ME 451C: Compressible Turbulence, Spring 2017

Stanford University
Midterm Exam
Tuesday, May 9

Guidelines: Please turn in *neat* and *clean* exam solutions that give all the formulae that you have used as well as details that are required for the grader to understand your solution. Attach these sheets to your solutions. Assume $\gamma = 1.4$ and $c_p = 1.0$ KJ/KgK for all problems.

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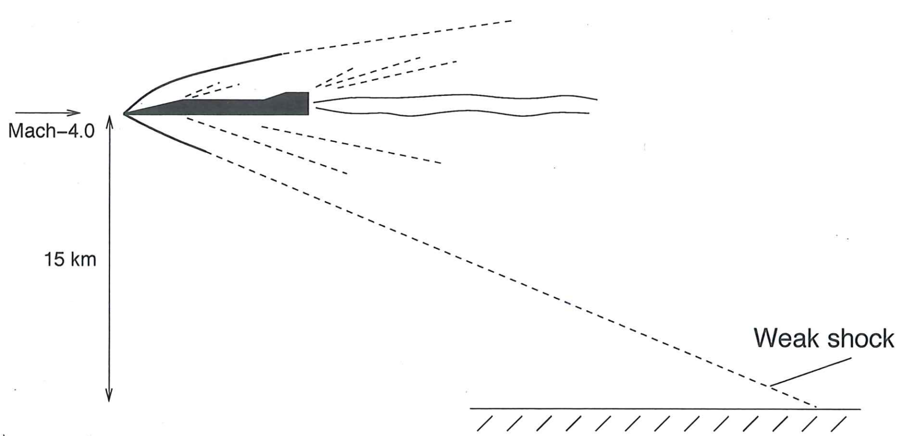
PART I: Closed books, closed notes, calculators allowed
Time: 30 mins

Questions (40 pts)

1. - Explain the basis of Lighthill's analogy for computing the sound radiated from a turbulent flow, including the associated conservation equations for the acoustic field.
   - If the integral size of the eddies is $\ell \sim 1$ cm and the corresponding turnover velocity is $u_\ell \sim 1$ m/s, estimate the characteristic wavelength L_a of the acoustic radiation (assume $T = 300$ K).
   - Estimate the turbulent dissipation, ϵ, and compare it with the amount of acoustic energy emitted per unit time and per unit mass by the turbulent flow, ϵ_s.

2. During flight at an altitude of $h = 15$ km in the stratosphere, where $T = 216.5$ K, a pilot of a supersonic airplane reads Mach-4.0 in the Mach-meter, which is an instrument that approximately provides the flight speed in units of the local speed of sound in the undisturbed air environment around the aircraft.
   - Estimate the Mach number based on ground conditions, $T = 288$ K.
   - Assuming that the nose shock wave around the aircraft has become a weak shock near the ground with a small streamline deflection angle $\delta \sim 10^{-4}$ rad, compute the characteristic angle formed by the shock and the horizontal ground surface, along with the corresponding sound-pressure level (SPL) caused by the sonic boom by taking $P_{ref} = 20$ μPa as the reference pressure.



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PART II: Open books, open notes, calculators allowed

Time: 50 mins

Problem 1 (60 pts)

A turbulent compressible turbulent shear layer is formed by two parallel gaseous streams (denoted by the subindexes $_1$ and $_2$ as shown in the Fig. 1a) with free-stream velocities U_1 and $U_2 < U_1$ downstream from a splitter plate. The corresponding densities are ρ_1 and ρ_2. The molecular weights of the gases are W_1 and W_2, while their adiabatic coefficients are γ_1 and γ_2. The static pressure is the same on both sides of the mixing layer and equals P_∞. As a result of vortex pairing, the thickness of the mixing layer is observed to increase with increasing distances from the splitter plate, as sketched in Fig. 1a.

a) An important parameter that characterizes the relative importance of compressibility and its effects on the growth of the mixing layer is the convective Mach number, $\mathrm{Ma}_{C,1} = (U_1 - U_C)/a_1$, which represents a Mach number in a relative frame moving at the convective speed U_C of the disturbances in the mixing layer (see Fig. 1b) [1]. In this formulation, a_1 and a_2 are the speed of the sound waves in each stream. A method to compute the convective Mach number is outlined below.

