ECONOMIC THEORY OF DEPLETABLE RESOURCES: AN INTRODUCTION

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I. BACKGROUND

A. A Classification of Resources

One can think of a two-way classification of natural resources, based on 1) physical properties of the resource and 2) the time scale of the relevant adjustment processes.

Based on physical characteristics, we can divide resources into biological, non-energy mineral, energy, and environmental resources. Each of these categories could be broken down further if useful for purposes of analysis or information collection. As examples, biological resources would include fish, wild animals, flowers, whales, insects, and most agricultural products. Non-energy minerals could include gold, iron ore, salt, or soil. Energy would include solar radiation, wood used for burning, and natural gas. Environmental resources could include air, water, forests, the ozone layer, or a virgin wilderness.

Based on the time scale of the relevant adjustment processes, we can also classify resources as expendable, renewable, or depletable. Depletable resources are those whose adjustment speed is so slow that we can meaningfully model them as made available once and only once by nature. Crude oil or natural gas deposits provide prototypical examples, but a virgin wilderness, an endangered species, or top soil also can well be viewed as depletable resources. Renewable resources adjust more rapidly so that they are self renewing within a time scale important for economic decisionmaking. But actions in one time period which alter the stock of the resource can be expected to have consequences in subsequent time periods. For example, populations of fish or wild animals can well be viewed as renewable as can be water in reservoirs or in many ground water deposits. Expendable resources are those whose adjustment speed is so fast that impacts on the resource in one time period have little or no effects in subsequent periods. For example, noise pollution and particulates in the air, solar radiation, as well as much agricultural production can be thought of as expendable.

Although there is a correlation between the physical properties and the time scale of adjustment, the correlation is far from perfect. Table 1 illustrates the two-way categorization, giving examples of resources based on both classifications. Each physical class of resources includes examples of each adjustment speed. For example, while most non-energy mineral resources can be viewed as depletable, salt evaporated from the San Francisco Bay can be viewed as expendable since the cordoning off of an area of seawater has no perceptible impact on the total availability of seawater in the Bay. Energy resources include solar radiation (expendable), hydropower and wood (renewable), and petroleum (depletable).

Volumes I and II of the *Handbook of Energy and Natural Resource Economics* deal with the economics of renewable and environmental resources, including biological resources. Volume III focuses attention on depletable resources and energy resources. While particular attention is given to desirable, depletable, energy resources in Volume III, we focus attention on the bottom row -- depletable resources -- and on the third column -- energy resources.

This chapter focuses on the bottom row, providing an introduction to the economic theory of depletable resources. The introduction is designed to make accessible fundamental theoretical models of depletable resource supply and of market equilibrium and to provide the reader with an understanding of basic methods underlying the theory. It is meant to present theoretical economic models in a self contained document and to provide a background useful for the papers that follow.

Table 1
Natural Resource Examples

	BIOLOGICAL	NON-ENERGY MINERAL	ENERGY	ENVIRONMENTAL
EXPENDABLE	Most Agricultural Products Corn Grains	Salt	Solar Radiation Hydropower Ethanol	Noise Pollution Non-Persistent: Air Pollution (NOx, SOx, Particulates) Water Pollution
RENEWABLE	Forest Products Fish Livestock Harvested Wild Animals Wood Whales Flowers Insects		Wood for burning Hydropower Geothermal	Ground Water Air Persistent: Air Pollution Water Pollution: Carbon Dioxide Toxics Animal Populations Forests
DEPLETABLE	Endangered Species	Most Minerals Gold Iron Ore Bauxite Salt Top Soil	Petroleum Natural Gas Coal Uranium Oil Shale	Virgin Wilderness Ozone Layer Water in Some Aquifers

B. The Depletability Concept

The depletable resources indicated in Table 1 all have adjustment speeds so slow that we can think of them as made available once and only once by nature. Their consumptive[2] use can be allocated over time, but once they are used up, they are gone forever, or for such a long time that the possibility of their eventual renewal has no current economic significance. In particular, there initially exists some stock (or stocks) of the resource in various deposits. As the resource in a given deposit is used, stock declines. The greater the consumptive use, the more rapid the decline in remaining resource stock. No processes increase the stock in any deposit, although the number of deposits available for use could increase. If stock ever declines to zero, then no further use is possible and for some positive stock level, further use may be uneconomic. These characteristics will be taken to define depletable resources.

Definition: <u>Depletable Resource</u>. A resource is depletable if 1) its stock decreases over time whenever the resource is being used, 2) the stock never increases over time, 3) the rate of stock decrease is a monotonically increasing function of the rate of resource use, and 4) no use is possible without a positive stock.

Let S_t denote stock at the end of time period t for the particular deposit and let E_t denote the quantity of the resource extracted from that deposit during time period t. E_t is generally be referred to as the "extraction rate", but its units are physical quantities, such as tons or barrels, and not physical quantities per unit of time. Then the depletable resource definition implies the following relationships in a discrete time model:

$$S_t = S_{t-1} - h(E_t) \quad (1)$$
$$h(E_t) \geq 0 \quad and \quad E_t > 0 \rightarrow h(E_t) > 0 \quad (2)$$
$$E_t' > E_t \quad \rightarrow \quad h(E_t') > h(E_t) \quad (3)$$
$$S_t \geq 0 \quad (4)$$

Several examples can illustrate the underlying concept. A deposit of natural gas or oil many remain under the ground with its stock unchanged until the resource is discovered. Then as it is extracted, the stock declines at the rate of one Btu[3] for every Btu of natural gas or oil extracted from the deposit. In this case, $h(E_t) = E_t$. However, if oil is extracted very rapidly, some is left trapped in the mineral media and that oil cannot be extracted. Thus the resource available for extraction may decline by more than one Btu for each Btu extracted. In this case, $h(E_t) > E_t$. If extraction stops, the stock will remain constant, unless there is some leakage from the deposit, in which case the stock will continue to decline. Once there is nothing left in the deposit, no more can be extracted. However, it may become virtually impossible to extract any more of the stock once the pressure driving the resource to the well declines enough, that is, once stock is below some critical level.

A virgin wilderness can remain unspoiled forever, absent human intervention, although its precise composition will change over time. We can consider many different uses of the resource, only some of which would be the consumptive use envisioned under the definition above. At one extreme of non-consumptive use, small groups can backpack through the wilderness, having no more impact than that of grazing deer. At the other extreme of consumptive use, the forest can be clear cut for timber. It is the latter type of activity -- consumptive use -- that would be considered "use" under the depletable resource definition. The greater the area that was used by clear cutting in each decade, the less the remaining stock of virgin wilderness, and the more rapid the rate of stock decrease.

Top soil may be eroded as a result of agricultural activity and differing crops may lead to differing rates of top soil erosion from cultivated lands. In this case we may have a vector of agricultural activities, E_t, with the amount of annual erosion as a complex function of this vector of activities. The function $h(E_t)$ would indicate the amount of top soil eroded away as a function of this vector of agricultural activities. The variable S_t would measure the remaining quantity of top soil remaining at the end of time t.

Note that none of these examples, in fact, none of the resources characterized as depletable in Table 1, perfectly meets the definition, but that each approximately meets it. Oil and natural gas are derived from the transformation of organic material underground. This process continues today, so that strictly, the stock of oil in some locations <u>is</u> increasing, although at an infinitesimally small rate. Leakage from a deposit may involve migration to another deposit, which then may be increasing over time. If we were to harvest a virgin forest but then allowed the land to remain undisturbed for 10,000 years, the forest would revert to a virgin state. We can reinject natural gas back into a well and thereby increase stock of natural gas in that deposit. Thus the definition must be viewed as a mathematical abstraction, but an abstraction that approximates many situations so closely that it is a useful analytical construct.[4]

For most analysis, we will not require as much generality as allowed in (2) and (3). In particular, it will normally be appropriate to assume that every unit of the resource extracted reduces the remaining stock by a single unit.   We will refer to such an assumption as "linear stock dynamics". Although the assumption of linear stock dynamics is not always valid, most insights from depletable resource theory can be developed without requiring the greater generality allowed in (2) and (3).

