ME 355: Compressible Flows, Spring 2016

Stanford University

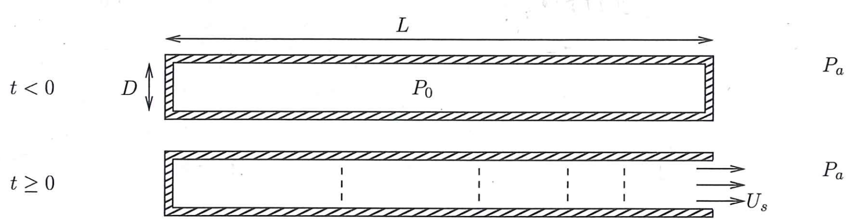
Homework 4: Wave motion

Due Tuesday, May 31, in class.

Guidelines: Please turn in a *neat* and *clean* homework that gives all the formulae that you have used as well as details that are required for the grader to understand your solution. Attach these sheets to your solutions. Assume $\gamma = 1.4$ and $c_p = 1$ KJ/KgK for all problems.

Student's Name:.....JAVIER URZAY..... **Student's ID:**............................

A cylinder of diameter D and length L initially closed at both ends is filled with an stagnant gas at pressure P_0 and density ρ_0. The cylinder is surrounded by air at pressure $P_a < P_0$. At $t = 0$, a pyrotechnic system breaks the right-end wall of the duct, so that the gas starts flowing out of the cylinder. This problem addresses the dynamics shortly after rupture of the right-end wall during times of order L/a_0, where a_0 is the speed of sound in the gas. Depending on the value of the overpressure $\epsilon = (P_0 - P_a)/P_0$, the disturbances produced in the gas may be small ($\epsilon \ll 1$), so that the linear-acoustics theory is applicable, or large ($\epsilon = O(1)$), in which case the linear approximation is no longer valid and the non-linear acoustics formulation needs to be employed.



Consider first the case in which the air pressure differs slightly from the pressure of the gas in the cylinder, $\epsilon = (P_0 - P_a)/P_0 \ll 1$.

20 **a)** Compute the velocity of the gas at the cylinder exit U_s.

10 **b)** Calculate the force on the cylinder.

20 **c)** Characterize the spatial profiles of velocity and pressure inside the cylinder during the time interval $0 < t < 2L/a_0$.

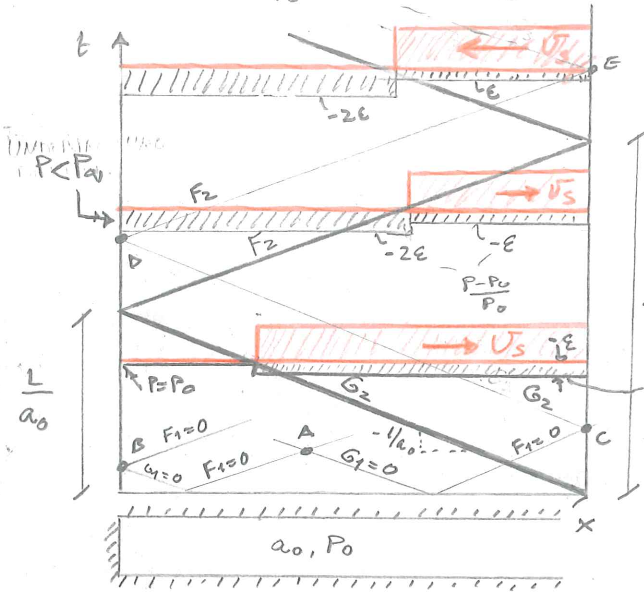
Secondly, consider the case in which the air pressure differs from the pressure of the gas by an amount of order unity, $\epsilon = (P_0 - P_a)/P_0 = O(1)$.

30 **d)** Compute the velocity of the gas at the cylinder exit U_s. Check that U_s becomes the one obtained in part a) in the limit $\epsilon \ll 1$.

40 **e)** Calculate the force on the cylinder.

