bar{\mu}\bar{\sigma})}$$

- Equivalently:

$$\frac{\hat{p}_t}{1-\delta\beta} = E_t \sum_{k=0}^{\infty} (\delta\beta)^k \left[\frac{\left(\widehat{mc}_{t+k} + \hat{\mu}_{t+k}\right)}{(1+\phi\bar{\mu}\bar{\sigma})} + \hat{\Pi}_{t,k}\right]$$

- Quasi-Differencing:

$$\hat{p}_t - \delta\beta E_t \hat{p}_{t+1} = (1-\delta\beta)\left[\frac{\widehat{mc}_t + \hat{\mu}_t}{(1+\phi\bar{\mu}\bar{\sigma})}\right] + \delta\beta E_t \hat{\pi}_{t+1} \qquad (41)$$

6.2 The Price Index

- Price Level:

$$P_t = \left[(1-\delta)\sum_{s=0}^{\infty} \delta^s \left[V_{t-s}\Pi^*_{t-s,s}\right]^{1-\sigma_t}\right]^{\frac{1}{1-\sigma_t}}$$

- Deflate by current target price level P_t^* and write out terms ($\check{P}_t = P_t/P_t^*$, $\check{V}_t = V_t/P_t^*$):

$$\left(\check{P}_t\right)^{1-\sigma_t} = (1-\delta)\check{V}_t^{1-\sigma_t} + (1-\delta)\delta\check{V}_{t-1}^{1-\sigma_t} + (1-\delta)\delta^2\check{V}_{t-2}^{1-\sigma_t} + (1-\delta)\delta^3\check{V}_{t-3}^{1-\sigma_t} + ...$$

- Divide by $\check{P}_{t-1}$:

$$\begin{aligned}\left(\frac{\check{P}_t}{\check{P}_{t-1}}\right)^{1-\sigma_t} &= (1-\delta)\left(\frac{\check{V}_t}{\check{P}_t}\right)^{1-\sigma_t}\left(\frac{\check{P}_t}{\check{P}_{t-1}}\right)^{1-\sigma_t} \\ &+(1-\delta)\delta\left(\frac{\check{V}_{t-1}}{\check{P}_{t-1}}\right)^{1-\sigma_t} \\ &+(1-\delta)\delta^2\left(\frac{\check{V}_{t-2}}{\check{P}_{t-2}}\right)^{1-\sigma_t}\left(\frac{\check{P}_{t-2}}{\check{P}_{t-1}}\right)^{1-\sigma_t} \\ &+(1-\delta)\delta^3\left(\frac{\check{V}_{t-3}}{\check{P}_{t-3}}\right)^{1-\sigma_t}\left(\frac{\check{P}_{t-3}}{\check{P}_{t-2}}\right)^{1-\sigma_t}\left(\frac{\check{P}_{t-2}}{\check{P}_{t-1}}\right)^{1-\sigma_t} + ...\end{aligned}$$

- Use $\check{P}_t/\check{P}_{t-1} = \check{\pi}_t = \frac{\pi_t}{\pi_t^*}$:

$$\begin{aligned}(\check{\pi}_t)^{1-\sigma_t} &= (1-\delta)(p_t)^{1-\sigma_t}(\check{\pi}_t)^{1-\sigma_t} \\ &+(1-\delta)\delta(p_{t-1})^{1-\sigma_t} \\ &+(1-\delta)\delta^2(p_{t-2})^{1-\sigma_t}\left(\frac{1}{\check{\pi}_{t-1}}\right)^{1-\sigma_t} \\ &+(1-\delta)\delta^3(p_{t-3})^{1-\sigma_t}\left(\frac{1}{\check{\pi}_{t-2}}\right)^{1-\sigma_t}\left(\frac{1}{\check{\pi}_{t-1}}\right)^{1-\sigma_t} + ...\end{aligned}$$

- Divide through by $(\check{\pi}_t)^{1-\sigma_t}$:

$$\begin{aligned}1 &= (1-\delta)p_t^{1-\sigma_t} \\ &+(1-\delta)\delta p_{t-1}^{1-\sigma_t}\left(\frac{1}{\check{\pi}_t}\right)^{1-\sigma_t} \\ &+(1-\delta)\delta^2 p_{t-2}^{1-\sigma_t}\left(\frac{1}{\check{\pi}_{t-1}\check{\pi}_t}\right)^{1-\sigma_t} \\ &+(1-\delta)\delta^3 p_{t-3}^{1-\sigma_t}\left(\frac{1}{\check{\pi}_{t-2}\check{\pi}_{t-1}\check{\pi}_t}\right)^{1-\sigma_t} + ...\end{aligned}$$

- Linearize:

$$0 = \hat{p}_t + \delta(\hat{p}_{t-1} - \hat{\pi}_t) + \delta^2(\hat{p}_{t-2} - \hat{\pi}_{t-1} - \hat{\pi}_t) + \delta^3(\hat{p}_{t-3} - \hat{\pi}_{t-2} - \hat{\pi}_{t-1} - \hat{\pi}_t) + ...$$

- Cancel terms and bring $\hat{\pi}_t$ onto left-hand side:

$$\hat{\pi}_t\left(\delta + \delta^2 + \delta^3 + ...\right) = \hat{p}_t + \delta\hat{p}_{t-1} + \delta^2(\hat{p}_{t-2} - \hat{\pi}_{t-1}) + \delta^3(\hat{p}_{t-3} - \hat{\pi}_{t-2} - \hat{\pi}_{t-1}) + ...$$

- Equivalently:

$$\hat{\pi}_t = \frac{1-\delta}{\delta}\hat{p}_t + (1-\delta)\hat{p}_{t-1} + (1-\delta)\delta(\hat{p}_{t-2} - \hat{\pi}_{t-1}) + (1-\delta)\delta^2(\hat{p}_{t-3} - \hat{\pi}_{t-2} - \hat{\pi}_{t-1}) + ...$$