Assumption: <u>Linear Stock Dynamics</u>:
The stock is reduced by one unit for every unit of the resource extracted. This reduction is independent of the rate of extraction and of the remaining stock:

$$h(E_t) = E_t$$

Under the assumption of linear stock dynamics, equations (1), (2) and (3) translate to:

$$S_t = S_{t-1} - E_t \qquad (5)$$

Equations (1) through (4) describe the most fundamental constraints underlying a theory of depletable resources. In addition, linear stock dynamics will be assumed, so that equation (5) will be used as a more specific form of relationships (1) through (3). Therefore, equations (4) and (5) will provide the fundamental mathematical constraints underlying depletable resource theory in this chapter.

Under the assumption of linear dynamics, equations (4) and (5) can be combined to imply a simple form of the depletability condition. Equation (6) will always hold, but includes less information than obtainable from equations (4) and (5).

$$\sum_{t=\tau+1}^{\infty} E_t \le S_\tau \qquad (6)$$

The total extraction of the resource over all time can be no larger than the initial stock of that resource, or more generally, the total extraction of that resource over all time beginning from an arbitrary starting point can be no greater than the stock remaining at that starting point.

Each equation presented above has assumed a discrete time representation. We will use such discrete time representations throughout this chapter, although a continuous time formulation could be utilized, as in most published theoretical literature. Continuous time formulations can be seen as discrete time formulations in which the length of each time interval converges to zero. As such all equations presented in this chapter can be readily translated to their continuous time counterparts. Our discrete time models can have arbitrarily short intervals, so no loss of generality is entailed by using discrete time representations. And whenever empirical work is conducted or computational models are constructed, discrete time formulations are the only ones possible. In addition, discrete time models allow the analyst to avoid some mathematical subtleties of infinite dimensional spaces at points required by continuous time market models. For these reason we have chosen to use discrete time representations.

Even though a discrete time representation is used, we can envision an underlying continuous time model such that the discrete time variables are equal to integrals of the corresponding continuous time variables. Let the continuous time extraction rate be denoted by ε(t) and let the length of each time interval be denoted by L. Then the discrete time extraction rate, E_t and the ε(t) would be related:

$$E_t = \int_t^{t+L} \varepsilon(\gamma)\, d\gamma$$

E_t will be roughly proportional to L, in the sense that if each interval were partioned into smaller intervals, the sum of the E_t over these smaller intervals would equal the original E_t.

$$dS/dt = -\varepsilon(t)$$

For the underlying continuous time model, let stock be denoted by S or by S(t). Then equation (5) would become:
Other variables and functions to be presented can be related to underlying continuous time models in a like manner. At a later point we will show that the existence of an underlying continuous time model imposes constraints on the functions in the discrete time model.

For depletable resources such as energy and other mineral resources, there is a typical sequence of activities, each governed by economic considerations. Initially there may be preliminary exploration of a broad geological area and later of specific tracts. At some point the land may be offered for leasing, perhaps through a competitive bidding process. After a period of more focused exploration the deposit may be discovered. Only then can extraction begin. Further activities may be devoted to delineating the extent of the resource and these activities may lead to further discoveries. The resource, once extracted, is then transported to some location for further processing and then to final users. Some resources might be recycled for further rounds of processing and consumption.

The timing and magnitude of each process is governed by human decisions and typically by economic forces. But the amount and quality of the deposit discovered and ultimately extracted are constrained by the natural endowment. Thus the basic patterns of depletable resource use are governed by an interplay of economic forces and natural constraints.

The combination of processes can be very complex. Yet economic models of depletable resources -- including those discussed in this chapter -- typically abstract away from most processes and focus attention on the elements in the definition: the rate of use (extraction) of the resource and the resultant change in the quantity of the resource stock. This abstraction allows insights about the economic forces, insights which may not be available from more detailed analyses. But the abstraction does present a fairly bare bones image of a complex set of processes, an image which could well be usefully expanded. For example, Harris, in this volume, brings in a richer understanding of the interplay between physical constraints and human choices.

Depletable resource theory typically addresses several broad classes of questions, either in a normative manner ("should") or in a positive manner ("would") for a particular set of economic conditions:

> Should a specific resource ever be extracted? Would it under competitive markets?
> How much of that resource should or would ultimately be extracted?
> What would be the timing of extraction with competitive markets?
> What would be market price pattern over time under competitive forces?
> What timing of extraction should be best for society as a whole?
> How do market determined and socially optimal rates compare?
>> Can we expect overuse, underuse, or correct use with competitive markets? Under monopolistic conditions?
>
> How would various market changes -- higher interest rates, changed expectations, varying market structures, taxes -- change patterns of extraction?
> What is the nature of the supply function for depletable resources?

This chapter will address positive questions in the context of a sequence of depletable resource models. Heal addresses normative questions in a separate chapter.

Section II presents a sequences of models of extraction from one resource stock when prices are exogenously determined. Section III presents intertemporal market equilibrium conditions and analyzes markets in which prices are determined endogenously from the interplay of supply and demand. Finally, Section IV provides concluding thoughts.

II. EXTRACTION WITH PRICES DETERMINED EXOGENOUSLY

The simplest depletable resource models are those applicable to the competitive owner of a resource stock as that owner chooses the time path of its extraction. We address such models in this section. We assume that the firm takes selling prices of the extracted commodity as fixed in the marketplace, not influenced by his or her actions. These prices may be varying over time but their future path is assumed to be known with certainty. Although the assumption of uncertainty is very strong, particularly when we consider the long term future evolution of economic parameters, we will not address uncertainty per se within this chapter.

A. General Problem Formulation

1. Objective and Constraints

If P_t and E_t represent price and extraction rate at time t, the revenue at time t (R_t) obtained from selling the extracted commodity is a linear function of extraction rate:

$$R_t = P_t E_t \tag{7}$$

Total cost incurred by the resource owner during a time period will depend upon total extraction during that period, perhaps upon the stock remaining from the last period, and on time: $C_t(E_t,S_{t-1})$. We will assume that this time dependant cost function will be known to the resource owner with perfect certainty.

Given revenue and cost functions plus constraints defined by resource depletability, the resource owner will be assumed to choose a time path of extraction so as to maximize present value of profit. Equivalently, the owner is assumed to select an extraction path so as to maximize the deposit value, where value is determined as a discounted present value of revenues minus costs. For our analysis we will assume a finite time horizon of T, where T is arbitrarily long[5]. If Π denotes discounted present value of profit, the firm faces the following maximization problem, where r represents the instantaneous interest rate facing the owner of the resource deposit:

$$\left.\begin{aligned} &Max \;\; \Pi = \sum_{t=1}^{T} [P_t E_t - C_t(E_t, S_{t-1})]\, e^{-rt} \\ &Under: \quad S_t = S_{t-1} - E_t \quad for\ all\ t \\ &\qquad\qquad S_T \geq 0 \\ &\qquad\qquad E_t \geq 0 \end{aligned}\right\} \qquad (8)$$

2. Characteristics of the Discrete Time Cost Function

The cost function in such a discrete time model should in principle be derivable as the integral of cost in an underlying continuous time representation. Let $g(\varepsilon(\gamma),S(\gamma))$ be the underlying continuous time cost function. Then the discrete time cost function will be the minimum feasible integral of cost[6] over the interval from t to t+L, given that total extraction is E_t and stock at t is S_{t-1}:

$$\left.\begin{aligned} &C_t(E_t,S_{t-1}) = \min \int_t^{t+L} g(\varepsilon(\gamma),S(\gamma))\, d\gamma \\ &\quad such\ that \int_t^{t+L} \varepsilon(\gamma)\, d\gamma = E_t \quad and \\ &\quad S(\gamma) = S_{t-1} - \int_t^{\gamma} \varepsilon(\gamma)\, d\gamma \end{aligned}\right\} \qquad (9)$$

The discrete time cost function depends on properties of $g(\varepsilon(\gamma), S(\gamma))$, and on L, S_{t-1}, and E_t.

