ME 355: Compressible Flows, Spring 2016

Stanford University

Homework 1

Due Tuesday, April 19, in class.

Guidelines: Please turn in a *neat* and *clean* homework that gives all the formulae that you have used as well as details that are required for the grader to understand your solution. Attach these sheets to your solutions.

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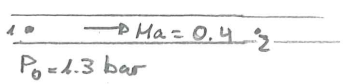
Questions (40 pts)

1. Air flows isentropically through a constant-area duct at Ma = 0.4. The stagnation pressure at point 1 is $P_0 = 1.3$ bar. What is the stagnation pressure at a downstream point 2?



SINCE THE FLOW IS ISENTROPIC, THE STAGNATION PRESSURE REMAINS CONSTANT. $P_{0_1} = P_{0_2} = 1.3$ bar

✓ 10/10

2. In addition to the conditions described in the previous question, it is also known that the stagnation density at point 1 is $\rho_0 = 1.0$ kg/m^3. What is the static enthalpy at the downstream point 2?

$$\frac{P_1}{P_0} = \left(1+\frac{\gamma-1}{2}Ma^2\right)^{\frac{\gamma}{1-\gamma}} = 0.89 = \left(\frac{\rho_1}{\rho_0}\right)^{\gamma} = \frac{P_2}{P_0} = \left(\frac{\rho_2}{\rho_0}\right)^{\gamma} \Rightarrow h_2 = \frac{h_2}{h_0} h_0 = \frac{T_2}{T_0} h_0 = \left(\frac{P_2}{P_0}\right)^{\frac{\gamma-1}{\gamma}} \frac{\gamma}{\gamma-1}\frac{P_0}{\rho_0} = 3.38\frac{P_0}{\rho_0} = 440 \text{ kJ/kg}$$

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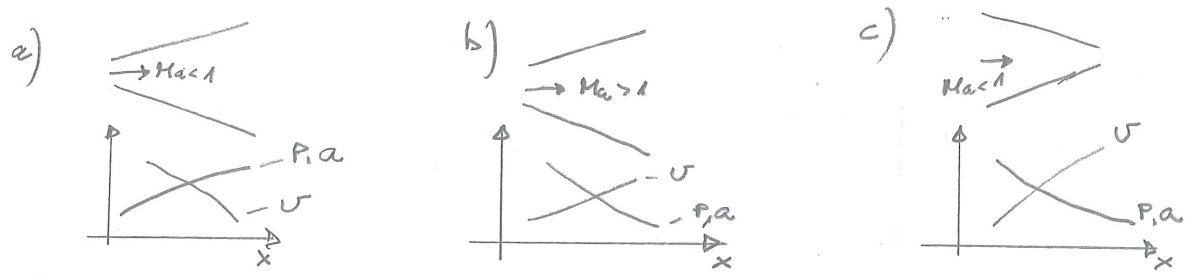
3. Sketch the streamwise variations of pressure, speed of sound and flow velocity for a) a subsonic flow through a divergent nozzle, b) a supersonic flow through a divergent nozzle, and c) a subsonic flow through a convergent nozzle.



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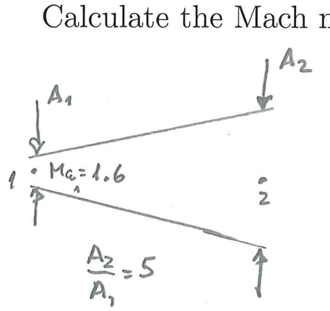
4. A gas with $\gamma = 1.4$ flows supersonically through a diverging duct. At section 1, the cross-sectional area is A_1 and the Mach number is $Ma_1 = 1.6$. At section 2, the cross-sectional area is $A_2 = 5A_1$. Calculate the Mach number Ma_2 at section 2.



