ME 355: Compressible Flows
Stanford University

Warmup Problem Set

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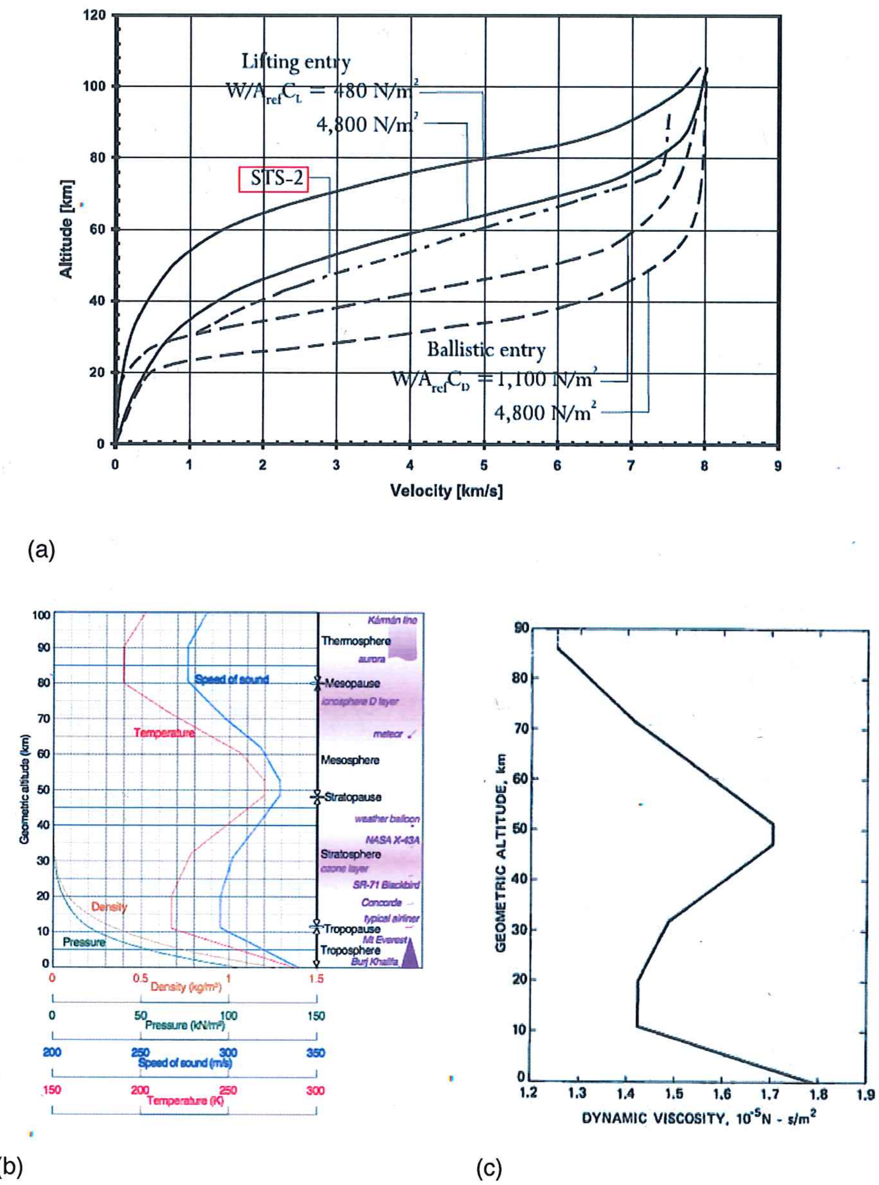
Problem 1

The altitude-velocity fightpath of the Space Shuttle Columbia during descent is shown in Fig. 1(a), where it is denoted by the label "STS-2". Variations of the properties and thermodynamic variables in the atmosphere are provided in Fig. 1(b,c). Assume that the density decreases with altitude as $\rho = \rho_A \exp(-h/\alpha)$ where $\alpha = 7.2$ km and $\rho_A = 1.2$ kg/m^3.



Assuming that the characteristic length of the Space Shuttle's thermal shield is $L = 32.82$ m, calculate at points A (h =80 km altitude), B (60 km) and C (20 km) of the trajectory the following parameters:

a) Knudsen number
b) Reynolds number
c) Mach number

KNUDSEN #: $Kn = \frac{\lambda}{L}$, $\lambda = \frac{m}{\sqrt{2}\pi d^2 \rho} = \frac{m}{\sqrt{2}\pi d^2 \rho_A} e^{+h/\alpha}$ WITH $\left\{ \begin{array}{l} m = 4.79 \cdot 10^{-26} \frac{kg}{molecule} \\ d = 390 \cdot 10^{-12} m \end{array} \right.$

AIR

REYNOLDS #: $Re_L = \frac{\rho v L}{\mu}$, WITH $\rho = \rho_A e^{-h/\alpha}$

MACH #: $Ma = \frac{v}{a}$.