- Lag the last equation and multiply by δ:

$$\delta\hat{\pi}_{t-1} = (1-\delta)\,\hat{p}_{t-1} + (1-\delta)\,\delta\hat{p}_{t-2} + (1-\delta)\,\delta^2\,(\hat{p}_{t-3} - \hat{\pi}_{t-2}) +$$

- Deduct the last equation from the preceding one:

$$\hat{\pi}_t = \frac{1-\delta}{\delta}\hat{p}_t \tag{42}$$

6.3 Final Inflation Dynamics

- Plug (42) into (41):

$$\frac{\delta}{1-\delta}\hat{\pi}_t - \delta\beta\frac{\delta}{1-\delta}E_t\hat{\pi}_{t+1} = (1-\delta\beta)\left[\frac{\widehat{mc}_t + \hat{\mu}_t}{(1+\phi\bar{\mu}\bar{\sigma})}\right] + \delta\beta E_t\hat{\pi}_{t+1}$$

- Simplify:

$$\hat{\pi}_t = \beta E_t\hat{\pi}_{t+1} + \frac{(1-\delta\beta)(1-\delta)}{\delta}\left[\frac{\widehat{mc}_t + \hat{\mu}_t}{(1+\phi\bar{\mu}\bar{\sigma})}\right] \tag{43}$$

- This is the New Keynesian Phillips Curve for the Calvo-Yun case, which replaces the three equation pricing block of the optimal indexation system.

7 RIGIDITIES IN FINANCIAL INTERMEDIATION

7.1 Market Structure and Intermediation Demand

- Intermediaries z:
  - Competitive in their input market, renting capital K_t from households at rental rate r_t^k.
  - Monopolistically competitive in their output market, lending capital varieties $k_t(z)$ to firms at rental rates $u_t(z)$.
- Firms j:
  - Need all varieties of intermediated capital, because they must use the composite:

$$k_t(j) = \left(\int_0^1 \left(k_t(j,z)\right)^{\frac{\sigma^k - 1}{\sigma^k}} dz \right)^{\frac{\sigma^k}{\sigma^k - 1}} \quad , \quad \text{therefore } k_t = \left(\int_0^1 \left(k_t(z)\right)^{\frac{\sigma^k - 1}{\sigma^k}} dz \right)^{\frac{\sigma^k}{\sigma^k - 1}}$$

  - Cost minimization yields demands:

$$k_t(z) = k_t \left(\frac{u_t(z)}{u_t} \right)^{-\sigma^k}$$

  - The overall user cost to firms is:

$$u_t = \left(\int_0^1 \left(u_t(z)\right)^{1-\sigma^k} dz \right)^{\frac{1}{1-\sigma^k}}$$

7.2 Optimization Problem

- Profit Maximization:

$$\underset{V_t^k, v_t^k}{Max} E_t \sum_{i=0}^{\infty} \left(\delta_k \beta\right)^i \lambda_{t+i} \left[V_t^k \left(v_t^k\right)^i k_{t+i}(z) - r_{t+i}^k k_{t+i}(z) - \frac{\phi_k}{2} u_{t+i} \frac{\left(k_{t+i}(z) - k_{t+i}\right)^2}{k_{t+i}} \right] \; s.t.$$

$$k_t(z) = k_t \left(\frac{u_t(z)}{u_t} \right)^{-\sigma^k}$$

  - Cost of deviating from "normal" activity level of other intermediaries:
    - ∗ Quadratic in the % deviation from "normal" intermediation activity $\frac{\phi_k}{2} \frac{\left(k_{t+i}(z) - k_{t+i}\right)^2}{k_{t+i}^2}$.
    - ∗ Proportional to aggregate intermediation revenue $u_{t+i} k_{t+i}$.

- Define terms:
  - Front loading term: $p_t^k = V_t^k / u_t$.
  - Gross intermediation spread: $s_t = u_t / r_t^k$.
  - Gross rate of change of user cost: $\pi_t^k = u_t / u_{t-1}$, with cumulative rate of change $\Pi_{t,i}^k$, as in (15).
  - Mark-up: $\mu_k = \sigma^k / (\sigma^k - 1)$ (assumed not time-varying).

7.3 First-Order Conditions

- First-order conditions rescaled by technology ($\check{k}_t(z) = k_t(z)/S_t^y$).
  - FOC for V_t^k:

$$
\begin{aligned}
& p_t^k E_t \sum_{i=0}^{\infty} (\delta_k \beta)^i \check{\lambda}_{t+i} \check{k}_{t+i}(z) u_{t+i} \left(\frac{\left(v_t^k\right)^i}{\Pi_{t,i}^k} \right) \\
= \quad & \mu_k E_t \sum_{i=0}^{\infty} (\delta_k \beta)^i \check{\lambda}_{t+i} \check{k}_{t+i}(z) u_{t+i} \left[\frac{1}{s_{t+i}} + \phi_k \left(\frac{\check{k}_{t+i}(z) - \check{k}_{t+i}}{\check{k}_{t+i}} \right) \right]
\end{aligned}
$$

  - FOC for v_t^k:

$$
\begin{aligned}
& p_t^k E_t \sum_{i=0}^{\infty} (\delta_k \beta)^i \, i \check{\lambda}_{t+i} \check{k}_{t+i}(z) u_{t+i} \left(\frac{\left(v_t^k\right)^i}{\Pi_{t,i}^k} \right) \\
= \quad & \mu_k E_t \sum_{i=0}^{\infty} (\delta_k \beta)^i \, i \check{\lambda}_{t+i} \check{k}_{t+i}(z) u_{t+i} \left[\frac{1}{s_{t+i}} + \phi_k \left(\frac{\check{k}_{t+i}(z) - \check{k}_{t+i}}{\check{k}_{t+i}} \right) \right]
\end{aligned}
$$

- The linearized FOC for this problem are exactly analogous to those for firm price setting, with three modifications:
  - The term $\widehat{mc}_t$ is replaced with $-\hat{s}_t$.
  - The mark-up μ_k is assumed to not be time-varying.
  - The user cost of capital and return to capital are real returns rather than nominal price levels, therefore the unit root in the inflation target has no effect on the inertial component of pricing.