CONTINUITY: $\dot{m} = \rho U A = \text{CONST.}$

$$\Rightarrow \dot{m} = \rho_0 a_0 \frac{U}{a}\frac{a}{a_0}\frac{\rho}{\rho_0} A = \rho_0 a_0 A\, Ma\left[1+\frac{(\gamma-1)}{2}Ma^2\right]^{-\frac{\gamma+1}{2(\gamma-1)}}$$

THEREFORE
$$\frac{A_1}{A_2} = \frac{Ma_2\left[1+\frac{(\gamma-1)}{2}Ma_2^2\right]^{-\frac{\gamma+1}{2(\gamma-1)}}}{Ma_1\left[1+\frac{(\gamma-1)}{2}Ma_1^2\right]^{-\frac{\gamma+1}{2(\gamma-1)}}} = \frac{1}{5} \Rightarrow \text{OBTAIN } Ma_2 \quad (Ma_2 > 1)$$

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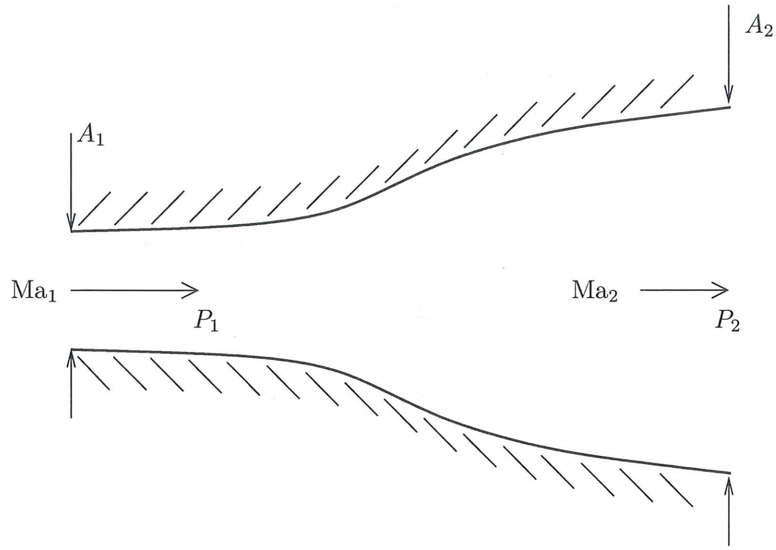
THIS EQ. CAN BE SOLVED BY DOING $\frac{A_2}{A^*} = \frac{A_2}{A_1}\cdot\frac{A_1}{A^*} = 5\underbrace{\frac{A_1}{A^*}}_{1/0.79 \text{ FROM TABLE I}} = 6.28 \Rightarrow \underbrace{Ma_2 = 3.4}_{\text{FROM TABLE I}}$

Problem 1 (30 pts)

Consider the subsonic diffuser shown in the figure where the inlet Mach number $\mathrm{Ma}_1 < 1$ and the area ratio $A_2/A_1 > 1$ are known parameters.



a) Derive an expression to compute the static-pressure ratio P_2/P_1 as a function of Ma_1, A_2/A_1 and γ.

b) Calculate the static-pressure ratio P_2/P_1 for $\mathrm{Ma}_1 = 0.4$ and $A_2/A_1 = 4$.

c) Prove that the expression

$$\frac{P_0}{P_1} = \left[1 + \frac{(\gamma-1)}{2}\mathrm{Ma}_1^2\right]^{\gamma/(\gamma-1)}$$

leads to the incompressible Bernoulli equation

$$P_1 + \rho U_1^2/2 = P_0$$

in the incompressible limit $\mathrm{Ma}^2 \ll 1$, where P_0 is the stagnation pressure, U_1 is the flow velocity, and ρ is the density.