Properties of $C_t(E_t,S_{t-1})$ must derive from the optimization problem (9). First, $C_t(E_t,S_{t-1})$ must be roughly proportional to L in the sense that if the interval L were partitioned into N intervals, the N costs must add to the value of the original $C_t(E_t,S_{t-1})$.

The existence of an underlying continuous time representation implies restrictions on the partial derivatives of allowable discrete time cost functions. Consider first the partial derivative of cost with respect to initial stock[7]:

$$\partial C_t/\partial S_{t-1} = \int_t^{t+L} \partial g/\partial S \, d\gamma = \partial g/\partial S \, L$$

where $\partial g/\partial S$ is evaluated at some point between t and t+L. The partial derivative of cost with respect to initial stock must be approximately proportional to the length of the underlying time interval and must have the same sign as does $\partial g/\partial S$.

The existence of an underlying continuous time representation imposes more rigid restrictions on the marginal extraction cost, $\partial C_t/\partial E_t$. As E_t varies, the instantaneous extraction rates, $\varepsilon(\gamma)$, must vary so that their sum remains equal to E_t. $\partial C_t/\partial E_t$ then is the integral of the changes in costs associated with the changes in the $\varepsilon(\gamma)$, accounting for the changes in $S(\gamma)$ induced by the changes in $\varepsilon(\gamma)$. Because $C_t(E_t,S_{t-1})$ is defined as the result of an optimization, the impact on total cost for a small increase in $\varepsilon(\gamma)$ will be the same for all γ. Thus, in order to assess the integral, we can evaluate the marginal cost of a change in instantaneous extraction rates at any time, including at $\gamma = t$. Evaluating at $\gamma = t$, the partial derivative $\partial C_t/\partial E_t$ is thus:

$$\partial C_t/\partial E_t \;=\; \partial g/\partial \varepsilon \;-\; \int_t^{t+L} \partial g/\partial S \; d\gamma \tag{10}$$

Equation (10) is derived more rigorously in the Appendix.

By equation (10), marginal cost of extraction during a discrete interval consists of two components. The first, $\partial g/\partial \varepsilon$, is simply the additional cost directly associated with additional extraction at t. The second term captures the incremental cost of lower stock for the rest of the interval, associated with more extraction at the beginning of the period.

The second term in equation (10) is identically equal to the derivative of cost with respect to initial stock. Thus equation (10) can be combined with the previous equation to show:

$$\partial C_t/\partial E_t \;+\; \partial C_t/\partial S_{t-1} \;=\; \partial g/\partial \varepsilon \;>\; 0 \tag{11}$$

where where $\partial g/\partial \varepsilon$ is evaluated at $\gamma = t$.

Equation (11) imposes an important restriction on the specification of a discrete time cost function. The restriction must hold even if no analogous restrictions exist for the underlying continuous cost function. Thus in using discrete time models, it is important to recognize that continuous time underlying cost functions do not translate precisely to discrete time cost functions having the same functional form, and that not all cost functions appropriate for a continuous time model are also appropriate for a discrete time model.

Equation (11) can be differentiated with respect to E_t in order to derive an expression relating second partial derivatives:

$$\partial^2 C_t/\partial E_t^2 \;+\; \partial^2 C_t/\,\partial S_{t-1}\,\partial E_t \;=\; \partial^2 c/\,\partial e^2\,\partial e(t)/\,\partial E_t \;> \qquad (12)$$

Expression (12) implies that the derivative of marginal cost with respect to extraction rate will be the sum of two effects. Increasing total extraction during the interval (E_t) increases the instantaneous extraction rate for all time, including at t, and an increase in instantaneous extraction rate increases marginal cost (the right hand side of equation (12)). Thus the right hand side of equation is positive. In addition, increasing extraction rate at the beginning of the time period reduces stock during the remainder of the period. This stock reduction further increases marginal cost if $\partial^2 C/\partial S\partial E$ is negative (on the left hand side of equation (12)).

Many discrete time models improperly start with a discrete time cost function without so restricting its properties. For this chapter, however, we will always assume that the discrete cost function is consistent with the existence of an underlying continuous cost function and will thus always assume that equations (11) and (12) will be valid[8].

<u>Assumption: Dominance of extraction rate on marginal cost</u>.
Marginal cost is more sensitive to extraction rate than to stock level:
$\partial^2 C_t/\partial E_t^2 \;>\; -\,\partial^2 C_t/\partial E_t \partial S_{t-1}$

Even meeting the necessary restrictions, the cost function in problem (8) could have many different characteristics. The marginal cost of extraction ($\partial C_t/\ \partial E_t$) could be decreasing, constant, or increasing in extraction rate. Similarly, the marginal extraction cost and the total cost could be increasing or decreasing in remaining stock or it could be independent of stock remaining from the last period. The characteristics of the optimal solution will depend upon which of these combinations is appropriate for a given problem.

Much depletable resource literature assumes that the cost function at each time is independent of the remaining stock of the resource. The initial works by Hotelling and by Grey assumed, in addition, that marginal extraction cost was independent of extraction rate. And this set of assumptions has been followed by many researchers. Other work maintains the assumption that the extraction cost function is independent of the remaining stock but assumes that the marginal extraction cost is an increasing function of extraction rate.

Alternatively, one might assume that marginal extraction costs do vary with remaining stock. Typically one might expect marginal extraction cost to increase as the resource is depleted. This relationship could hold for physical reasons. For example, as oil or natural gas deposits are depleted, the driving pressure in the deposit declines and extraction rates decline. Reestablishing the previous extraction rate could be very costly. In addition, it will typically be optimal to extract high quality low cost portions of deposits before low quality, higher cost grades[9]. In that case, the smaller the remaining stock, the higher the unit extraction costs.

But the reverse situation can occur, at least in early stages of extraction. The extraction process itself can lead to technological improvements which reduce extraction costs. This "learning by doing" phenomenon would imply that for a range of stock levels, the lower the stock, the lower the marginal cost.

Another common approach reflects these possibilities in a simple way. Marginal extraction cost of the underlying continuous time cost function might be assumed to depend on remaining stock, but not on extraction rate. Total cost would be linear in extraction rate for the underlying continuous time model. However, these linearity assumptions for a continuous time model lead to a discrete time model in which marginal extraction cost is an increasing function of extraction rate and a decreasing function of the remaining stock.

Total extraction cost could be an increasing or decreasing function of remaining stock independent of whether marginal cost increased or decreased with stock. For example, subsidence of land overlying an aquifer may be a function of the stock of water in the aquifer and not a function of the extraction rate. Total environmental costs associated with clear cutting a virgin forest depend upon the amount of the forest which has been clear cut although the marginal costs of additional harvest may be virtually independent of the remaining stock. The costs of global climate change may depend upon the cumulative extraction of fossil fuels and thus upon the remaining stock.