- In the moving reference frame depicted in Fig. 1b, the point O is a saddle point between the structures and represents a common stagnation point for both streams. As a result, the stagnation pressures in that frame are the same for both streams. If, additionally, the deceleration at the stagnation point is assumed to be isentropic, find an expression that relates $\mathrm{Ma}_{C,1} = (U_1 - U_C)/a_1$, $\mathrm{Ma}_{C,2} = (U_C - U_2)/a_2$, γ_1 and γ_2.

- Prove that the expression found above leads to the convective velocity

$$U_C = \frac{a_2 U_1 + a_1 U_2}{a_2 + a_1} \tag{1}$$

when $\gamma_1 = \gamma_2$ and the convective Mach numbers are moderately small. Note that Eq. (1) implies that both convective Mach numbers are equal, $\mathrm{Ma}_{C,1} = \mathrm{Ma}_{C,2} = \mathrm{Ma}_C$. Using expression Eq. (1) for U_C, compute the characteristic convective Mach number when stream 1 is air at $\mathrm{Ma}_1 = 4$, $T_1 = 1300$ K, and stream 2 is molecular hydrogen at $\mathrm{Ma}_2 = 0.5$ and $T_2 = 500$ K, which correspond to conditions typical of fuel-air mixing layers in scramjets.

[1] Note that compressibility effects necessarily emerge from the amount of relative motion between both streams.

b) In order to reduce the computational cost of simulating the spatially development of the mixing layer, most approaches involve simulation of the problem in a reference frame moving at the mean velocity $U_m = (U_1 + U_2)/2$, with the inflow/outflow streamwise boundary conditions of the original problem being replaced by periodic boundary conditions. The boundary conditions at the top and bottom boundaries become free-stream conditions with $u_2 = u_3 = 0$. In this way, the problem reduces to two supersonic free streams flowing in opposite directions at the same velocity U (see Fig. 1c).

- Derive an expression for U as a function of U_1 and U_2, and show that the resulting convective Mach number is

$$\mathrm{Ma}_C = 2U/(a_1 + a_2). \tag{2}$$

- Given U and the speeds of sound a_1 and a_2 as the only input data provided in the problem, would it be possible to retrieve the values of the absolute velocities U_1, U_2, the mean velocity U_m and the convective speed U_C? If the answer is affirmative, provide details about how such retrieval would be done[2].

c) In the relative reference frame described in part b), the solution to the compressible shear layer involves a momentum thickness δ_θ that grows in time -rather than space as in the spatially-evolving mixing layer-, thereby leading to unsteady statistics. In particular, δ_θ is defined as

$$\delta_\theta(t) = \frac{1}{4\rho_m U^2} \int_{-\infty}^{+\infty} \overline{\rho}\left(U^2 - \widetilde{u}_1^2\right) dx_2, \tag{3}$$

where $\rho_m = (\rho_1 + \rho_2)/2$ is the mean density. In this formulation, $\overline{\rho}$ and $\widetilde{u}_1$ represent, respectively, the Reynolds-averaged density and the Favre-averaged streamwise velocity (refer to the coordinate system in Fig. 1c). All types of averaging in this problem are performed along planes x_2 =constant. The rate of growth of the mixing layer can be computed from

$$\dot{\delta}_\theta = \frac{d\delta_\theta}{dt}. \tag{4}$$

Figure 2a shows the ratio of $\dot{\delta}_\theta$ to the incompressible growth rate, as a function of the convective Mach number Ma_C for a number of simulations and experiments. The results suggest that δ_θ rapidly decreases with increasing convective Mach numbers as a result of compressibility effects, a phenomenon that is related to the decrease of the turbulent production, as shown in Fig. 2b. To see this, consider the Favre-averaged mass and momentum equations

$$\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial}{\partial x_i}(\overline{\rho}\widetilde{u}_i) = 0, \tag{5}$$

$$\frac{\partial}{\partial t}(\overline{\rho}\widetilde{u}_i) + \frac{\partial}{\partial x_j}(\overline{\rho}\widetilde{u}_i\widetilde{u}_j) = -\frac{\partial \overline{P}}{\partial x_i} + \frac{\partial \overline{\tau'}_{ij}}{\partial x_j} - \frac{\partial}{\partial x_j}\left(\overline{\rho}\widetilde{u_i''u_j''}\right). \tag{6}$$