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a) FROM MASS CONSERVATION: $\dot{m} = \rho U A = \frac{\rho}{\rho_0}\frac{U}{a}\frac{a}{a_0}\rho_0 a_0 A = \text{CONST} \Rightarrow \frac{\rho_1}{\rho_0} Ma_1 \left(\frac{T_1}{T_0}\right)^{1/2} A_1 = \frac{\rho_2}{\rho_0} Ma_2 \left(\frac{T_2}{T_0}\right)^{1/2} A_2$

AND SINCE $\rho_1 = \frac{P_1}{R_g T_1}$ AND $\rho_2 = \frac{P_2}{R_g T_2}$, THEN $\frac{P_2}{P_1}\frac{Ma_2}{Ma_1}\left(\frac{T_1}{T_2}\right)^{1/2}\frac{A_2}{A_1} = 1$. (1)

SUBSTITUTING THE RELATION $\frac{T_2}{T_1} = \frac{T_2}{T_0}\frac{T_0}{T_1} = \left(1 + \frac{(\gamma-1)}{2}Ma_1^2\right) \Big/ \left(1 + \frac{(\gamma-1)}{2}Ma_2^2\right)$ INTO EQ (1)

YIELDS: $\frac{P_2}{P_1}\frac{Ma_2}{Ma_1}\left(\frac{1+\frac{(\gamma-1)}{2}Ma_2^2}{1+\frac{(\gamma-1)}{2}Ma_1^2}\right)^{1/2}\frac{A_2}{A_1} = 1$, WHICH GIVES THE QUADRATIC EQUATION

$$Ma_2^2\left(1 + \frac{(\gamma-1)}{2}Ma_2^2\right) = Ma_1^2\left(1 + \frac{(\gamma-1)}{2}Ma_1^2\right)\left(\frac{A_1}{A_2}\right)^2\left(\frac{P_1}{P_2}\right)^2$$

FOR Ma_2 ·

$\Longrightarrow$

THE SOLUTION IS

$$(\gamma-1)Ma_2^2 = -1 + \left[1 + \frac{2(\gamma-1)Ma_1^2\left(1+\frac{(\gamma-1)}{2}Ma_1^2\right)}{\left(\frac{P_2}{P_1}\right)^2\left(\frac{A_2}{A_1}\right)}\right]^{1/2}$$

SUBSTITUTING THIS EXPRESSION INTO THE ISENTROPIC RELATIONS

$$\frac{P_2}{P_1} = \frac{P_2}{P_0}\frac{P_0}{P_1} = \left(\frac{1+\frac{(\gamma-1)}{2}Ma_2^2}{1+\frac{(\gamma-1)}{2}Ma_1^2}\right)^{\frac{\gamma}{1-\gamma}}$$

GIVES

$$\boxed{\frac{P_2}{P_1} = \left\{\frac{\frac{1}{2}+\frac{1}{2}\left[1+\frac{2(\gamma-1)Ma_1^2\left(1+\frac{(\gamma-1)}{2}Ma_1^2\right)}{\left(\frac{P_2}{P_1}\right)^2\left(\frac{A_2}{A_1}\right)^2}\right]^{1/2}}{1+\frac{(\gamma-1)}{2}Ma_1^2}\right\}^{\frac{\gamma}{1-\gamma}}} \qquad (2)$$

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WHICH REPRESENTS THE IMPLICIT RELATION BETWEEN $\frac{P_2}{P_1}$, $\frac{A_2}{A_1}$ AND Ma_1.

b) FOR $Ma_1 = 0.4$ AND $\frac{A_2}{A_1} = 4$, EQ. (2) CAN BE SOLVED ITERATIVELY TO GIVE $\boxed{\frac{P_2}{P_1} \approx 1.11}$