In this chapter, we do not seek full generality. Although we will examine several different derivatives of total cost with respect to stock and several different derivatives of marginal costs with respect to stock, we will always assume that the cost function is weakly convex in its arguments. Therefore the second order conditions for optimality will be satisfied, the set of optimal choices must be convex, and multiple local unconnected optima cannot exist[10].

Assumption: Weak convexity. We assume that the cost function is weakly convex.

A function is weakly convex if and only if the Hessian matrix -- the matrix of second partial derivatives -- is positive semidefinite at every point. A matrix is positive semidefinite if and only if its principal minor determinants are all positive or zero. The principal minor determinants of the Hessian matrix from the cost function are: $\partial^2 C_t/\partial E_t^2$, $\partial^2 C_t/\partial S_{t-1}^2$, and $[\partial^2 C_t/\partial E_t^2 \;\; \partial^2 C_t/\partial S_{t-1}^2] - [\partial^2 C_t/\partial E_t \partial S_{t-1}]^2$, all of which must be non-negative everywhere, given the convexity assumption.

With this background, we can now turn to simplified versions of these models, versions useful for deriving insight into the behavior of optimizing suppliers of depletable resources. We will examine various cost assumptions in turn, starting with models in which extraction rate is independent of remaining stock. We will discuss the Hotelling assumption that marginal cost is independent of both stock and extraction rate as a special case of the more general model in which marginal cost in non-decreasing in extraction rate. We will only then turn to models in which remaining stock influences marginal extraction cost.

B. Optimizing Models Without Stock Effects

A fundamental distinction among depletable resource models is whether the remaining stock influences cost of extraction from a given deposit. The initial stock may typically influence the cost structure. However, the distinction is whether current extraction decisions influence future costs through their impacts on stock remaining at those future times. We will refer to such impacts as "stock effects".

In this section we deal with models in which there are no stock effects: in which the remaining stock has no influence on extraction costs.

Assumption: No Stock Effects
The remaining stock does not appear as an argument in the cost function.

Under the assumption that total (and marginal) extraction cost is independent of the remaining stock of the resource, problem (8) reduces to problem (13):

$$\left.\begin{aligned} &Max\ \Pi = \sum_{t=1}^{T} [P_t E_t - C_t(E_t)]e^{-rt} \\ &Under: \quad S_t = S_{t-1} - E_t \quad for\ all\ t \\ &\qquad\qquad S_T \geq 0 \\ &\qquad\qquad E_t \geq 0 \end{aligned}\right\} \tag{13}$$

Several different optimization methods can be used to solve problem (13). In what follows we use the Kuhn-Tucker conditions to develop first order necessary conditions for optimality. In addition, we show that the same conditions can be obtained in a more insightful manner by examining feasible variations from the optimal path.

1. Necessary Conditions for Optimality: Kuhn-Tucker Conditions

a. Kuhn-Tucker Theorem

The Kuhn-Tucker theorem aids solution of constrained optimization problems by providing first order necessary conditions for optimality[11]. Consider the optimization problem:

$$\begin{aligned} &\max f(x) \\ &under\ G_i(x) \leq 0 \qquad i = 1,\ldots,k \end{aligned} \tag{M}$$

where x is a vector of variables to be selected, f(x) is the objective function, and $G_i(x)$ is one of k inequality constraints on the components of x. Let x^* denote the optimum value of the x vector and let $\nabla G_i(x^*)$ denote the gradient of $G_i(x)$ evaluated at x^*. Number the constraints so that constraints 1 through n are binding, where $n \leq k$. The <u>constraint qualification</u> will be said to hold if the set of gradients $\nabla G_i(x^*)$, for i = 1,....,n is linearly independent.

<u>Kuhn-Tucker Theorem</u>. Assume that the constraint qualification holds at x^*. If x^* solves problem (M) then there exists a set of dual variables λ_i for i = 1,...,k, such that:

$$\nabla f(x^*) = \sum_{i=1}^{k} \lambda_i \nabla G_i(x^*) \tag{KT}$$

and the complementary slackness conditions hold for every i:

$$\begin{aligned} \lambda_i &\geq 0 \\ \lambda_i G_i(x^*) &= 0 \end{aligned} \qquad \textbf{(CS)}$$

The Kuhn-Tucker condition is <u>necessary</u> for optimality. In one special case, corresponding to the weak convexity assumption, the condition is <u>necessary and sufficient</u>:

<u>Kuhn-Tucker Sufficiency Condition</u>: **Suppose that f(x) is a concave function and $G_i(x)$ is a convex function for all i. If x^* is a feasible point and if we can find dual variables which satisfy (KT) and (CS), then x^* solves the maximization problem (M).**

Lagrange multipliers can be thought of as a convenient mnemonic for using the Kuhn-Tucker theorem. Define the Lagrangian as:

$$\mathcal{L}(x,\lambda) = f(x) - \sum_{i=1}^{k} \lambda_i G_i(x)$$

Necessary conditions for interior optimality can be obtained by finding the stationary point of the Lagrangian, the point at which the gradient of the Lagrangian, with respect to the vector x, is equal to zero.

$$\nabla_x \mathcal{L}(x^* \lambda) = \nabla f(x^*) - \sum_{i=1}^{k} \lambda_i \nabla G_i(x^*) = 0$$

At the optimal point, x^*, condition (KT) is satisfied. Then as long as condition (CS) is satisfied as well, the necessary conditions for optimality are met. Thus Lagrange multipliers can be seen as providing a convenient method for using the Kuhn-Tucker theorem.

b. Optimality Conditions for Depletable Resources without Stock Effects

Necessary conditions for interior point optimality can be obtained by finding the stationary point of the Lagrangian formed from problem (13). A dual variable is defined for each constraint in the optimization problem. The depletability condition, equation (5), defines one constraint for and thus requires one dual variable for each t. The dual variable for the constraint at time t will be denoted by λ_t. We will use standard economic interpretations of dual variables; as such we will often refer to λ_t as the <u>present</u>

value shadow price or as present value opportunity cost. In addition, the constraint that $S_T \geq 0$ leads to a dual variable, to be denoted as μ.

In order to use the Kuhn-Tucker theorem we must assure that the constraint qualification holds: that the set of gradients $\nabla[S_t - S_{t-1} - E_t]$ is linearly independent. Linear independence in this system is easily established, since the derivative with respect to E_t is equal to 1 for the t^{th} constraint and 0 for all other constraints. Thus no gradient can be expressed as a linear combination of the other gradients[12]. Since the constraint qualification is satisfied, the Kuhn-Tucker theorem can be used.

Denoting the Lagrangian by $\mathcal{L}$ we can convert problem (13) to the following unconstrained optimization problem.

$$Max\ \mathcal{L} = \sum_{t=1}^{T} [P_t E_t - C_t(E_t)]e^{-rt} - \sum_{t=1}^{T} [S_t - S_{t-1} + E_t]\,\lambda_t\ + \tag{14}$$

The Kuhn-Tucker theorem shows that at the optimal point, the Lagrangian must be a stationary point with respect to each S_t and each E_t. Differentiating $\mathcal{L}$ with respect to each variable gives the first order necessary conditions:

$$\partial\mathcal{L}/\partial E_t = [P_t - dC_t/dE_t]e^{-rt} - \lambda_t \quad \begin{cases} = 0 & if\ E_t > 0 \\ \leq 0 & if\ E_t = 0 \end{cases}$$

$$\partial\mathcal{L}/\partial S_t = -\lambda_t + \lambda_{t+1} = 0 \quad for\ t < T$$

$$\partial\mathcal{L}/\partial S_T = -\lambda_T + \mu = 0$$

$$\mu\, S_T = 0$$

These equations show that the present value shadow price is time independent for models in which extraction cost does not depend on the remaining stock. Therefore we can drop the time index from λ_t

and denote λ as the present value shadow price. Combining these equations gives the fundamental first order necessary conditions for optimality:

$$P_t \begin{matrix} = \\ \le \end{matrix} \; dC_t/dE_t + \lambda \, e^{rt} \quad \begin{cases} \text{if } E_t > 0 \\ \text{if } E_t = 0 \end{cases} \tag{19}$$

$$\lambda \, S_T = 0 \tag{15}$$

The right-hand-side of equation (15) consist of two terms: the marginal extraction cost plus the <u>current value opportunity cost</u>. Thus extraction rate in a competitive depletable resource market is chosen so that marginal cost <u>plus</u> the current value opportunity cost equals the price; price <u>exceeds</u> marginal extraction cost.