FROM THE FIGURES: AND THE DENSITY VS ALTITUDE EQUATION:

	h (km)	v (km/s)	ρ (kg/m³)	λ (m)	μ ($\frac{Ns}{m^3}$)	a (m/s)	Kn	Re_L	Ma
A	80	7.4	$1.79 \cdot 10^{-5}$	$3.9 \cdot 10^{-3}$	$1.28 \cdot 10^{-5}$	~275	$1.2 \cdot 10^{-4}$	$3.4 \cdot 10^{5}$	26.9
B	60	5.0	$2.88 \cdot 10^{-4}$	$0.2 \cdot 10^{-3}$	$1.6 \cdot 10^{-5}$	~325	$6.1 \cdot 10^{-6}$	$2.9 \cdot 10^{6}$	15.3
C	20	~0.9	0.074	$0.95 \cdot 10^{-6}$	$1.42 \cdot 10^{-5}$	~290	$2.8 \cdot 10^{-8}$	$1.5 \cdot 10^{8}$	3.1

NOTE THAT AN ESTIMATION OF THESE NUMBERS AT $h \sim 130$ km (NOT SHOWN IN FIGURES) GIVES

$\rho \sim 1.7 \cdot 10^{-8}$ kg/m³

$\mu \sim 10^{-5} \frac{Ns}{m^2}$

$\lambda \sim 4.1$ m

$a \sim 310$ m/s

$v \sim 8$ km/s

$Re_L \sim 446$ ⟵ LOW REYNOLDS

$Kn \sim 0.12$ ⟵ SLIP / NON-CONTINUUM FLOW

$Ma \sim 25.8$

Problem 2

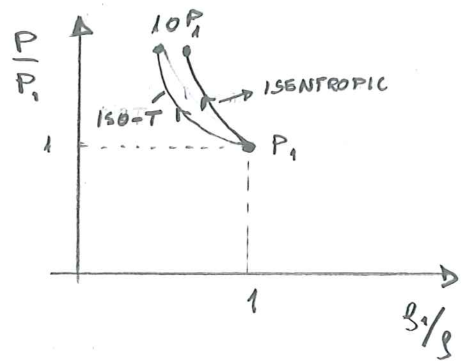
An ideal gas at pressure $P_1 = 1$ atm and density $\rho_1 = 1$ kg/m^3 is compressed isothermally to $P_2 = 10P_1$. The adiabatic constant of the gas is $\gamma = 1.4$ and its molecular weight is $W = 10$ g/mol.

a) What is the density of the compressed gas?

b) Calculate the increments of thermal energy, entropy and enthalpy in the gas

c) Compute the ratio of the mean distances between molecules in the gas before and after compression

d) Answer to parts a-c if the compression process is isentropic instead of isothermal.



GAS CONSTANT: $R_g = \frac{R^o}{W} = \frac{8.31}{10^{-2}} = 831\ J/kg\,K$

SPECIFIC HEATS: $c_p = \frac{\gamma R_g}{\gamma - 1} = 2.9\ kJ/kg\,K$

$c_v = c_p/\gamma = 2.1\ kJ/kg\,K$

* ISOTHERMAL TRAJECTORY: $T = \frac{P_1}{\rho_1 R_g} = \frac{P_2}{\rho_2 R_g} \Rightarrow \frac{\rho_2}{\rho_1} = \frac{P_2}{P_1} = 10 \Rightarrow \boxed{\rho_2 = 10\ \frac{kg}{m^3}}$

* INCREMENT OF THERMAL ENERGY: $e_2 - e_1 = c_v(\cancel{T_2 - T_1}) = 0$

" ENTHALPY: $h_2 - h_1 = c_p(\cancel{T_2 - T_1}) = 0$

" ENTROPY: $\left[s_2 - s_1 = c_v \ln\left[\frac{P_2}{P_1}\left(\frac{\rho_1}{\rho_2}\right)^{\gamma}\right]\right] = c_v \ln\left\{\frac{10}{10^{1.4}}\right\} \boxed{= -1.9\ \frac{kJ}{K}}$

* MEAN INTERMOLECULAR DISTANCE: $\delta \sim n^{-1/3}$, WHERE $n = \frac{\rho N_A}{W}$ IS THE NUMBER DENSITY AND N_A IS THE AVOGADROS' NUMBER.