✓ 10/10

c) $\frac{P_0}{P_1} = \left(1+\frac{(\gamma-1)}{2}Ma_1^2\right)^{\frac{\gamma}{1-\gamma}}$ IS OF THE FORM $(1+\varepsilon)^b$, WHICH CAN BE EXPANDED IN TAYLOR SERIES AS $(1+\varepsilon)^b \sim 1 + b\varepsilon$ FOR $\varepsilon \ll 1$. IN THIS WAY,

$$\frac{P_0}{P_1} = \left(1+\frac{(\gamma-1)}{2}Ma_1^2\right)^{\frac{\gamma}{1-\gamma}} \sim 1 + \frac{\gamma}{2}Ma_1^2 = 1 + \frac{\gamma U_1^2}{2\gamma P_1/\rho} = 1 + \frac{\rho U_1^2}{2P_1}$$

$$\Rightarrow \boxed{P_0 = P_1 + \frac{\rho U_1^2}{2}}$$

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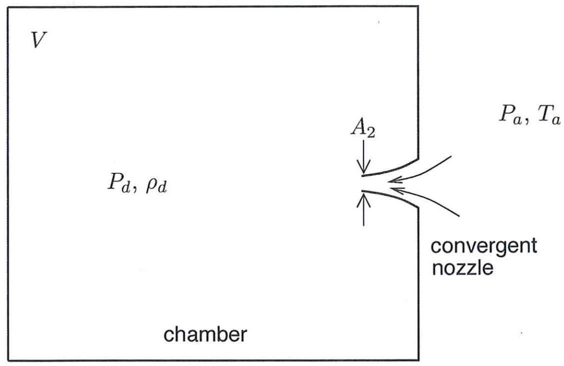
Problem 2 (30 pts)

A small hole is perforated on the wall of a chamber of volume V that is initially void of gas. The hole resembles a convergent nozzle of minimum cross-sectional area $A_2 \ll V^{2/3}$. Air from the atmosphere flows through the hole filling up the chamber, as shown in the figure below. The atmosphere is at pressure P_a and temperature T_a.



a) Show that mass and total internal-energy conservation constraints require that the time evolution of the density ρ_d and pressure P_d of the gas in the chamber are given by

$$\frac{d\rho_d}{dt} = \frac{\dot{m}}{V} \tag{1}$$

and

$$\frac{dP_d}{dt} = \frac{\dot{m}}{V} a_a^2 \tag{2}$$

where $\dot{m}$ is the mass flow rate and a_a is the speed of sound in the atmospheric air.

b) Obtain an expression for the time t^* during which the hole remains choked as a function of γ, V, A_2, T_a and the gas constant R_g.

c) Plot the time evolution of the density ρ_d/ρ_a and pressure P_d/P_a of the gas in the chamber during the interval $0 < t/t^* < 5$ for $\gamma = 1.4$.

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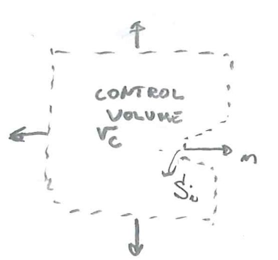
a)



* CONTINUITY: $\underbrace{\frac{d}{dt}\int_{V_c} \rho\, dV}_{V\frac{d\rho_d}{dt}} + \underbrace{\int_{S_c} \rho \vec{v}\cdot d\vec{S}}_{-\rho_2 U_2 A_2 = -\dot{m}} = 0 \Rightarrow \boxed{V\frac{d\rho_d}{dt} = \dot{m}}$ (1)

✓ 10/10

* TOTAL INTERNAL ENERGY: $\underbrace{\frac{d}{dt}\int_{V_c} \rho\left(e + \cancel{\frac{|\vec{v}|^2}{2}}\right) dV}_{\substack{\frac{d}{dt}(\rho_d e_d) = \\ = c_v V \frac{d}{dt}(\rho_d T_d) = \\ = \frac{V}{\gamma - 1}\frac{dP_d}{dt}}} + \underbrace{\int_{S_c} \rho\left(e + \frac{|\vec{v}|^2}{2} + \frac{P}{\rho}\right)\vec{v}\cdot d\vec{S}}_{\substack{-\dot{m}\left(h_2 + \frac{U_2^2}{2}\right) = -\dot{m} h_a = \\ = -\frac{\dot{m} a_a^2}{\gamma - 1}}} = 0$