7.4 Final User Cost Dynamics

$$\hat{\psi}_t^k = \delta_k \hat{\psi}_{t-1}^k + (1-\delta_k)\hat{v}_{t-1}^k \tag{44}$$

$$E_t \hat{v}_{t+1}^k = \hat{v}_t^k + \frac{(1-\delta_k\beta)^2}{(\delta_k\beta)^2}\frac{\delta_k}{1-\delta_k}\hat{\psi}_t^k - \frac{(1-\delta_k\beta)^2}{(\delta_k\beta)^2}\frac{\delta_k}{1-\delta_k}\hat{\pi}_t^k - \frac{(1-\delta_k\beta)^2}{(\delta_k\beta)^2}\frac{\hat{s}_t}{(1+\phi_k\bar{\mu}_k\bar{\sigma}_k)} \tag{45}$$

$$\begin{aligned} E_t \hat{\pi}_{t+1}^k = \hat{\pi}_t^k \left(\frac{2}{\beta} - \delta_k\right) + \hat{v}_t^k \left((1-\delta_k)(1+\delta_k)\right) \\ + \hat{\psi}_t^k \left(\delta_k(1+\delta_k) - \frac{2}{\beta}\right) + \frac{2(1-\delta_k)(1-\delta_k\beta)}{(\delta_k\beta)(1+\phi_k\bar{\mu}_k\bar{\sigma}_k)}\hat{s}_t \end{aligned} \tag{46}$$

$$\hat{u}_t = \hat{u}_{t-1} + \hat{\pi}_t^k \tag{47}$$

$$\hat{s}_t = \hat{u}_t - \hat{r}_t^k \tag{48}$$

8 HOUSEHOLD WAGE SETTING

8.1 Utility Maximization

- Wage setting policy of a worker i that reoptimizes at t, choosing V_t^w and v_t^w (a gross wage inflation rate):

$$W_{t+k}(i) = V_t^w \left(v_t^w\right)^k$$

- Define terms:
  - Front-loading term:

$$p_t^w \equiv \frac{V_t^w}{W_t}$$

  - Real wage:

$$w_t = \frac{W_t}{P_t}$$

  - Wage inflation rates re-scaled by the inflation target, with $\breve{\pi}_t^w = \pi_t^w / \pi_t^*$.
  - Cumulative aggregate rescaled wage inflation:

$$\breve{\Pi}_{t,k}^w = \Pi_{j=1}^k \breve{\pi}_{t+j}^w \text{ for } k \geq 1 \;\; (\equiv 1 \text{ for } k = 0)$$

  - Cumulative aggregate rescaled wage inflation deviation:

$$\hat{\Pi}_{t,k}^w = \Sigma_{j=1}^k \hat{\pi}_{t+j}^w \text{ for } k \geq 1 \;\; (\equiv 0 \text{ for } k = 0)$$

- Utility maximization - relevant part of the problem:

$$\underset{V_t^w, v_t^w}{Max} \quad E_t \sum_{k=0}^{\infty} \left(\delta_w \beta\right)^k \left[-\psi S_t^L \frac{\left(L_{t+k}(i)\right)^{1+\frac{1}{\gamma}}}{1+\frac{1}{\gamma}} + \lambda_{t+k} \frac{V_t^w \left(v_t^w\right)^k}{W_{t+k}} \frac{W_{t+k}}{P_{t+k}} L_{t+k}(i) \right.$$

$$\left. -\lambda_{t+k} \frac{\phi_w}{2} \frac{W_{t+k}}{P_{t+k}} \frac{\left(L_{t+k}(i) - \ell_{t+k}\right)^2}{\ell_{t+k}} \right] \quad , \; s.t.$$

$$L_{t+k}(i) = \ell_{t+k} \left(\frac{V_t^w \left(v_t^w\right)^k}{W_{t+k}} \right)^{-\sigma_w}$$

8.2 First-Order Conditions

- Real wage rescaled by technology, with $\breve{w}_t = w_t / S_t^y$.
- Firm specific wage inflation rescaled by the inflation target $\breve{v}_t^w = v_t^w / \pi_t^*$.
  - By the unit root property we can ignore future changes in the inflation target (see discussion above for firms):

$$E_t \hat{v}_{t+k}^w = E_t \left(\ln \left(v_{t+k}^w \right) - \ln(\pi_{t+k}^*) \right) = E_t \left(\ln \left(v_{t+k}^w \right) - \ln(\pi_t^*) \right)$$

- FOC for V_t^w:

$$\begin{aligned} & E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k \check{\lambda}_{t+k} L_{t+k}(i) \left(\breve{w}_{t+k} \left(\sigma_{t+k}^w - 1 \right) p_t^w \frac{\left(\breve{v}_t^w\right)^k}{\breve{\Pi}_{t,k}^w} - \phi_w \breve{w}_{t+k} \sigma_{t+k}^w \left(\frac{L_{t+k}(i) - \ell_{t+k}}{\ell_{t+k}} \right) \right) \\ = \; & E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k \psi \sigma_{t+k}^w S_{t+k}^L \left(L_{t+k}(i) \right)^{1+\frac{1}{\gamma}} \end{aligned} \tag{49}$$

- FOC w.r.t. v_t^w:

$$\begin{aligned} & E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k k \check{\lambda}_{t+k} L_{t+k}(i) \left(\breve{w}_{t+k} \left(\sigma_{t+k}^w - 1 \right) p_t^w \frac{\left(\breve{v}_t^w\right)^k}{\breve{\Pi}_{t,k}^w} - \phi_w \breve{w}_{t+k} \sigma_{t+k}^w \left(\frac{L_{t+k}(i) - \ell_{t+k}}{\ell_{t+k}} \right) \right) \\ = \; & E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k k \psi \sigma_{t+k}^w S_{t+k}^L \left(L_{t+k}(i) \right)^{1+\frac{1}{\gamma}} \end{aligned} \tag{50}$$