$$\rightarrow \frac{\delta_2}{\delta_1} = \left(\frac{n_2}{n_1}\right)^{-1/3} = \left(\frac{\rho_2}{\rho_1}\right)^{-1/3} = 10^{-1/3} = \boxed{0.46}$$

IF THE PROCESS IS ISENTROPIC INSTEAD:

$\frac{P_2}{P_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} = 10 \Rightarrow \frac{\rho_2}{\rho_1} = 10^{1/\gamma} = 5.2$, AND $\frac{T_2}{T_1} = \frac{P_2}{P_1}\frac{\rho_1}{\rho_2} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma - 1} = 1.9$

$\Rightarrow e_2 - e_1 = c_v(T_2 - T_1) = c_v T_1 (T_2/T_1 - 1) = \frac{P_1}{(\gamma - 1)\rho_1}\left[\left(\frac{\rho_2}{\rho_1}\right)^{\gamma - 1} - 1\right] = \boxed{230\ kJ/kg}$

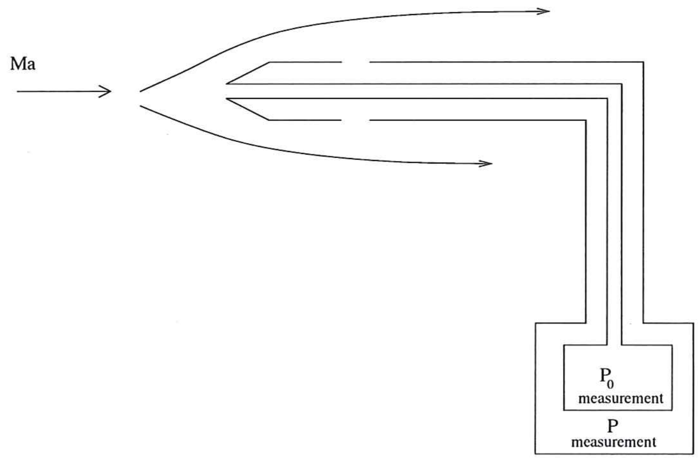
$h_2 - h_1 = \gamma(e_2 - e_1) = \boxed{322\ kJ/kg}$

$\boxed{s_2 - s_1 = 0}$

AND $\frac{\delta_2}{\delta_1} = \left(\frac{\rho_2}{\rho_1}\right)^{-1/3} = \boxed{0.57}$

Problem 3

Consider the subsonic Pitot tube in the figure below. Knowing that the static and stagnation pressures measured are, respectively, $P_0 = 2.45$ bar and $P = 2$ bar, compute the velocity of the free stream assuming that the gas is air at a stagnation temperature $T_0 = 550$K.



RECALL FROM BERNOULLI'S EQUATION THAT $h + \frac{|\vec{U}|^2}{2} = \text{CONST.}$

$$\Rightarrow \quad c_p T + \frac{U^2}{2} = c_p T_0 \Rightarrow \frac{T_0}{T} = 1 + \frac{(\gamma-1)}{2} Ma^2$$

SINCE $\frac{P_0}{P} = \left(\frac{\rho_0}{\rho}\right)^\gamma = \left(\frac{T_0}{T}\right)^{\frac{\gamma}{\gamma-1}} = \left[1 + \frac{(\gamma-1)}{2} Ma^2\right]^{\frac{\gamma}{\gamma-1}} = \frac{2.45}{2} = 1.22.$

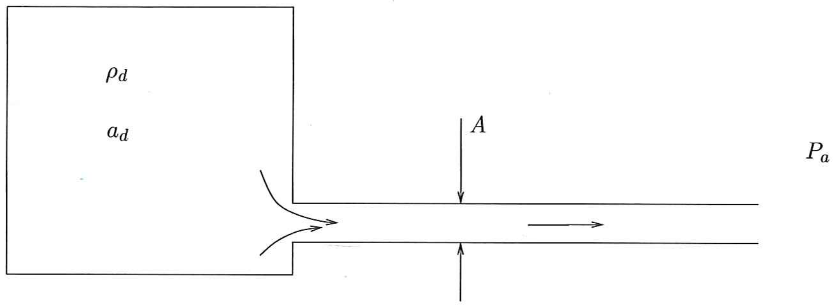
THEN $\boxed{Ma = 0.55}$ (SEE ALSO TABLE I)

THE STATIC TEMPERATURE IS $T = \frac{T_0}{1 + \frac{(\gamma-1)}{2} Ma^2} = 517$ K

AND $U = Ma\sqrt{\gamma R_g T} = \boxed{250 \text{ m/s}}$

Problem 4

A straight duct of cross sectional area A is attached to a large gas reservoir as shown in the figure. The atmospheric pressure is P_a. Assuming that the values of the density ρ_d and speed of sound a_d in the reservoir are known, provide the expressions to compute the pressure, velocity, speed of sound, Mach number, mass flow rate and density in the duct. Consider the cases $P_d/P_a < [(\gamma+1)/2]^{\gamma/(\gamma-1)}$ and $P_d/P_a > [(\gamma+1)/2]^{\gamma/(\gamma-1)}$ separately in your calculations.