40 **f)** Characterize the spatial profiles of velocity and pressure inside the cylinder during the time interval $0 < t < L/a_0$.

a) SINCE $\frac{P_0 - P_a}{P_0} = \varepsilon \ll 1 \Rightarrow \frac{\Delta P}{P_0} \sim \gamma Ma^2 \ll 1 \Rightarrow$ USE LINEAR THEORY



THE CONDITIONS AT $t=0$ ARE $u' = P' = 0$

FROM LINEAR THEORY: $\frac{u'}{a_0} = F + G$, $\frac{P'}{\rho_0 a_0^2} = F - G$

• AT POINT A: $F_1 = 0$, $G_1 = 0$ $\Big\}$ $u'_A = P'_A = 0 \Rightarrow$ THE FLOW IS AT ITS INITIAL STATE

• AT POINT B: $G_1 = 0$, $u'_B = 0$ (BC) $\Big\}$ THE REBOUND WAVE IS $F = F_1 = 0 \Rightarrow P'_B = 0$

• AT POINT C: $F_1 = 0$, $P = P_a$ (BC) $\Big\}$ $\frac{P_a - P_0}{\rho_0 a_0^2} = \cancel{F_1} - G_2$

$\frac{u'_C}{a_0} = G_2 = \frac{P_0 - P_a}{\rho_0 a_0^2}$

$\Rightarrow \boxed{U_S = u'_C = \frac{P_0 - P_a}{\rho_0 a_0}} = \frac{\varepsilon P_0}{\rho_0 a_0} = \frac{\varepsilon a_0}{\gamma} \ll a_0$

• AT POINT D: $G_2 = \frac{P_0 - P_a}{\rho_0 a_0^2}$, $u'_D = 0$ (BC) $\Big\}$ $F_2 = -G_2$, $\frac{P'_D}{\rho_0 a_0^2} = \frac{2(P_a - P_0)}{\rho_0 a_0^2} = -\frac{2\varepsilon P_0}{\rho_0 a_0^2}$

• AT POINT E: $F_2 = \frac{P_a - P_0}{\rho_0 a_0^2}$, $P = P_a$ (BC) $\Big\}$ $\frac{P_a - P_0}{\rho_0 a_0^2} = \frac{P_a - P_0}{\rho_0 a_0^2} - G_3 \Rightarrow G_3 = 0$

$\frac{u'_E}{a_0} = F_2 = -\frac{U_S}{a_0}$

b) SEE PROFILES ABOVE.

VELOCITY u'

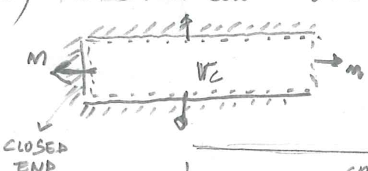
PRESSURE $(P - P_0)/P_0$

NOTE: YOU COULD ALSO COMPUTE $\vec{F}$ IN A MORE DIFFICULT WAY:

$\vec{F} = -\frac{d}{dt}\int_{V_c} \rho \vec{u}\, dV - \int_{S_c} \rho \vec{u}\, \vec{u} \cdot d\vec{S} = -\frac{d}{dt}\left(\frac{\pi D^2}{4}\rho_0\left(1-\frac{\varepsilon}{\gamma}\right)U_S a_0 t\right) - \rho_0\left(1-\frac{\varepsilon}{\gamma}\right)U_S^2 \frac{\pi D^2}{4}\vec{e}_x$

$= -\frac{\pi D^2}{4}\rho_0\left(1-\frac{\varepsilon}{\gamma}\right)\frac{\varepsilon a_0^2}{\gamma}\left[\frac{\varepsilon}{\gamma}+1\right]\vec{e}_x \underset{\varepsilon \ll 1}{\approx} -\frac{\pi D^2}{4}\rho_0 a_0^2 \frac{\varepsilon}{\gamma}\vec{e}_x = -\frac{\pi D^2}{4}\varepsilon P_0 \vec{e}_x$

c) MOMENTUM CONSERVATION EQUATION:



$\frac{d}{dt}\int_{V_c} \rho \vec{u}\, dV + \int_{S_c} \rho \vec{u}\, \vec{u} \cdot d\vec{S} = -\int_{S_c} P\, d\vec{S}$