8.3 Linearization

8.3.1 Linearization for V_t^w

- First step:

$$E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k \left[\hat{\lambda}_{t+k} + \hat{L}_{t+k}(i) + \hat{w}_{t+k} + \left(\hat{p}_t^w + k\hat{v}_t^w - \hat{\Pi}_{t,k}^w\right)\right]$$

$$= E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k \left[\hat{\mu}_{t+k}^w + \hat{S}_{t+k}^L + \left(1 + \frac{1}{\gamma}\right) \hat{L}_{t+k}(i) + \phi_w \bar{\mu}_w \left(\hat{L}_{t+k}(i) - \hat{\ell}_{t+k}\right)\right]$$

- Linearization of labor demand:

$$\left(\hat{L}_{t+k}(i) - \hat{\ell}_{t+k}\right) = -\bar{\sigma}_w \left(\hat{p}_t^w + k\hat{v}_t^w - \hat{\Pi}_{t,k}^w\right)$$

- Marginal rate of substitution:
  - MRS in levels:

$$mrs_t = \frac{S_t^L \psi L_t(i)^{\frac{1}{\gamma}}}{\lambda_t}$$

  - Log-linearized:

$$\widehat{mrs}_t = \frac{1}{\gamma} \hat{L}_t(i) + \hat{S}_t^L - \hat{\lambda}_t$$

  - Combine with the expression for labor demand (note that, for contemporaneous terms, $k = 0$):

$$\widehat{mrs}_t = \frac{1}{\gamma} \hat{\ell}_t - \frac{\bar{\sigma}_w}{\gamma} \hat{p}_t^w + \hat{S}_t^L - \hat{\lambda}_t$$

- Combine the above:

$$E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k \left(\hat{p}_t^w + k\hat{v}_t^w - \hat{\Pi}_{t,k}^w\right) = E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k \frac{\left(\widehat{mrs}_{t+k} - \hat{w}_{t+k} + \hat{\mu}_{t+k}^w\right)}{(1 + \phi_w \bar{\mu}_w \bar{\sigma}_w)}$$

- Apply formulas (19) and (20):

$$\frac{\hat{p}_t^w}{1 - \delta_w \beta} + \frac{\hat{v}_t^w \delta_w \beta}{(1 - \delta_w \beta)^2} = E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k \left[\frac{\left(\widehat{mrs}_{t+k} - \hat{w}_{t+k} + \hat{\mu}_{t+k}^w\right)}{(1 + \phi_w \bar{\mu}_w \bar{\sigma}_w)} + \hat{\Pi}_{t,k}^w\right] \tag{51}$$

- With the appropriate change in notation this is exactly identical to equation (23) for price setting, after replacing $\left(\widehat{mc}_{t+k} + \hat{\mu}_{t+k}\right) / (1 + \phi \bar{\mu} \bar{\sigma})$ with $\left(\widehat{mrs}_{t+k} - \hat{w}_{t+k} + \hat{\mu}_{t+k}^w\right) / (1 + \phi_w \bar{\mu}_w \bar{\sigma}_w)$.

8.3.2 Linearization for v_t^w

- First step:

$$E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k k \left[\hat{\lambda}_{t+k} + \hat{L}_{t+k}(i) + \hat{w}_{t+k} + \left(\hat{p}_t^w + k\hat{v}_t^w - \hat{\Pi}_{t,k}^w \right) \right]$$

$$= E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k k \left[\hat{\mu}_{t+k}^w + \hat{S}_{t+k}^L + \left(1 + \frac{1}{\gamma}\right) \hat{L}_{t+k}(i) + \phi_w \bar{\mu}_w \left(\hat{L}_{t+k}(i) - \hat{\ell}_{t+k} \right) \right]$$

- Combine with the above:

$$E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k k \left(\hat{p}_t^w + k\hat{v}_t^w - \hat{\Pi}_{t,k}^w \right) = E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k k \frac{\left(\widehat{mrs}_{t+k} - \hat{w}_{t+k} + \hat{\mu}_{t+k}^w \right)}{(1 + \phi_w \bar{\mu}_w \bar{\sigma}_w)}$$

- Apply formulas (19) and (20):

$$\begin{aligned} & \frac{\hat{p}_t^w \delta_w \beta}{(1-\delta_w \beta)^2} + \frac{\hat{v}_t^w \delta_w \beta (1+\delta_w \beta)}{(1-\delta_w \beta)^3} \\ = \quad & E_t \sum_{k=0}^{\infty} (\delta_w \beta)^k k \left[\frac{\left(\widehat{mrs}_{t+k} - \hat{w}_{t+k} + \hat{\mu}_{t+k}^w \right)}{(1 + \phi_w \bar{\mu}_w \bar{\sigma}_w)} + \hat{\Pi}_{t,k}^w \right] \end{aligned} \tag{52}$$

- With the appropriate change in notation this is exactly identical to equation (24) for price setting, after replacing $\left(\widehat{mc}_{t+k} + \hat{\mu}_{t+k}\right) / (1 + \phi\bar{\mu}\bar{\sigma})$ with $\left(\widehat{mrs}_{t+k} - \hat{w}_{t+k} + \hat{\mu}_{t+k}^w\right) / (1 + \phi_w \bar{\mu}_w \bar{\sigma}_w)$.