NEGLIGIBLE

$\Rightarrow \boxed{V\frac{dP_d}{dt} = \dot{m} a_a^2}$ (2)

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b) INITIALLY THERE IS NO GAS IN THE CHAMBER $\Rightarrow \rho_d = P_d = 0$ AT $t=0$. AS GAS FLOWS THROUGH THE HOLE, THE PRESSURE P_d AND DENSITY ρ_d INCREASE ACCORDING TO EQS (1) + (2). IT IS HOWEVER NECESSARY TO OBTAIN AN EXPRESSION FOR THE MASS FLOW RATE $\dot{m}$...

AT THE BEGINNING, THE PRESSURE IN THE CHAMBER IS SMALL AND THE NOZZLE IS CHOKED, WITH $Ma_2 = 1$ AND

$$\dot{m} = \dot{m}^* = P_a \sqrt{\frac{\gamma}{R_g T_a}}\, A_2 \left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}} \qquad (3)$$

USING (3) IN (2) LEADS TO

$$\left.\begin{aligned} \frac{dP_d}{dt} &= \frac{P_a\sqrt{\gamma^3 R_g T_a}\, A_2}{V}\left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}} \\ P_d(0) &= 0 \end{aligned}\right\} \quad (4)$$

SO THAT $\dfrac{P_d(t)}{P_a} = \dfrac{\sqrt{\gamma^3 R_g T_a}}{V} A_2 \left(\dfrac{\gamma+1}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}} t$. THE NOZZLE REMAINS CHOKED AS LONG AS $\dfrac{P_a}{P_d} \geqslant \left(\dfrac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}}$, SO THAT $\left(\dfrac{\gamma+1}{2}\right)^{-\frac{\gamma}{\gamma-1}} = \dfrac{\sqrt{\gamma^3 R_g T_a}}{V} A_2 \left(\dfrac{\gamma+1}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}} t^*$

$$\Rightarrow \boxed{t^* = \left(\frac{2}{\gamma+1}\right)^{1/2} \frac{V}{\sqrt{\gamma^3 R_g T_a}\, A_2}} \qquad (5)$$

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c) FOR $0 < t < t^*$ THE NOZZLE IS CHOKED $(\dot{m} = \dot{m}^*)$ AND

$$\left.\begin{aligned} \frac{d\rho_d}{dt} &= \frac{P_a}{V}\sqrt{\frac{\gamma}{R_g T_a}}\, A_2 \left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}}, & \rho_d(0) = 0 \\ \frac{dP_d}{dt} &= \frac{P_a}{V}\sqrt{\gamma^3 R_g T_a}\, A_2 \left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}}, & P_d(0) = 0 \end{aligned}\right\} \quad (6)$$

FOR $t > t^*$ THE NOZZLE CEASES TO BE CHOKED AND $\dot{m} = P_a \sqrt{\dfrac{\gamma}{R_g T_a}}\, A_2 \left(\dfrac{2}{\gamma-1}\right)^{1/2} \left[\left(\dfrac{P_a}{P_d}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]^{1/2} \left(\dfrac{P_a}{P_d}\right)^{-\frac{\gamma+1}{2(\gamma-1)}}$

SO THAT

$$\left.\begin{aligned} \frac{d\rho_d}{dt} &= \frac{P_a}{V}\sqrt{\frac{\gamma}{R_g T_a}}\, A_2 \left(\frac{2}{\gamma-1}\right)^{1/2}\left[\left(\frac{P_a}{P_d}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]^{1/2}\left(\frac{P_a}{P_d}\right)^{-\frac{\gamma+1}{2\gamma}}, & \rho_d(t^*) = \rho_d^* \\ \frac{dP_d}{dt} &= \frac{P_a}{V}\sqrt{\gamma^3 R_g T_a}\, A_2 \left(\frac{2}{\gamma-1}\right)^{1/2}\left[\left(\frac{P_a}{P_d}\right)^{\frac{(\gamma-1)}{\gamma}} - 1\right]^{1/2}\left(\frac{P_a}{P_d}\right)^{-\frac{\gamma+1}{2\gamma}}, & P_d(t^*) = P_d^* \end{aligned}\right\} \quad (7)$$