At some points it will be convenient to denote the current value opportunity cost as ϕ_t, where:

$$\phi_t = \lambda_t \, e^{rt} \tag{16}$$

Since present value opportunity cost is independent of time, current value opportunity cost grows at the interest rate for models without stock effects.

Equation (16) shows that unless the entire stock is depleted within the time horizon, the opportunity cost is zero for models with no stock effects. In that case, price equals marginal cost, the identical condition to that for a competitive producer of a conventional commodity.
The Kuhn-Tucker conditions in this case are sufficient as well as necessary for optimality. If $C_t(E_t)$ is a convex function, then $-C_t(E_t)$ is a concave function and the objective function in problem (13) is a concave function. The constraints are linear and therefore define a convex feasible set. Thus the Kuhn-Tucker sufficiency condition is satisfied and the extraction path that satisfies equations (15) and (16) is the optimal path.

Since we assume no stock effects, the cost function is convex as long as the following second order condition holds for all extraction rates:

$$d^2C_t/dE_t^2 \ge 0$$

As long as marginal cost is a non-decreasing function of extraction rate at the optimal point, first order necessary conditions for optimality are sufficient conditions as well.

The optimal choice of extraction rate for opportunity cost fixed, based on equation (15), is diagrammed in Figure 1. Optimal extraction rate occurs where marginal cost plus opportunity cost equals price of the extracted commodity.

In Figure 1, because marginal cost is an increasing function of extraction rate, the higher the opportunity cost or the marginal cost function, the lower the optimal extraction rate. The higher the price, all else equal, the higher the optimal extraction rate.