The last term in Eq. (6) involves the unclosed Reynolds stresses, $\overline{\rho}\widetilde{u_i''u_j''}$. Note that all averaged quantities are, by definition of the averaging operators, functions of t and x_2.

- Substituting (3) into the right-hand side of (4) and using (5)-(6), find an expression for $\dot{\delta}_\theta$ as a function of the streamwise turbulent production $-\overline{\rho}\widetilde{u_1''u_2''}(\partial\widetilde{u}_1/\partial x_2)$.

[2] Note that retrieving U_m would allow one to compute the approximate streamwise distance $x \sim U_m t$ traveled by the moving frame and therefore link space in the absolute frame with the simulation time t in the relative frame.

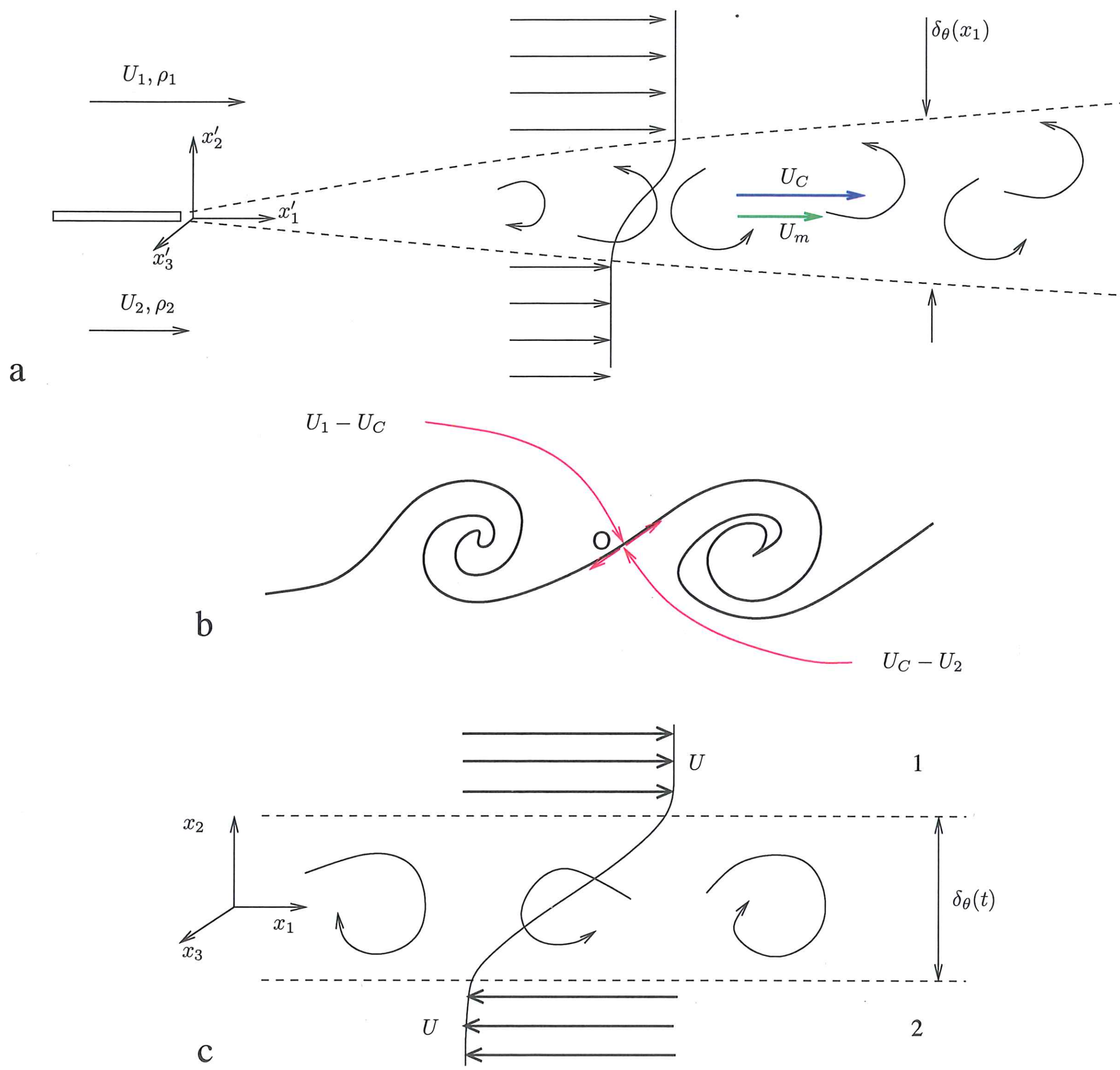


Figure 1: (a) Spatially developing mixing layer. (b) Sketch of streamlines in the convective frame of reference. (c) Description of the problem in a reference frame moving with the mean speed U_m.

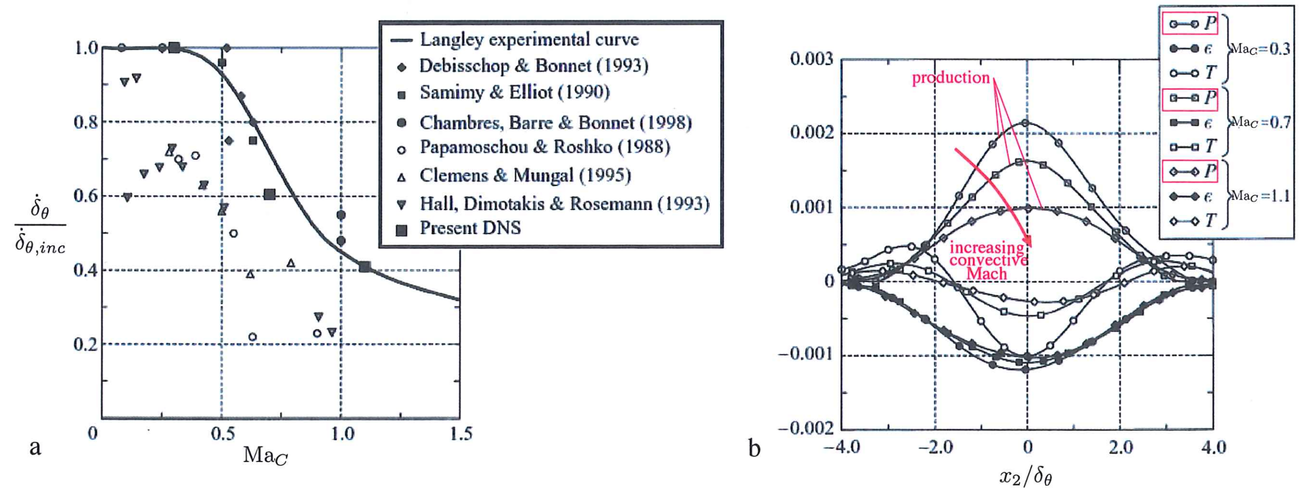


Figure 2: (a) Dependence of shear-layer growth on Ma$_C$. (b) Turbulent production as a function of the x_2 coordinate for three values of Ma$_C$. Figures adapted from Pantano & Sarkar, JFM (2002).

SOLUTIONS

QUESTIONS

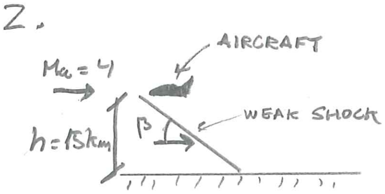
1. – SEE PAGES 35-37 FROM CLASS NOTES.