WHERE

$$\left\{\begin{aligned} \rho_d^* &= \frac{P_a}{\cancel{V}}\sqrt{\frac{\gamma}{R_g T_a}}\, \cancel{A_2}\left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}}\left(\frac{2}{\gamma+1}\right)^{1/2}\frac{\cancel{V}}{\sqrt{\gamma^3 R_g T_a}\,\cancel{A_2}} = \left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma}{\gamma-1}}\frac{P_a}{\gamma R_g T_a} \\ P_d^* &= \left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma}{\gamma-1}} P_a \end{aligned}\right.$$

ARE THE CRITICAL VALUES OF DENSITY AND TEMPERATURE. NORMALIZING $\tilde{\rho} = \rho_d/\rho_a = \dfrac{\rho_d R_g T_a}{P_a}$, $\tilde{P} = P_d/P_a$ AND $\tau = t/t^*$, EQUATIONS (6) AND (7) BECOME

$$\boxed{\left.\begin{aligned} \frac{d\tilde{\rho}}{d\tau} &= \frac{1}{\gamma}\left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma}{\gamma-1}} \\ \frac{d\tilde{P}}{d\tau} &= \left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma}{\gamma-1}} \end{aligned}\right\} (0 < \tau < 1), \text{ SUBJECT TO } \tilde{P}(0) = \tilde{\rho}(0) = 0 \quad \text{CHOKED FLOW } (\dot{m} = \text{CONST.})} \qquad (8)$$

✓

→

AND

$$\frac{d\tilde{\rho}}{d\tau} = \frac{1}{\gamma}\left(\frac{2}{\gamma+1}\right)^{1/2}\left(\frac{2}{\gamma-1}\right)^{1/2}\left(\tilde{P}^{\frac{1-\gamma}{\gamma}}-1\right)^{1/2}\tilde{P}^{\frac{\gamma+1}{2\gamma}}$$

$$\frac{d\tilde{P}}{d\tau} = \left(\frac{2}{\gamma+1}\right)^{1/2}\left(\frac{2}{\gamma-1}\right)^{1/2}\left(\tilde{P}^{\frac{1-\gamma}{\gamma}}-1\right)^{1/2}\tilde{P}^{\frac{\gamma+1}{2\gamma}}$$

$(1<\tau<\infty)$ (9)

SUBJECT TO $\tilde{\rho} = \tilde{\rho}^{*} = \frac{1}{\gamma}\left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma}{\gamma-1}}$ AND $\tilde{P} = \tilde{P}^{*} = \gamma\tilde{\rho}^{*}$ AT $\tau = 1$

NON-CHOKED FLOW ($\dot{m} \neq$ CONST)

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THE INTEGRATION OF EQS. (8) IS TRIVIAL AND YIELDS THE LINEAR VARIATIONS

$$\tilde{\rho} = \frac{1}{\gamma}\left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma}{\gamma-1}}\tau \quad , \qquad \tilde{P} = \gamma\tilde{\rho} \qquad (10).$$

THE INTEGRATION OF EQS. (9) DETERMINES THE TIME EVOLUTIONS $\tilde{P}(\tau)$ AND $\tilde{\rho}(\tau)$ WHEN THE CHAMBER IS FILLED ENOUGH SUCH THAT THE FLOW IS NOT CHOKED ANYMORE. THE INTEGRATION IS NOT STRAIGHTFORWARD, BUT A SKETCH OF THE SOLUTION IS GIVEN BELOW