8.4 Final Wage Inflation Dynamics

- Given the identical forms of (51)/(23) and (52)/(24), we obtain after quasi-differencing as the end result an equation analogous to (31):

$$\left(E_t\hat{v}^w_{t+1} - \hat{v}^w_t\right) = \frac{(1-\delta_w\beta)^2}{(\delta_w\beta)^2}\left(\frac{(\widehat{mrs}_t - \hat{w}_t + \hat{\mu}^w_t)}{(1+\phi_w\bar{\mu}_w\bar{\sigma}_w)} - \hat{p}^w_t\right) \tag{53}$$

- Furthermore, the derivation of the wage index is identical to that for the price index above. The correct expression for wage inflation is:

$$\hat{\pi}^w_t = \hat{w}_t - \hat{w}_{t-1} + \hat{g}_t - \hat{\pi}_t \tag{54}$$

- Furthermore, the derivation of the wage index is identical to that for the price index above. We get:

$$\hat{\pi}^w_t = \frac{1-\delta_w}{\delta_w}\hat{p}^w_t + \hat{\psi}^w_t \tag{55}$$

$$\hat{\psi}^w_t = \delta_w\hat{\psi}^w_{t-1} + (1-\delta_w)\hat{v}^w_{t-1} - \hat{\varepsilon}^{\pi^*}_t \tag{56}$$

- Combining this with the results of quasi-differencing we obtain the equations for $E_t\hat{v}^w_{t+1}$ and $E_t\hat{\pi}^w_{t+1}$:

$$\begin{aligned} E_t\hat{v}^w_{t+1} &= \hat{v}^w_t + \frac{(1-\delta_w\beta)^2}{(\delta_w\beta)^2}\frac{\delta_w}{1-\delta_w}\hat{\psi}^w_t - \frac{(1-\delta_w\beta)^2}{(\delta_w\beta)^2}\frac{\delta_w}{1-\delta_w}\hat{\pi}^w_t \\ &\quad + \frac{(1-\delta_w\beta)^2}{(\delta_w\beta)^2}\frac{(\widehat{mrs}_t - \hat{w}_t + \hat{\mu}^w_t)}{(1+\phi_w\bar{\mu}_w\bar{\sigma}_w)} \end{aligned} \tag{57}$$

$$\begin{aligned} E_t\hat{\pi}^w_{t+1} &= \hat{\pi}^w_t\left(\frac{2}{\beta} - \delta_w\right) + \hat{v}^w_t\left((1-\delta_w)(1+\delta_w)\right) \\ &\quad + \hat{\psi}^w_t\left(\delta_w(1+\delta_w) - \frac{2}{\beta}\right) - \frac{2(1-\delta_w)(1-\delta_w\beta)}{(\delta_w\beta)}\frac{(\widehat{mrs}_t - \hat{w}_t + \hat{\mu}^w_t)}{(1+\phi_w\bar{\mu}_w\bar{\sigma}_w)} \end{aligned} \tag{58}$$

- We can also use (55) to get a final expression for the marginal rate of substitution:

$$\widehat{mrs}_t = \frac{1}{\gamma}\hat{\ell}_t - \frac{\overline{\sigma^w}}{\gamma}\frac{\delta_w}{1-\delta_w}\hat{\pi}^w_t + \frac{\overline{\sigma^w}}{\gamma}\frac{\delta_w}{1-\delta_w}\hat{\psi}^w_t + \hat{S}^L_t - \hat{\lambda}_t \tag{59}$$

8.5 The Calvo-Yun Case

By analogy with the derivation under price setting we obtain the following for wage setting:

$$\hat{\pi}^w_t = \beta E_t\hat{\pi}^w_{t+1} + \frac{(1-\delta_w\beta)(1-\delta_w)}{\delta_w}\left(\frac{\widehat{mrs}_t - \hat{w}_t + \hat{\mu}^w_t}{(1+\phi_w\bar{\mu}_w\bar{\sigma}_w)}\right) \tag{60}$$

9 GOVERNMENT AND MARKET CLEARING

- Interest rate rule in levels:
  - Re-scaled by dividing through by π_t^*.
  - Remember that $\check{i}_t = i_t/\pi_t^*$.
  - Let $r_g = \bar{g}/\beta$.

$$\left(\frac{i_t}{\pi_t^*}\right)^4 = \left[\left(\frac{i_{t-1}}{\pi_{t-1}^*}\frac{\pi_{t-1}^*}{\pi_t^*}\right)^4\right]^{\xi^{int}} \left[(r_g)^4 \frac{\pi_t\pi_{t-1}\pi_{t-2}\pi_{t-3}}{(\pi_t^*)^4}\right]^{1-\xi^{int}} \left[\frac{\pi_{t+1}\pi_t\pi_{t-1}\pi_{t-2}}{(\pi_t^*)^4}\right]^{\xi^\pi} \left[\frac{\check{Y}_{t-1}}{\bar{Y}}\right]^{\xi^y} \left[\frac{\check{Y}_{t+1}}{\check{Y}_{t-3}}\right]^{\xi^{\Delta y}}$$

  - Equivalently:

$$\left(\check{i}_t\right)^4 = \left[\left(\frac{\check{i}_{t-1}}{\varepsilon_t^{\pi^*}}\right)^4\right]^{\xi^{int}} \left[(r_g)^4 \frac{\check{\pi}_t\check{\pi}_{t-1}\check{\pi}_{t-2}\check{\pi}_{t-3}}{\left(\varepsilon_t^{\pi^*}\right)^3\left(\varepsilon_{t-1}^{\pi^*}\right)^2\left(\varepsilon_{t-2}^{\pi^*}\right)}\right]^{1-\xi^{int}} \left[\frac{\check{\pi}_{t+1}\check{\pi}_t\check{\pi}_{t-1}\check{\pi}_{t-2}}{\left(\varepsilon_t^{\pi^*}\right)^2\varepsilon_{t-1}^{\pi^*}}\right]^{\xi^\pi} \left[\frac{\check{Y}_{t-1}}{\bar{Y}}\right]^{\xi^y} \left[\frac{\check{Y}_{t+1}}{\check{Y}_{t-3}}\right]^{\xi^{\Delta y}}$$