– $\ell \sim 1\,cm$, $U_\ell \sim 1\,m/s$, $T = 300\,K$

$$L_a \sim \frac{\ell}{U_\ell}\, a = \frac{\ell}{U_\ell}\sqrt{\gamma R_g T} = \underline{\underline{3.4\ m}}$$

– $\varepsilon \sim U_\ell^3/\ell \sim 100\ \frac{m^2}{s^3}$, $\quad \varepsilon_s \sim \varepsilon \left(\frac{U_\ell}{a}\right)^5 \sim \underline{\underline{2.2\cdot 10^{-11}\ m^2/s^3}}$

2.



– ON THE GROUND: $Ma_G = \frac{U}{a_G} = \frac{a_h}{a_G}\frac{U}{a_h} = \sqrt{\frac{216.5}{288}}\cdot 4 = \underline{\underline{3.46}}$

– THE INCIDENCE ANGLE OF A WEAK SHOCK IS VERY CLOSE TO THE MACH ANGLE IN THE FIRST APPROXIMATION,

$$\beta \approx \mu' = \arctan\left(\frac{1}{\sqrt{Ma_G^2 - 1}}\right) = 16.8^\circ$$

FOR $\delta = 10^{-4}$ rad, THE CARACTERISTIC PRESSURE VARIATION ACROSS THE SHOCK IS

$$\frac{P_2 - P_1}{P_1} = \frac{\gamma Ma_G^2}{\sqrt{Ma_G^2 - 1}}\,\delta \simeq 5.05\cdot 10^{-4} \Rightarrow P' = P_2 - P_1 \approx 50.6\ Pa$$

THE CORRESPONDING SPL IS $20\log_{10}\left(\frac{P'}{P_{ref}}\right) \approx \underline{\underline{128\ dB}}$

PROBLEMS

1. a) – $P_{0_1} = \cancel{P_\infty}\left(1 + \left(\frac{\gamma_1 - 1}{2}\right) Ma_{c_1}^2\right)^{\frac{\gamma_1}{\gamma_1 - 1}} = \cancel{P_\infty}\left(1 + \left(\frac{\gamma_2 - 1}{2}\right) Ma_{c_2}^2\right)^{\frac{\gamma_2}{\gamma_2 - 1}} = P_{0_2}$

SO THAT $\underline{\underline{\left[1 + \left(\frac{\gamma_1 - 1}{2}\right) Ma_{c_1}^2\right]^{\frac{\gamma_1}{\gamma_1 - 1}} = \left[1 + \left(\frac{\gamma_2 - 1}{2}\right) Ma_{c_2}^2\right]^{\frac{\gamma_2}{\gamma_2 - 1}}}}$

– FOR $Ma_{c_1}, Ma_{c_2} \ll 1$, A SERIES EXPANSION OF THE ABOVE EXPRESSION LEADS TO

$$1 + \left(\frac{\gamma_1 - 1}{2}\right)\frac{\gamma_1}{\gamma_1 - 1} Ma_{c_1}^2 + O\left(Ma_{c_1}^2\right) \sim$$

$$\sim 1 + \left(\frac{\gamma_2 - 1}{2}\right)\frac{\gamma_2}{\gamma_2 - 1} Ma_{c_2}^2 + O\left(Ma_{c_2}^2\right)$$

AND CANCELLING TERMS; $\quad Ma_{c_1} = \left(\frac{\gamma_2}{\gamma_1}\right)^{1/2} Ma_{c_2}$

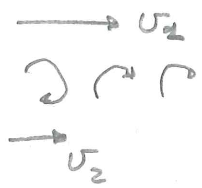
IF $\gamma_1 = \gamma_2$, THEN THIS EXPRESSION REDUCES TO $Ma_{c_1} = Ma_{c_2}$, OR EQUIVALENTLY;

$$\frac{U_1 - U_c}{a_1} = \frac{U_c - U_2}{a_2} \Rightarrow U_c = \frac{a_1 U_2 + a_2 U_1}{a_1 + a_2}$$

REWRITING U_c AS $\quad \frac{U_c}{a_1} = \frac{Ma_2 + Ma_1}{1 + a_1/a_2} = \frac{Ma_2 + Ma_1}{1 + \sqrt{\frac{W_2 T_1}{W_1 T_2}}} = 3.15$