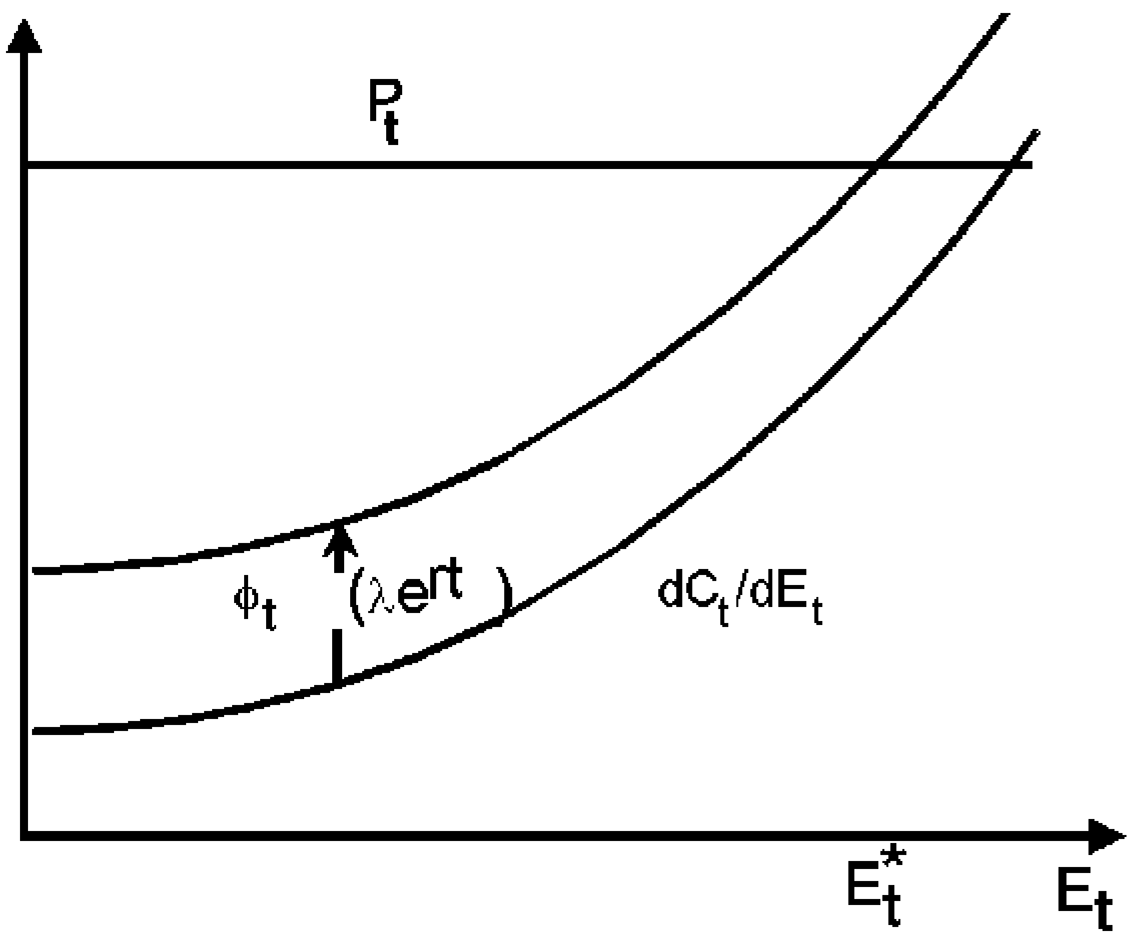

Figure 1
Optimal Extraction Rate for λ Fixed

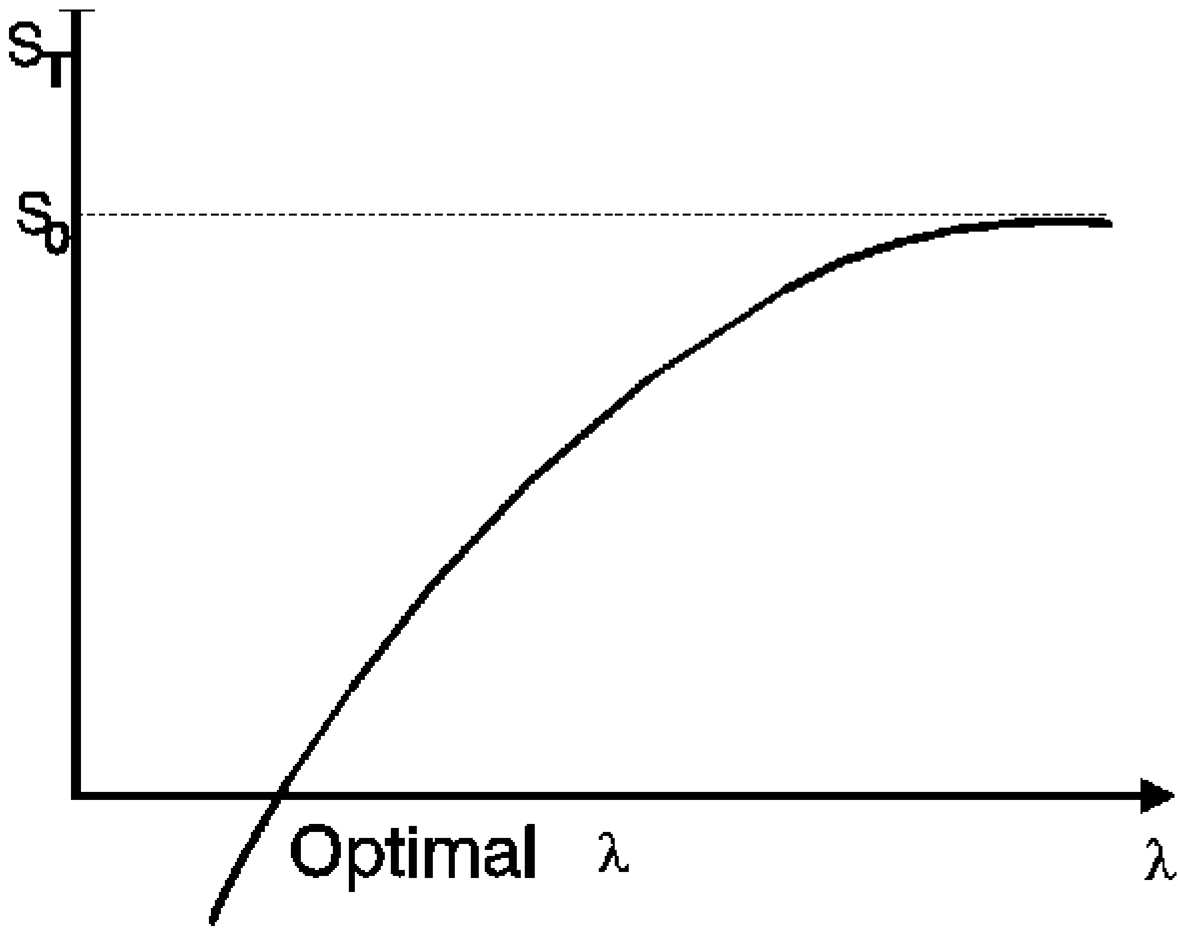

Figure 2
Choice of λ for Optimal Extraction Path

Figure 2 diagrams the choice of λ as determined by equation (16). The upward sloping line is the stock of the resource remaining at the time horizon if equation (15) alone governed the extraction rate and if the non-negativity condition of problem (13) could be violated. The higher the opportunity cost, the lower the extraction rate at every time period, as illustrated in Figure 1. Thus the higher the opportunity cost, the greater the resource stock remaining at the time horizon (S_T). For a high enough λ, there would be no extraction at any time and the final stock would be identical to the initial stock, as illustrated in Figure 2. For a low enough λ, the non-negativity condition would be violated: S_T would be negative. The value of λ at the optimal extraction trajectory is that value which just equates S_T to zero (unless for all positive values of λ, S_T remains positive, in which case λ is reduced to zero.) Note that the optimal value of λ depends on the initial stock of the resource, the prices over time of the extracted commodity, and the marginal cost functions at each time period.

Figures 1 and 2, taken together, provide the key diagrammatic tools for analyzing the optimal extraction trajectories for depletable resources absent stock effects. These tools illustrate the fundamental ideas from depletable resource theory:

- Optimal extraction at any time requires the inclusion of an opportunity cost in addition to the out-of-pocket marginal extraction costs.
- Opportunity cost may be influenced by past, present, and future conditions and reflects the incremental revenue or cost implications of additional extraction.

The concept of opportunity cost here is perhaps the most fundamental organizing concept for depletable resource economics, a concept which will pervade this chapter as it has much of depletable resource literature.

2. Necessary Conditions for Optimality: Feasible Variations

The identical necessary conditions can be derived from a calculus of variations approach by recognizing that at an optimal solution, no feasible variation can increase the value of Π. Infinitesimally small feasible variations around the optimal solution can be postulated and these variations must lead to non-positive changes in Π. That exercise will provide a set of necessary conditions that characterize the optimal choice, conditions which will be identical to equations (15) and (16).

Under problem (13), from any feasible solution, including the optimal one, it is possible to increase extraction at some t. Iif the deposit would ultimately be fully depleted, it would be necessary to compensate by decreasing extraction by an identical amount at some other time, τ, at which extraction is positive. Let these variations be denoted as δE_t and δE_τ, where:

$$\delta E_t = -\delta E_\tau$$

By totally differentiating the objective function in problem (13), we can examine impacts on Π of the extraction changes. If the trajectory is optimal, this combination of feasible variations must not increase Π.

$$\delta \Pi = [P_t - dC_t/dE_t]\, e^{-rt}\, \delta E_t + [P_\tau - dC_\tau/dE_\tau]\, e^{-r\tau}\, \delta E_\tau \leq 0$$

The inequality must hold no matter what the sign of δE_t and δE_τ, as long as $\delta E_t = -\delta E_\tau$. Therefore, it follows that equation (18) must hold for any values of t and τ as long as extraction rate is positive at both times:

$$[P_t - dC_t/dE_t]\, e^{-rt} = [P_\tau - dC_\tau/dE_\tau]\, e^{-r\tau} \qquad (18)$$

Equation (18) shows that a necessary condition for optimality is that the present value of price minus marginal cost must be identical at each moment in time. If this were not true, a feasible variation which increased extraction at one period and decreased it equivalently at another could increase the present value of profits[13]. We will interpret the right-hand side of equation (18) as the **present value opportunity cost** of extracting a unit of the resource at any other time, and we will denote this present value opportunity cost as λ.

If both sides of equation (18) are multiplied by e^{rt}, we obtain an equation equivalent to equation (15).

$$P_t - dC_t/dE_t \; \overset{=}{\underset{\leq}{}} \; \lambda\, e^{rt} \quad \begin{cases} \text{if } E_t > 0 \\ \text{if } E_t = 0 \end{cases} \tag{19}$$

The opportunity cost in equation (19) reflects the complete depletion of the resource: if the resource ultimately would be totally depleted, then a decision to extract more of the resource at some time implies the absolute necessity to reduce extraction by an equivalent amount at some other time, either earlier or later. That necessity gives rise to an opportunity cost for additional extraction, equal to the discounted present value of the price minus the marginal cost of extracting resources at that other time.

If the resource would ultimately not be totally depleted, then opportunity cost would be zero. This result can be seen by postulating an increase or decrease in extraction at one time period, say t, but not at any other time. Without total depletion, either variation would be feasible. If the original trajectory were optimal, neither variation could increase present value of profit and the right hand side of equation (18) would equal zero: there would be no opportunity cost. Equation (20), which is equivalent to equation (16), expresses this result:

$$[P_t - dC_t/dE_t] \quad = \quad 0 \quad \text{for all } t \quad \text{if } S_T > 0 \tag{20}$$

Either Lagrange multipliers or direct application of feasible variations gives identical necessary conditions for optimality. The first method is mathematically more efficient, but the second more fully shows the role of λ the opportunity cost. In subsequent sections of this chapter the Kuhn-Tucker conditions are used. However, in general, analysis of feasible variations will remain possible throughout.

The basic necessary conditions for optimality and the properties of optimal trajectories can be analyzed more once we more fully postulate properties of the cost function. We do so in what follows.

3. Solutions for the Hotelling case of Fixed Marginal Costs

We now turn to the simplest of the cases, that underlying the original works by Gray and by Hotelling. The assumption here is that the marginal extraction cost depends neither on extraction rate nor on remaining stock. While that assumption will greatly simplify the optimal choices, it should be recognized that this Hotelling assumption is very restrictive.