FORCE IS ONLY IN X DIRECTION

$-\underbrace{\int P\, d\vec{S}}_{\text{CLOSED END}} - \underbrace{\cancel{\int P\, d\vec{S}}}_{\text{LATERAL SURFACES}} - \underbrace{\int P\, d\vec{S}}_{\text{EXIT}} =$ (0 SYMMETRY)

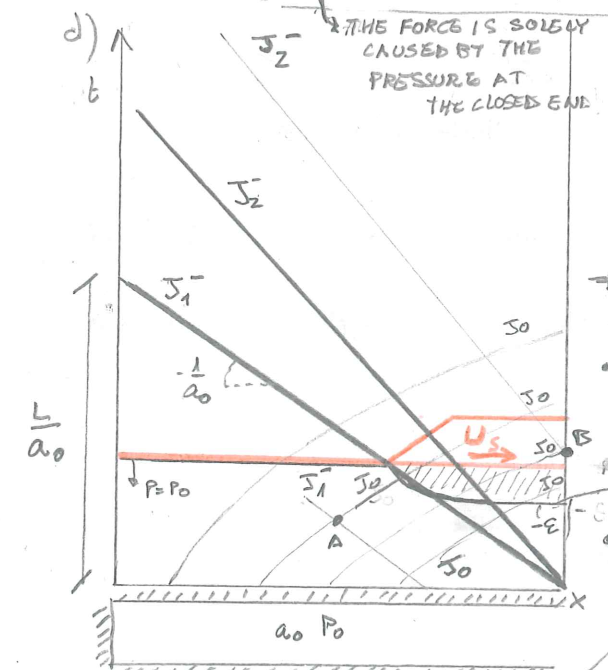
$-F_{\text{FLUID} \to \text{CYLINDER}}$ $\quad -P_a\frac{\pi D^2}{4}$

∗ EASY WAY: $\vec{F} = \int_{\text{CLOSED END}} (P - P_a)\, d\vec{S} = \begin{cases} -\varepsilon P_0 \frac{\pi D^2}{4}\vec{e}_x & \left(0 < t < \frac{L}{a_0}\right) \\ +\varepsilon P_0 \frac{\pi D^2}{4}\vec{e}_x & \left(\frac{L}{a_0} < t < \frac{2L}{a_0}\right) \end{cases}$

CONTRIBUTION OF THE ATMOSPHERE

THE FORCE IS SOLELY CAUSED BY THE PRESSURE AT THE CLOSED END

d)



SINCE $\frac{P_0 - P_a}{P_0} \sim 1 \Rightarrow \frac{\Delta P}{P_0} \sim \gamma Ma^2 \sim O(1) \Rightarrow$ USE NON-LINEAR THEORY

AGAIN, CONDITIONS AT $t=0$ ARE $u = 0$, $P = P_0$.

FROM THE THEORY: $J^+ = a + u\left(\frac{\gamma-1}{2}\right)$, $J^- = a - u\left(\frac{\gamma-1}{2}\right)$ $\Big\}$ $u = \frac{J^+ - J^-}{\gamma - 1}$, $a = \frac{J^+ + J^-}{2}$ $\Big\}$ AND $\left(\frac{a_0}{a}\right)^2 = \left(\frac{P_0}{P}\right)^{\frac{\gamma-1}{\gamma}} = \left(\frac{T_0}{T}\right) \neq 1 + \frac{\gamma-1}{2}Ma^2$

$\Rightarrow$ FROM THE INITIAL CONDITIONS: $J^+ = J_0 = a_0 \Rightarrow$ NON-LINEAR SIMPLE WAVES (C_- CHARACTERISTICS ARE STRAIGHT LINES)

• AT POINT A: $J^+ = J_0 = a_0$, $J_1^- = a_0$ $\Big\}$ $u_A = 0 \Rightarrow$ THE GAS IS AT REST AND HAS NOT NOTICED THE EFFECT OF THE OPEN END BELOW THE FIRST CRITICAL CHARACTERISTIC EMERGING FROM $x = L$ AT $t = 0$

• AT POINT B (EXIT): THE PRESSURE CONDITION DEPENDS ON WHETHER THE DUCT IS CHOKED OR NOT. FOR $P_a \gtrsim P_S^*$ (P^* THE CALCULATED BELOW)

$\rightarrow$ THE DUCT CANNOT BE CHOKED AND $P_B = P_a$ (SUBSONIC FLOW)

RECALL A STRAIGHT DUCT CAN ONLY ACHIEVE SONIC CONDITIONS AT EXIT (AT MOST)!