AND THEREFORE $\quad Ma_c = \frac{U_1 - U_c}{a_1} = Ma_1 - \frac{U_c}{a_1} = 4 - 3.15 = \underline{0.85}$

b) — ABSOLUTE FRAME



FRAME MOVING AT MEAN SPEED U_m

$\longrightarrow U_1 - U_m = U_1 - \frac{1}{2}(U_1 + U_2) = \frac{1}{2}(U_1 - U_2)$

$\longleftarrow U_m - U_2 = \frac{1}{2}(U_1 + U_2) - U_2 = \frac{1}{2}(U_1 - U_2)$

$$\Rightarrow U = \frac{1}{2}(U_1 - U_2)$$

THE CONVECTIVE MACH NUMBER IS $Ma_c = \frac{U_1 - U_c}{a_1} = \frac{U_1}{a_1} - \frac{1}{a_1}\left(\frac{a_1 U_2 + U_1 a_2}{a_1 + a_2}\right) =$

$$= \frac{U_1}{a_1} - \frac{U_2 + U_1 a_2/a_1}{a_1 + a_2} = \frac{(a_1 + a_2)U_1 - a_1 U_2 - U_1 a_2}{a_1(a_1 + a_2)} = \frac{U_1 - U_2}{a_1 + a_2} = \frac{2U}{a_1 + a_2}$$

— GIVEN U, a_1 AND a_2, ONE CAN COMPUTE THE CONVECTIVE MACH NUMBER FROM $Ma_c = 2U/(a_1 + a_2)$ AND THE VELOCITY DIFFERENCE $U_1 - U_2 = 2U$, BUT IT IS NOT POSSIBLE TO RECOVER THE VALUES OF U_1, U_2, U_m NOR U_c.

c) $\dot{\delta}_\theta = \frac{d\delta_\theta}{dt} = \frac{1}{4\rho_m U^2}\int_{-\infty}^{+\infty}\left(\frac{\partial \bar{\rho}}{\partial t} U^2 - \frac{\partial}{\partial t}(\bar{\rho}\tilde{u}_1^2)\right) dx_2 =$

$$= \frac{-1}{4\rho_m U}\underbrace{\int_{-\infty}^{+\infty}\frac{\partial}{\partial x_2}(\bar{\rho}\tilde{u}_2)\,dx_2}_{0} - \frac{1}{4\rho_m U^2}\int_{-\infty}^{+\infty}\frac{\partial}{\partial t}(\bar{\rho}\tilde{u}_1^2)\,dx_2$$

IF $\tilde{u}_2 = 0$ AT $x_2 \to \pm\infty$

IN ORDER TO SIMPLIFY THE LAST TERM, MULTIPLY THE x_1-COMPONENT OF THE MOMENTUM EQUATION (6) BY $\tilde{u}_1$:

$$\tilde{u}_1 \frac{\partial}{\partial t}(\bar{\rho}\tilde{u}_1) + \tilde{u}_1 \frac{\partial}{\partial x_2}(\bar{\rho}\tilde{u}_1\tilde{u}_2) = -\tilde{u}_1 \overbrace{\frac{\partial \bar{p}}{\partial x_1}}^{0} + \tilde{u}_1 \frac{\partial \bar{\tau}_{12}}{\partial x_2} - \tilde{u}_1 \frac{\partial}{\partial x_2}\left(\bar{\rho}\widetilde{u_1'' u_2}\right)$$

ALSO NOTE THAT

$$\tilde{u}_1 \frac{\partial}{\partial t}(\bar{\rho}\tilde{u}_1) + \tilde{u}_1 \frac{\partial}{\partial x_2}(\bar{\rho}\tilde{u}_1\tilde{u}_2) = \bar{\rho}\tilde{u}_1 \frac{\partial \tilde{u}_1}{\partial t} + \bar{\rho}\tilde{u}_1\tilde{u}_2 \frac{\partial \tilde{u}_1}{\partial x_2} = \bar{\rho}\frac{\partial}{\partial t}\left(\frac{\tilde{u}_1\tilde{u}_1}{2}\right) + \bar{\rho}\tilde{u}_2 \frac{\partial}{\partial x_2}\left(\frac{\tilde{u}_1\tilde{u}_1}{2}\right)$$

$$= \frac{\partial}{\partial t}\left(\frac{\bar{\rho}\tilde{u}_1\tilde{u}_1}{2}\right) + \frac{\partial}{\partial x_2}\left(\bar{\rho}\tilde{u}_2 \frac{\tilde{u}_1\tilde{u}_1}{2}\right),$$