Assumption: <u>Hotelling Cost</u>:
Extraction costs are independent of remaining stock; marginal extraction costs are independent of the extraction rate.

We will denote the marginal cost of extraction under the Hotelling cost assumption as c_t, a parameter which may vary with time. Then equation (21) reduces to:

$$P_t \begin{matrix} = \\ \leq \end{matrix} c_t + \lambda e^{rt} \quad \begin{cases} \text{if } E_t > 0 \\ \text{if } E_t = 0 \end{cases} \tag{21}$$

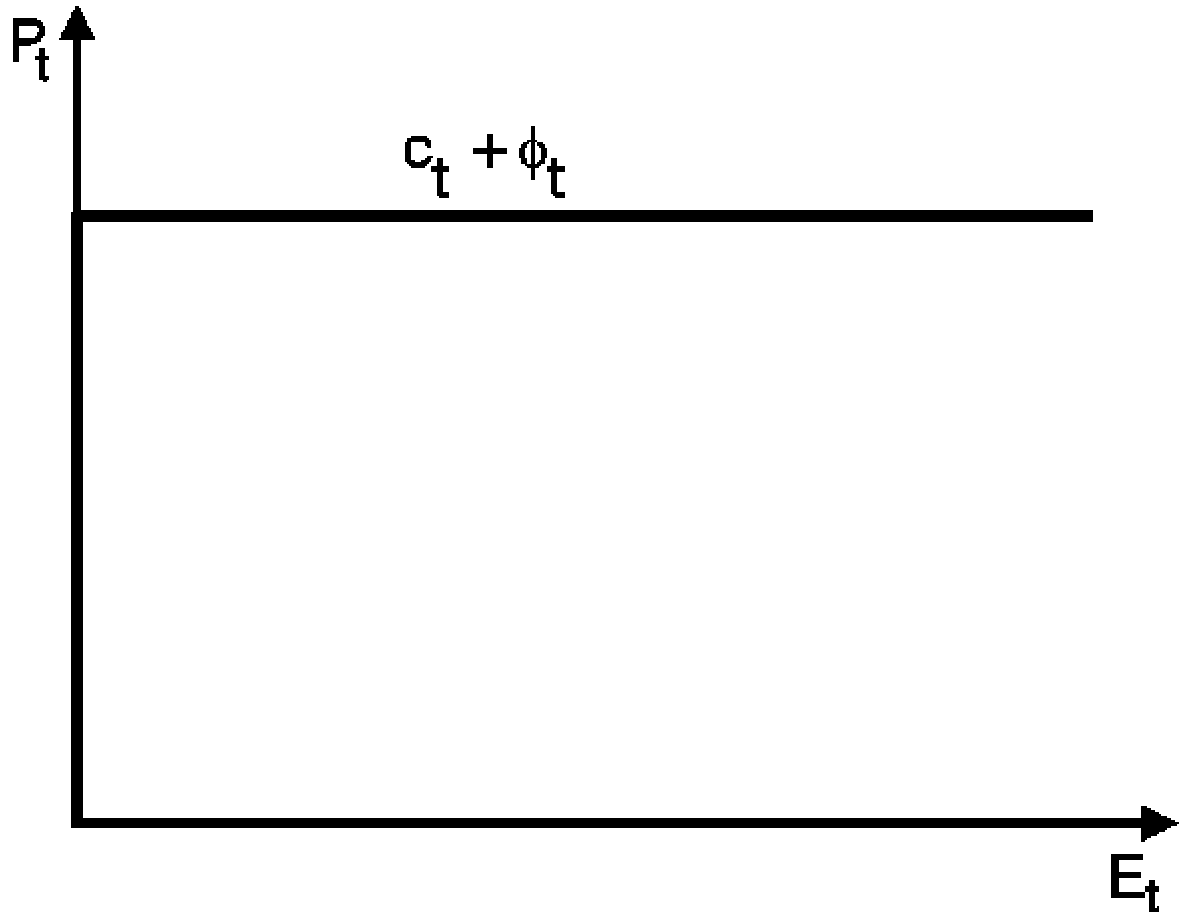

Figure 3
Optimal Extraction: Hotelling Cost Assumption

Equation (21) defines an optimal extraction rate that is zero if price is smaller than or equal to the marginal cost plus opportunity cost and is indeterminate for price equal to the fixed marginal cost plus opportunity cost. Price never exceeds marginal cost plus opportunity cost. This relationship is diagrammed in Figure 3.

Present value opportunity cost, λ, is determined using equation (16), which implies that price can never exceed $c_t + \lambda e^{rt}$. If price ever were to exceed $c_t + \lambda e^{rt}$, extraction would be infinite at that time and S_T would become minus infinity, a violation of equation (16). Thus λ must equal the maximum value (over time) of $[P_t - c_t] e^{-rt}$, as long as that maximum value is non-negative. If the maximum value of $[P_t - c_t] e^{-rt}$ is negative, then $\lambda = 0$ and no extraction ever occurs.

Under the Hotelling cost assumption, extraction can occur only when price is rising on a path denoted by the Hotelling rule: $P_t = c_t + \lambda e^{rt}$, at which time extraction rate is indeterminate. If price were to rise more rapidly during a time interval, it would be less profitable to extract during the interval than to wait until its end. If price were to rise more slowly, it would be more profitable to extract everything at the beginning of that interval than during it.

The Hotelling cost assumption leads naturally to a capital theory interpretation of the optimal extraction rule. By equation (21), present value (to time 0) of per unit revenue net of costs is just λ, independent of the extraction timing. Therefore the market value at time 0 of the entire resource deposit must be just equal to λS_0. If the optimal extraction rate is zero, then the value of the resource must be growing at the interest rate: the owner is earning returns from the investment through its capital gains, rather than through cash flows. In addition, the cash flows from extraction would be smaller than the value of the resource left in place.