$\Rightarrow P_B = P_a$, $J^+ = J_0 = a_0$ $\Big\}$ $\left(\frac{P_0}{P_a}\right)^{\frac{\gamma-1}{\gamma}} = \left(\frac{a_0}{a_B}\right)^2 \Rightarrow a_B = a_0\left(\frac{P_a}{P_0}\right)^{\frac{\gamma-1}{2\gamma}} = a_S$

AS A RESULT: $a_S = a_B = a_0\left(\frac{P_a}{P_0}\right)^{\frac{\gamma-1}{2\gamma}}$, $J^+ = J_0 = a_0$ $\Big\}$ $a_S = a_B = \frac{J_0 + J_2^-}{2} \Rightarrow J_2^- = a_0\left[2\left(\frac{P_a}{P_0}\right)^{\frac{\gamma-1}{2\gamma}} - 1\right]$

THE INVARIANT J_2^- RUNS ALONG A CHARACTERISTIC OF SLOPE $\frac{-1}{U_S + a_B}$, WITH $J_2^- = a_B - \left(\frac{\gamma-1}{2}\right)U_S$.

THEN $\left[U_S = \frac{J_0 - J_2^-}{\gamma-1} = \frac{a_0}{\gamma-1} - \frac{a_0}{\gamma-1}\left[2\left(\frac{P_a}{P_0}\right)^{\frac{\gamma-1}{2\gamma}} - 1\right] = \boxed{\frac{2}{\gamma-1}a_0\left[1 - \left(\frac{P_a}{P_0}\right)^{\frac{\gamma-1}{2\gamma}}\right]}\right.$

SINCE $\frac{P_a}{P_0} = \frac{P_a}{P_0} + 1 - 1 = -\varepsilon + 1 \Rightarrow U_S = \frac{2}{\gamma-1}a_0\left[1 - (1-\varepsilon)^{\frac{\gamma-1}{2\gamma}}\right] \overset{\varepsilon \ll 1}{\sim} \frac{\cancel{2}}{\cancel{\gamma-1}}a_0\left(\frac{\cancel{\gamma-1}}{\cancel{2}\gamma}\right)\varepsilon = a_0\frac{\varepsilon}{\gamma} = \frac{P_0 - P_a}{\rho_0 a_0}$

* NOTE THE SOLUTION ABOVE IS VALID AS LONG AS $P_a > P_S^*$, WITH P^* GIVEN BY THE CONDITION

$$Ma_S = \frac{U_S}{a_S} = \frac{2}{\gamma-1}\underbrace{\frac{a_0}{a_S}}_{\left(\frac{P_0}{P_S^*}\right)^{\frac{\gamma-1}{2\gamma}}}\left[1 - \left(\frac{P_S^*}{P_0}\right)^{\frac{\gamma-1}{2\gamma}}\right] = \frac{2}{\gamma-1}\left[\left(\frac{P_0}{P_S^*}\right)^{\frac{\gamma-1}{2\gamma}} - 1\right] = 1$$

$$\Rightarrow \boxed{P_S^* = \left(\frac{\gamma+1}{2}\right)^{-\frac{2\gamma}{\gamma-1}}P_0 \simeq 0.28\,P_0} \text{ FOR } \gamma = 1.4$$

THE PRESSURE P_S^* IS THE CRITICAL AMBIENT PRESSURE FOR WHICH $Ma_S = 1$. FOR $P_a > P_S^*$ THE FLOW REMAINS SUBSONIC EVERYWHERE, $Ma_S < 1$, AND THE SOLUTION IS THE ONE GIVEN ABOVE. THE VELOCITY AND PRESSURE PROFILES IN THE EXPANSION FAN ARE OBTAINED BY MAKING

CONSTANCY OF $J^+ = J_0$; CONSTANCY OF C_- SLOPES

$$J^+ = J_0 = a_0 = a + \left(\frac{\gamma-1}{2}\right)u \quad \text{AND} \quad \frac{dx}{dt} = u - a = \text{CONST} = \frac{x}{t}$$