- Linearized interest rate rule:

$$\hat{i}_t = \xi^{int}\left(\hat{i}_{t-1} - \hat{\varepsilon}_t^{\pi^*}\right) + (1-\xi^{int})E_t\left(\frac{\hat{\pi}_t + \hat{\pi}_{t-1} + \hat{\pi}_{t-2} + \hat{\pi}_{t-3} - 3\hat{\varepsilon}_t^{\pi^*} - 2\hat{\varepsilon}_{t-1}^{\pi^*} - \hat{\varepsilon}_{t-2}^{\pi^*}}{4}\right) \tag{61}$$

$$+\xi^\pi E_t\left(\frac{\hat{\pi}_{t+1} + \hat{\pi}_t + \hat{\pi}_{t-1} + \hat{\pi}_{t-2} - 2\hat{\varepsilon}_t^{\pi^*} - \hat{\varepsilon}_{t-1}^{\pi^*}}{4}\right) + \xi^y\left(\frac{\hat{y}_{t-1}}{4}\right) + \xi^{\Delta y}E_t\left(\frac{\hat{y}_{t+1} - \hat{y}_{t-3}}{4}\right) + \frac{\hat{S}_t^{int}}{4}$$

- Real Interest Rate:

$$\widehat{rr}_t = \hat{i}_t - E_t\hat{\pi}_{t+1} \tag{62}$$

- Steady State Spending = Fixed Fraction of GDP:

$$\begin{aligned} \overline{GOV} &= s_g\bar{Y} \\ GOV_t &= S_t^{gov}\overline{GOV} \end{aligned}$$

- Goods Market Clearing:
  - In levels:

$$Y_t = C_t + I_t + GOV_t$$

  - Rescaled by technology:

$$\check{Y}_t = \check{C}_t + \check{I}_t + G\check{O}V_t$$

  - Linearization:

$$\bar{Y}\hat{Y}_t = \bar{C}\hat{C}_t + \hat{I}_t + \overline{GOV}\hat{S}_t^g \tag{63}$$

- Capital Market Clearing (skipping the aggregation steps):

$$\hat{k}_t = \hat{x}_t + \hat{K}_t \tag{64}$$

10 STATIONARY EXOGENOUS SHOCKS

- Consumption:
$$\hat{S}_t^c = \rho^c \hat{S}_{t-1}^c + \hat{\varepsilon}_t^c \tag{65}$$
- Labor Supply:
$$\hat{S}_t^L = \rho^L \hat{S}_{t-1}^L + \hat{\varepsilon}_t^L \tag{66}$$
- Investment Demand:
$$\hat{S}_t^{inv} = \rho^{inv} \hat{S}_{t-1}^{inv} + \hat{\varepsilon}_t^{inv} \tag{67}$$
- Government Spending:
$$\hat{S}_t^{gov} = \rho^{gov} \hat{S}_{t-1}^{gov} + \hat{\varepsilon}_t^{gov} \tag{68}$$
- Monetary Policy:
$$\hat{S}_t^{int} = \rho^{int} \hat{S}_{t-1}^{int} + \hat{\varepsilon}_t^{int} \tag{69}$$
- Goods Mark-Up:
$$\hat{\mu}_t = \hat{\varepsilon}^{\mu} \tag{70}$$
- Labor Mark-Up:
$$\hat{\mu}_t^w = \hat{\varepsilon}^{\mu^w} \tag{71}$$

11 COMPLETE DYNAMIC SYSTEM

11.1 Steady State and Parameter Calibration

$$\bar{L} = 1/3 \quad \text{(fixed at the outset, } \psi \text{ is endogenous to that choice)} \tag{SS1}$$

$$s_L = 0.64 \text{ (labor share after adjusting for markups)} \tag{SS2}$$

$$\alpha = 1 - \bar{\mu} s_L \tag{SS3}$$

$$\beta_g = \frac{\beta}{\bar{g}} \tag{SS4}$$

$$\bar{q} = 1 \tag{SS5}$$

$$\bar{r}^k = \left(\frac{1 - \beta_g}{\beta_g} + \Delta \right) \tag{SS6}$$

$$\bar{s} = \bar{\mu}^k \tag{SS7}$$

$$\bar{u} = \bar{s} \bar{r}^k \tag{SS8}$$

$$\overline{mc} = \frac{1}{\bar{\mu}} \tag{SS9}$$

$$\bar{w} = \left(\frac{\overline{mc}}{A \bar{u}^{\alpha}} \right)^{\frac{1}{1-\alpha}} \tag{SS10}$$

$$\bar{Y} = \frac{\bar{w} \bar{L}}{(1 - \alpha) \overline{mc}} \tag{SS11}$$

$$\bar{K} = \frac{\alpha \overline{mc} \bar{Y}}{\bar{u}} \tag{SS12}$$

$$\bar{I} = (\bar{g} + \Delta - 1)\,\bar{K} \tag{SS13}$$

$$\overline{GOV} = s_g \bar{Y} \tag{SS14}$$

$$\bar{C} = \bar{Y} - \bar{I} - \overline{GOV} \tag{SS15}$$

11.2 Demand Block

$$\hat{\lambda}_t = \hat{\imath}_t + E_t\left(\hat{\lambda}_{t+1} - \hat{\pi}_{t+1} - \hat{g}_{t+1}\right) \tag{D1}$$

$$\hat{S}^c_t - \hat{H}_t = \hat{\lambda}_t \tag{D2}$$

$$\hat{H}_t = \frac{1}{1-\frac{\nu}{\bar{g}}}\hat{C}_t - \frac{\frac{\nu}{\bar{g}}}{1-\frac{\nu}{\bar{g}}}\left(\hat{C}_{t-1} - \hat{g}_t\right) \tag{D3}$$

$$\bar{K}\bar{g}\left(\hat{K}_t - \hat{g}_t\right) = (1-\Delta)\bar{K}\hat{K}_{t-1} + \hat{I}_{t-1} \tag{D4}$$

$$\hat{q}_t = \frac{\theta_k}{\bar{K}}\hat{I}_t - \frac{\theta_k \bar{I}}{\bar{K}}\hat{K}_t + \frac{\theta_i}{\bar{K}}\left(\hat{I}_t - \hat{I}_{t-1}\right) - \frac{\theta_i \bar{I}}{\bar{K}}\left(\hat{K}_t - \hat{K}_{t-1}\right) + \hat{S}^{inv}_t \tag{D5}$$