WHERE USE OF CONTINUITY $\left(\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_2}(\bar{\rho}\tilde{u}_2) = 0\right)$ HAS BEEN MADE. AS A RESULT,

$$\frac{\partial}{\partial t}(\bar{\rho}\tilde{u}_1\tilde{u}_1) = -\frac{\partial}{\partial x_2}(\bar{\rho}\tilde{u}_2\tilde{u}_1\tilde{u}_1) + \frac{\partial}{\partial x_2}(2\tilde{u}_1\bar{\tau}'_{12}) - 2\bar{\tau}'_{12}\frac{\partial \tilde{u}_1}{\partial x_2}$$
$$- \frac{\partial}{\partial x_2}\left(2\bar{\rho}\tilde{u}_1\widetilde{u_1''u_2''}\right) + 2\bar{\rho}\widetilde{u_1''u_2''}\frac{\partial \tilde{u}_1}{\partial x_2}.$$

SUBSTITUTING THIS EXPRESSION INTO $\dot{\delta}_\theta$:

$$\dot{\delta}_\theta = -\frac{1}{4\rho_m U^2}\int_{-\infty}^{+\infty} \frac{\partial}{\partial t}(\bar{\rho}\tilde{u}_1\tilde{u}_1)\,dx_2 = -\frac{1}{4\rho_m U^2}\left\{ -\cancelto{0 \text{ SINCE } \tilde{u}_2 \to 0 \text{ AT } x_2 \to \pm\infty}{\int_{-\infty}^{+\infty}\frac{\partial}{\partial x_2}(\bar{\rho}\tilde{u}_2\tilde{u}_1\tilde{u}_1)\,dx_2} + \right.$$
$$+ \cancelto{0 \text{ SINCE } \bar{\tau}'_{12} = 0 \text{ AT } x_2 \to \pm\infty}{\int_{-\infty}^{+\infty}\frac{\partial}{\partial x_2}(2\tilde{u}_1\bar{\tau}'_{12})\,dx_2} - \cancelto{0 \text{ SINCE } \widetilde{u_1''u_2''} = 0 \text{ AT } x_2 \to \pm\infty}{\int_{-\infty}^{+\infty}\frac{\partial}{\partial x_2}\left(2\bar{\rho}\tilde{u}_1\widetilde{u_1''u_2''}\right)dx_2}$$
$$\left. + \int_{-\infty}^{+\infty} 2\bar{\rho}\widetilde{u_1''u_2''}\frac{\partial \tilde{u}_1}{\partial x_2}\,dx_2 - \int_{-\infty}^{+\infty} 2\bar{\tau}'_{12}\frac{\partial \tilde{u}_1}{\partial x_2}\,dx_2 \right\}.$$

THE LAST TERM CORRESPONDS TO TWICE THE MEAN VISCOUS DISSIPATION,

$$\bar{\varepsilon} = +\bar{\tau}'_{12}\frac{\partial \tilde{u}_1}{\partial x_2} = +\bar{\tau}'_{12}\tilde{S}_{12}.$$

NOTE THIS INVOLVES THE PLANAR-AVERAGED GRADIENTS → NEGLIGIBLE

WHICH IS NEGLIGIBLE AT HIGH REYNOLDS NUMBERS.

AS A RESULT,

$$\dot{\delta}_\theta = -\frac{1}{2\rho_m U^2}\int_{-\infty}^{+\infty} \bar{\rho}\widetilde{u_1''u_2''}\frac{\partial \tilde{u}_1}{\partial x_2}\,dx_2$$