COMBINING THESE EQUATIONS: $u - \frac{x}{t} + \left(\frac{\gamma-1}{2}\right)u = a_0 \Rightarrow \begin{cases} u = \frac{2}{\gamma+1}\left(a_0 + \frac{x}{t}\right) \\ a = \frac{2a_0}{\gamma+1} + \frac{1-\gamma}{\gamma+1}\frac{x}{t} \end{cases}$ ($x=0$ IS TAKEN AT THE OPEN END) → IN THE EXPANSION FAN.

⇒ NOTE THAT THE VELOCITY IS LINEAR IN x ACROSS THE FAN.

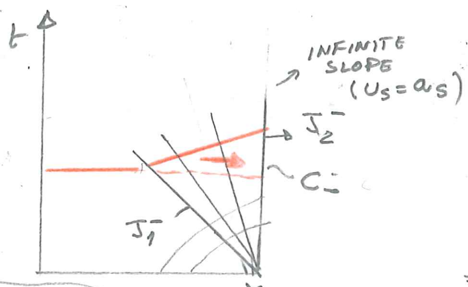
↳ $u = 0$, $a = a_0$ AT $\frac{x}{t} = -a_0$

THE CORRESPONDING PRESSURE VARIATIONS ACROSS THE FAN ARE $\left(\frac{P}{P_0}\right)^{\frac{\gamma-1}{\gamma}} = \left(\frac{a}{a_0}\right)^2 \Rightarrow \frac{P}{P_0} = \left(\frac{a}{a_0}\right)^{\frac{2\gamma}{\gamma-1}} = \left[\frac{2}{\gamma+1} + \frac{1-\gamma}{\gamma+1}\frac{x}{ta_0}\right]^{\frac{2\gamma}{\gamma-1}} = \left(\frac{2}{\gamma+1}\right)^{\frac{2\gamma}{\gamma-1}}\left[1 + \frac{1-\gamma}{2}\frac{x}{ta_0}\right]^{\frac{2\gamma}{\gamma-1}}$

WITH $P = P_0$ AT $\frac{x}{t} = -a_0$. SIMILARLY, $\frac{\rho}{\rho_0} = \left(\frac{P}{P_0}\right)^{1/\gamma} = \left(\frac{2}{\gamma+1}\right)^{\frac{2}{\gamma-1}}\left[1 + \left(\frac{1-\gamma}{2}\right)\frac{x}{ta_0}\right]^{\frac{2}{\gamma-1}}$

FOR $P_a < P_S^*$, THE DUCT IS CHOKED ($Ma_S = 1$), AND $P_S \neq P_a$, BUT $P_S = P_S^* = \left(\frac{\gamma+1}{2}\right)^{-\frac{2\gamma}{\gamma-1}}P_0$.

NOTE THIS VALUE IS LOWER THAN THE ONE PREDICTED FROM STEADY THEORY! $\left(\left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma}{\gamma-1}}P_0\right)$ → PAGE 18 NOTES.



THIS CASE IS SOLVED IN THE SAME WAY AS BEFORE, WITH THE DIFFERENCE THAT THE BOUNDARY CONDITION AT THE OPEN END IS $P_S = P_S^*$ INSTEAD OF $P_S = P_a$. THEN $\boxed{a_S = a_0\left(\frac{P_S^*}{P_0}\right)^{\frac{\gamma-1}{2\gamma}} = \left(\frac{\gamma+1}{2}\right)a_0 = U_S}$

⇒ NOTE THAT THE INVARIANT $J_2^- = a_S - \left(\frac{\gamma-1}{2}\right)U_S = \frac{3-\gamma}{2}$ RUNS ALONG A VERTICAL CHARACTERISTIC C^- SINCE THE FLOW IS SONIC AT THE EXIT.

e) FORCE ON CYLINDER: (SEE c))

$$F = \int (P - P_a)\,d\vec{A} = -\varepsilon P_0\frac{\pi D^4}{4}\vec{e}_x \quad \left(0 < t < \frac{L}{a_0}\right)$$