$$\begin{aligned} \hat{\lambda}_t + \hat{q}_t \quad = \quad & E_t\Big\{\hat{\lambda}_{t+1} - \hat{g}_{t+1} + \beta_g(1-\Delta)\hat{q}_{t+1} + \beta_g(r+\Delta)\hat{r}^k_{t+1} \\ & + \frac{\beta_g\theta_k\bar{I}}{\bar{K}^2}\hat{I}_{t+1} - \frac{\beta_g\theta_k\bar{I}^2}{\bar{K}^2}\hat{K}_{t+1} \\ & + \frac{\beta_g\theta_i\bar{I}}{\bar{K}^2}\left(\hat{I}_{t+1} - \hat{I}_t\right) - \frac{\beta_g\theta_i\bar{I}^2}{\bar{K}^2}\left(\hat{K}_{t+1} - \hat{K}_t\right)\Big\} \end{aligned} \tag{D6}$$

$$\hat{r}^k_t = \epsilon\hat{x}_t \tag{D7}$$

$$\widehat{INV}_t = \frac{\hat{I}_t}{\bar{I}} \tag{D8}$$

$$\hat{\psi}_t^w = \delta_w \hat{\psi}_{t-1}^w + (1-\delta_w)\hat{v}_{t-1}^w - \hat{\varepsilon}_t^{\pi^*} \tag{D9}$$

$$E_t \hat{v}_{t+1}^w = \hat{v}_t^w + \frac{(1-\delta_w\beta)^2}{(\delta_w\beta)^2}\frac{\delta_w}{1-\delta_w}\hat{\psi}_t^w - \frac{(1-\delta_w\beta)^2}{(\delta_w\beta)^2}\frac{\delta_w}{1-\delta_w}\hat{\pi}_t^w + \frac{(1-\delta_w\beta)^2}{(\delta_w\beta)^2}\frac{\widehat{mrs}_t - \hat{w}_t + \hat{\mu}_t^w}{(1+\phi_w\bar{\mu}_w\bar{\sigma}_w)} \tag{D10}$$

$$E_t \hat{\pi}_{t+1}^w = \hat{\pi}_t^w \left(\frac{2}{\beta} - \delta_w\right) + \hat{v}_t^w \left((1-\delta_w)(1+\delta_w)\right) \tag{D11}$$

$$+\hat{\psi}_t^w \left(\delta_w(1+\delta_w) - \frac{2}{\beta}\right) - \frac{2(1-\delta_w)(1-\delta_w\beta)}{(\delta_w\beta)}\frac{(\widehat{mrs}_t - \hat{w}_t + \hat{\mu}_t^w)}{(1+\phi_w\bar{\mu}_w\bar{\sigma}_w)}$$

$$\hat{w}_t = \hat{w}_{t-1} - \hat{g}_t + \hat{\pi}_t^w - \hat{\pi}_t \tag{D12}$$

$$\widehat{mrs}_t = \frac{1}{\gamma}\hat{L}_t - \frac{\overline{\sigma^w}}{\gamma}\frac{\delta_w}{1-\delta_w}\hat{\pi}_t^w + \frac{\overline{\sigma^w}}{\gamma}\frac{\delta_w}{1-\delta_w}\hat{\psi}_t^w + \hat{S}_t^L - \hat{\lambda}_t \tag{D13}$$

$$\hat{\psi}_t^k = \delta_k \hat{\psi}_{t-1}^k + (1-\delta_k)\hat{v}_{t-1}^k \tag{D14}$$

$$E_t \hat{v}_{t+1}^k = \hat{v}_t^k + \frac{(1-\delta_k\beta)^2}{(\delta_k\beta)^2}\frac{\delta_k}{1-\delta_k}\hat{\psi}_t^k - \frac{(1-\delta_k\beta)^2}{(\delta_k\beta)^2}\frac{\delta_k}{1-\delta_k}\hat{\pi}_t^k - \frac{(1-\delta_k\beta)^2}{(\delta_k\beta)^2(1+\phi^k\bar{\mu}^k\bar{\sigma}^k)}\hat{s}_t \tag{D15}$$

$$E_t \hat{\pi}_{t+1}^k = \hat{\pi}_t^k \left(\frac{2}{\beta} - \delta_k\right) + \hat{v}_t^k \left((1-\delta_k)(1+\delta_k)\right) \tag{D16}$$

$$+\hat{\psi}_t^k \left(\delta_k(1+\delta_k) - \frac{2}{\beta}\right) + \frac{2(1-\delta_k)(1-\delta_k\beta)}{(\delta_k\beta)(1+\phi^k\bar{\mu}^k\bar{\sigma}^k)}\hat{s}_t$$

$$\hat{u}_t = \hat{u}_{t-1} + \hat{\pi}_t^k \tag{D17}$$

$$\hat{s}_t = \hat{u}_t - \hat{r}_t^k \tag{D18}$$

11.3 Supply Block

$$\hat{\psi}_t = \delta\hat{\psi}_{t-1} + (1-\delta)\hat{v}_{t-1} - \hat{\varepsilon}_t^{\pi^*} \tag{S1}$$

$$E_t\hat{v}_{t+1} = \hat{v}_t + \frac{(1-\delta\beta)^2}{(\delta\beta)^2}\frac{\delta}{1-\delta}\hat{\psi}_t - \frac{(1-\delta\beta)^2}{(\delta\beta)^2}\frac{\delta}{1-\delta}\hat{\pi}_t + \frac{(1-\delta\beta)^2}{(\delta\beta)^2}\frac{\widehat{mc}_t + \hat{\mu}_t}{(1+\phi\bar{\mu}\bar{\sigma})} \tag{S2}$$

$$E_t\hat{\pi}_{t+1} = \hat{\pi}_t\left(\frac{2}{\beta} - \delta\right) + \hat{v}_t\left((1-\delta)(1+\delta)\right) \tag{S3}$$
$$+\hat{\psi}_t\left(\delta(1+\delta) - \frac{2}{\beta}\right) - \frac{2(1-\delta)(1-\delta\beta)}{(\delta\beta)(1+\phi\bar{\mu}\bar{\sigma})}\left(\widehat{mc}_t + \hat{\mu}_t\right)$$