MASS FLOW RATE: $\dot{m} = \rho_c U_c A = \rho_d a_d \dfrac{\rho_c}{\rho_d} \dfrac{a_c}{a_d} \dfrac{U_c}{a_c} A =$

($\rho_c, U_c, P_c, a_c, Ma_c$ REFER TO THE DUCT)

$$= \rho_d a_d A \left(\frac{T_c}{T_d}\right)^{1/2} \frac{\rho_c}{\rho_d} Ma_c A$$

$$= \rho_d a_d A \left[1 + \frac{(\gamma-1)}{2} Ma_c^2\right]^{-\frac{\gamma+1}{2(\gamma-1)}} Ma_c \qquad (1)$$

FOR SUBSONIC FLOW IN THE DUCT, Ma_c IS CALCULATED BY IMPOSING $\boxed{P_c = P_a}$ (2)

$$\Rightarrow \frac{P_d}{P_a} = \left[1 + \frac{\gamma-1}{2} Ma_c^2\right]^{\frac{\gamma}{\gamma-1}} \Rightarrow \boxed{Ma_c = \left\{\frac{2}{\gamma-1}\left[\left(\frac{P_d}{P_a}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]\right\}^{1/2}} \quad (3)$$

WITH $P_d = \dfrac{\rho_d a_d^2}{\gamma}$ THE PRESSURE IN THE RESERVOIR. SUBSTITUTING (3) INTO (1):

$$\boxed{\dot{m} = \rho_d a_d A \left(\frac{P_d}{P_a}\right)^{-\frac{\gamma+1}{2(\gamma-1)}} \left[\left(\frac{P_d}{P_a}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]^{1/2} \left(\frac{2}{\gamma-1}\right)^{1/2}} \quad (4)$$

WHILE THE DENSITY AND SPEED OF SOUND IN THE DUCT ARE $\boxed{\dfrac{\rho_c}{\rho_d} = \left(\dfrac{P_a}{P_d}\right)^{1/\gamma} \text{ AND } \dfrac{a_c}{a_d} = \left(\dfrac{P_a}{P_d}\right)^{\frac{\gamma-1}{2\gamma}}}$ (5)

THAT $P_c = P_a$ ONLY HOLDS FOR $Ma_c < 1$, OR EQUIVALENTLY, $\dfrac{P_d}{P_a} < \left(\dfrac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}}$ ACCORDING TO EQ. (3)

FOR $\dfrac{P_d}{P_a} > \left(\dfrac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}}$, THEN $\boxed{Ma_c = 1}$ (6) AND $P_c \neq P_a$, WITH P_c GIVEN BY

$$\boxed{\frac{P_d}{P_c} = \left[1 + \frac{(\gamma-1)}{2} \cancel{Ma_c^2}^{\,1}\right]^{\frac{\gamma}{\gamma-1}} = \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}}} \quad (7).$$

SIMILARLY, $\dot{m}$ IS GIVEN BY

$$\boxed{\dot{m} = \rho_d a_d A \left[1 + \frac{(\gamma-1)}{2} \cancel{Ma_c^2}^{\,1}\right]^{-\frac{\gamma+1}{2(\gamma-1)}} \cancel{Ma_c}^{\,1} = \rho_d a_d A \left(\frac{(\gamma+1)}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}}} \quad (8)$$

WHILE $$\boxed{\frac{\rho_c}{\rho_d} = \left(\frac{P_c}{P_d}\right)^{1/\gamma} \quad \text{AND} \quad \frac{a_c}{a_d} = \left(\frac{P_c}{P_d}\right)^{\frac{\gamma-1}{2\gamma}}} \quad (9)$$

IN SUMMARY, (1) IS ALWAYS VALID,

(2), (3), (4) AND (5) ARE VALID IF $\frac{P_d}{P_a} < \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}}$

(6), (7), (8) AND (9) ARE VALID IF $\frac{P_d}{P_a} > \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}}$ (i.e. THE DUCT IS CHOKED)