$$\widehat{mc}_t = (1-\alpha)\hat{w}_t + \alpha\hat{u}_t \tag{S4}$$

$$\hat{L}_t = \widehat{mc}_t - \hat{w}_t + \hat{Y}_t \tag{S5}$$

$$\hat{k}_t = \widehat{mc}_t - \hat{u}_t + \hat{Y}_t \tag{S6}$$

11.4 Aggregate Relationships

$$\hat{\imath}_t = \xi^{int}\left(\hat{\imath}_{t-1} - \hat{\varepsilon}_t^{\pi^*}\right) + (1-\xi^{int})E_t\left(\frac{\hat{\pi}_t + \hat{\pi}_{t-1} + \hat{\pi}_{t-2} + \hat{\pi}_{t-3} - 3\hat{\varepsilon}_t^{\pi^*} - 2\hat{\varepsilon}_{t-1}^{\pi^*} - \hat{\varepsilon}_{t-2}^{\pi^*}}{4}\right) \tag{A1}$$
$$+\xi^{\pi}E_t\left(\frac{\hat{\pi}_{t+1} + \hat{\pi}_t + \hat{\pi}_{t-1} + \hat{\pi}_{t-2} - 2\hat{\varepsilon}_t^{\pi^*} - \hat{\varepsilon}_{t-1}^{\pi^*}}{4}\right) + \xi^{y}\left(\frac{\hat{y}_{t-1}}{4}\right) + \xi^{\Delta y}E_t\left(\frac{\hat{y}_{t+1} - \hat{y}_{t-3}}{4}\right) + \frac{\hat{S}_t^{int}}{4}$$

$$\widehat{rr}_t = \hat{\imath}_t - E_t\hat{\pi}_{t+1} \tag{A2}$$

$$\bar{Y}\hat{Y}_t = \bar{C}\hat{C}_t + \hat{I}_t + \overline{GOV}\hat{S}_t^{gov} \tag{A3}$$

$$\hat{k}_t = \hat{x}_t + \hat{K}_t \tag{A4}$$

11.5 Shocks

11.5.1 Unit Roots

$$\hat{g}_t = \hat{g}_t^{gr} + \hat{g}_t^{iid} \tag{U1}$$

$$\hat{g}_t^{gr} = \rho_g \hat{g}_{t-1}^{gr} + \hat{\varepsilon}_t^{gr} \tag{U2}$$

$$\hat{g}_t^{iid} = \hat{\varepsilon}_t^{iid} \tag{U3}$$

11.5.2 Stationary

$$\hat{S}_t^c = \rho^c \hat{S}_{t-1}^c + \hat{\varepsilon}_t^c \tag{X1}$$

$$\hat{S}_t^L = \rho^L \hat{S}_{t-1}^L + \hat{\varepsilon}_t^L \tag{X2}$$

$$\hat{S}_t^{inv} = \rho^{inv} \hat{S}_{t-1}^{inv} + \hat{\varepsilon}_t^{inv} \tag{X3}$$

$$\hat{S}_t^{gov} = \rho^{gov} \hat{S}_{t-1}^{gov} + \hat{\varepsilon}_t^{gov} \tag{X4}$$

$$\hat{S}_t^{int} = \rho^{int} \hat{S}_{t-1}^{int} + \hat{\varepsilon}_t^{int} \tag{X5}$$

$$\hat{\mu}_t = \hat{\varepsilon}^{\mu} \tag{X6}$$

$$\hat{\mu}_t^w = \hat{\varepsilon}^{\mu^w} \tag{X7}$$

12 REPORTING VARIABLES

- Remember that the ˆ-variables are the ones that come from the computer code.

12.1 Real Variables - Growth Rates

- Real GDP Growth:

$$DOT\_GDP_t = \ln Y_t - \ln Y_{t-1} = \ln \check{Y}_t - \ln \check{Y}_{t-1} + \ln g_t = \hat{Y}_t - \hat{Y}_{t-1} + \hat{g}_t + \ln(\bar{g})$$

- Real Consumption Growth:

$$DOT\_C_t = \ln C_t - \ln C_{t-1} = \ln \check{C}_t - \ln \check{C}_{t-1} + \ln g_t = \hat{C}_t - \hat{C}_{t-1} + \hat{g}_t + \ln(\bar{g})$$

- Real Investment Growth:

$$DOT\_INV_t = \ln I_t - \ln I_{t-1} = \ln \check{I}_t - \ln \check{I}_{t-1} + \ln g_t = \ln\left(1 + \widehat{INV}_t\right) - \ln\left(1 + \widehat{INV}_{t-1}\right) + \hat{g}_t + \ln(\bar{g})$$

- Real Wage Growth:

$$DOT\_W_t = \ln w_t - \ln w_{t-1} = \ln \check{w}_t - \ln \check{w}_{t-1} + \ln g_t = \hat{w}_t - \hat{w}_{t-1} + \hat{g}_t + \ln(\bar{g})$$

12.2 Nominal Variables - Growth Rates

- Inflation Growth:

$$DOT\_PAI_t = \ln(\pi_t) - \ln(\pi_{t-1}) = \ln(\check{\pi}_t) - \ln(\check{\pi}_{t-1}) + \ln\left(\varepsilon_t^{\pi^*}\right) = \hat{\pi}_t - \hat{\pi}_{t-1} + \hat{\varepsilon}_t^{\pi^*}$$

- Nominal Interest Rate Growth:

$$DOT\_INT_t = \ln(i_t) - \ln(i_{t-1}) = \ln(\check{i}_t) - \ln(\check{i}_{t-1}) + \ln\left(\varepsilon_t^{\pi^*}\right) = \hat{\imath}_t - \hat{\imath}_{t-1} + \hat{\varepsilon}_t^{\pi^*}$$

12.3 Real Variables - Stationary

- Labor Supply:

$$NODOT\_L_t = \ln(L_t) - \ln(\bar{L}) = \hat{L}$$