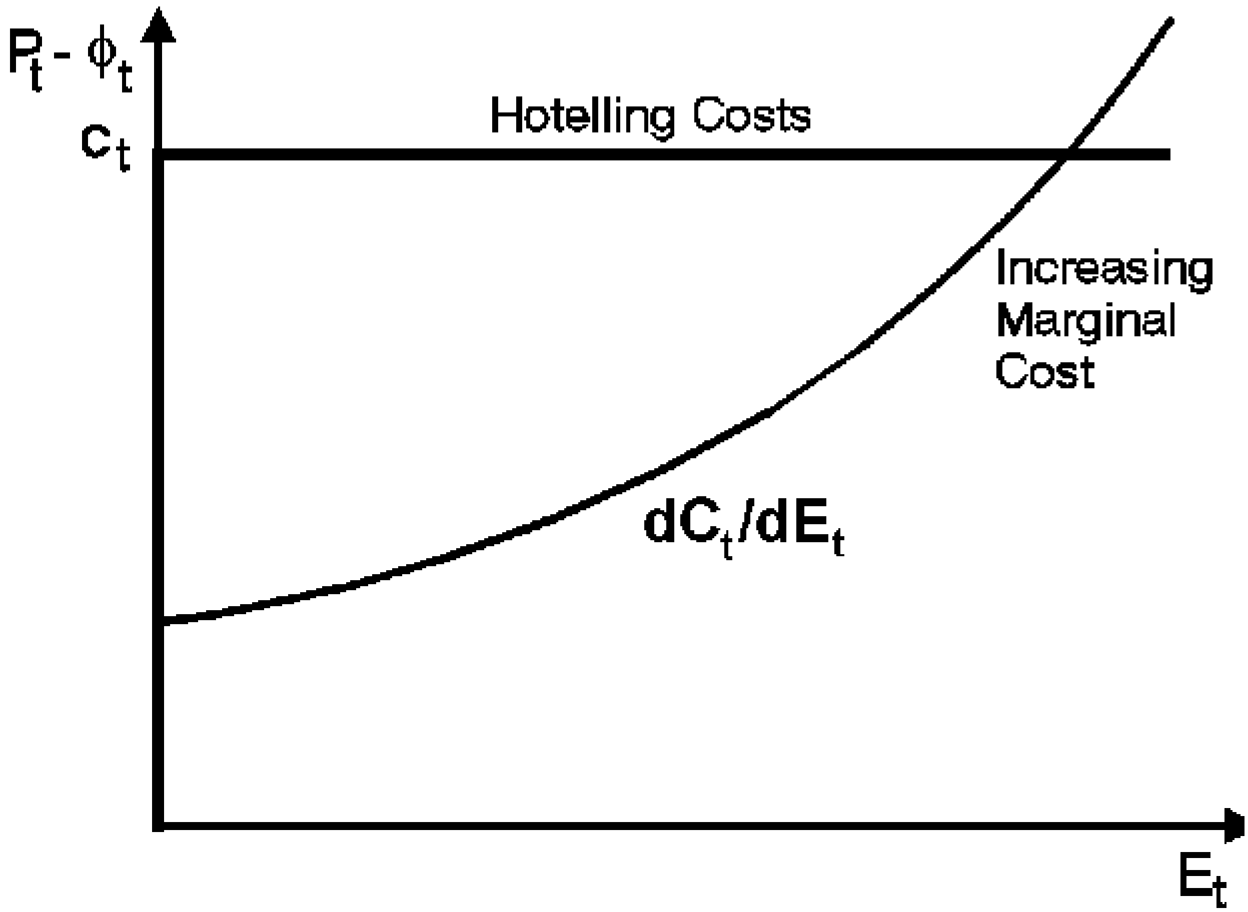

Figure 4
Supply as a Function of P_t - ϕ_t: Two Cases

When price is following the Hotelling rule ($P_t = C_t + \lambda e^{rt}$), the value per unit of the resource is just equal to per unit revenue net of extraction cost: $P_t - c_t$. The resource owner is indifferent between extracting the resource and still holding it. Capital gains would equal the normal rate of return on investment so that the value of the entire resource stock continues to grow at the interest rate.

Once the commodity price permanently stops following the Hotelling rule, and future prices promise to be below the Hotelling rule prices, the value of the resource at future times (t) will be less than $\lambda e^{rt} S_0$. Therefore, if he or she kept the unextracted resource, the owner would no longer receive a normal rate of return on invested capital and it would be optimal to extract any remaining quantities of the resource.

Although the Hotelling case is analytically very tractible, its rigid cost assumptions make this case less useful than more general cases for explaining of predicting depletable resource markets. We now turn to characteristics of supply functions for more general cost functions.

4. Depletable Resource Supply Functions

In much microeconomic theory, the supply function for a commodity can be written as a static function of price, with higher current prices implying higher optimal output. Although, in principle, future prices might influence current decisions, typically future prices are not seen as important arguments of the supply function.

For depletable resources, however, the quantity supplied at any time must be a function of prices at that time and prices and costs at all future times. Thus static supply functions, so typical in most economic analysis, are inconsistent with optimal extraction of depletable resources.

Although extraction at any time depends on future, as well as present, prices and costs, all information about the role of future prices and costs is embodied in a single unobservable variable: the opportunity cost, ϕ_t. Opportunity cost itself is dependent on the remaining stock and all future prices, or equivalently, on initial stock and on all past, present, and future prices. Therefore we can specify a supply function analogous to conventional supply functions, with the argument of the function $P_t - \phi_t$, rather than P_t. In principle this supply function is similar to a supply functions for conventional commodities except that ϕ_t is unobservable and is itself a function of other prices.

Such a supply function is plotted in Figure 4 for Hotelling costs and more normal, increasing marginal cost cases. For either cost assumption optimal extraction is an increasing function of $P_t - \phi_t$, but the character of the supply curves differs greatly between these situations. For the Hotelling case, small variations in $P_t - \phi_t$ can change optimal extraction from zero to an arbitrarily large rate. With increasing marginal cost of extraction, the optimal extraction rate increases continuously with increases in $P_t - \phi_t$, and can converge to some maximum level with capacity constraints or other factors limiting supply.

5. An Example

The analysis can be illustrated with a specific example of quadratic costs, an example which is easily solvable:

$$C_t(E_t) = M E_t^2 /2$$

This cost function gives optimal extraction rate as a function of price and opportunity cost:

$$E_t = (P_t - \lambda e^{rt}) /M \qquad for \ \lambda e^{rt} < P_t$$

For our analysis we will adopt the time invariance assumption: price remains constant over time and the terminal time is very far in the future. We drop the t subscript on price.

Let τ be the last time that $\lambda e^{rt} < P$. Then we can calculate S_τ as a function of λ:

$$S_\tau = S_0 - \{P \tau - \lambda \sum_{t=1}^{\tau} e^{rt}\} /M$$

If $\lambda > 0$, then $S_\tau = 0$. We obtain one equation and a pair of inequalities that can be solved simultaneously to determine τ, and λ:

$$M S_0 = P \tau - \lambda \sum_{t=1}^{\tau} e^{rt}$$

$$\lambda e^{r\tau} \leq P \leq \lambda e^{r(\tau+1)}$$

For very short L, we can approximate these equations to find a single equation for τ:

$$\frac{M S_0}{P} = \{\tau - [\frac{1 - e^{-r\tau}}{1 - e^{-r}}]\}$$

This set of conditions makes λ an increasing function and τ a decreasing function of price. Increases in M or in S_0 lead to decreases in λ and increases in τ.

We turn now to more general properties of solutions for more general cases.

6. Optimal Trajectories: Characteristics and Comparative Dynamics

The models can be used to examine characteristics of the optimal extraction path for a depletable resource governed by problem (13). We will present several examples to illustrate methods of analysis and conclusions that can be derived from use of these models. All of the examples maintain the no-stock-effects and the linear stock dynamics assumptions.

a. Extraction path under time invariant conditions.

Over a long history, many depletable resource prices have remained roughly constant or have fluctuated randomly. Thus it may be rational for the purposes of planning for the owner of a depletable resource stock to assume that future conditions will remain the same as current conditions.

Since both cost and prices remain constant, prices never follow a Hotelling path. Therefore, under the Hotelling cost assumption, the entire deposit will be extracted as soon as possible: equations (21) and (16) imply that extraction can only occur at the first moment.

With increasing marginal costs, the resource deposit will be extracted over time, with the greatest extraction in the first period and with extraction rates that decline from one period to the next. This conclusion stems directly from equation (15) and can be illustrated using Figure 4. Optimal extraction is an increasing function of $P_t - \lambda e^{rt}$, for a given cost function. Since price remains constant, $P_t - \lambda e^{rt}$ must decline from one time period to the next. With an unchanging cost function, extraction rate must decline over time.

b. The role of technological progress

Technological progress can lead over time to reductions in the cost function. If the firm anticipates reductions, that anticipation can influence extraction rates, including at times before the cost reduction occurs. It is possible for extraction to be small and increase over time, before it ultimately declined. This conclusion follows directly from equation (15). Although $P_t - \lambda e^{rt}$ must decline over time, because the cost function itself decreases, the extraction rate could either increase or decrease, depending upon the relative rates of change of the cost function and of its argument.

c. The role of price expectations

Expectations about future conditions may not remain stationary. Changes in technologies which substitute for depletable resources or which are complementary to their use can change market demand

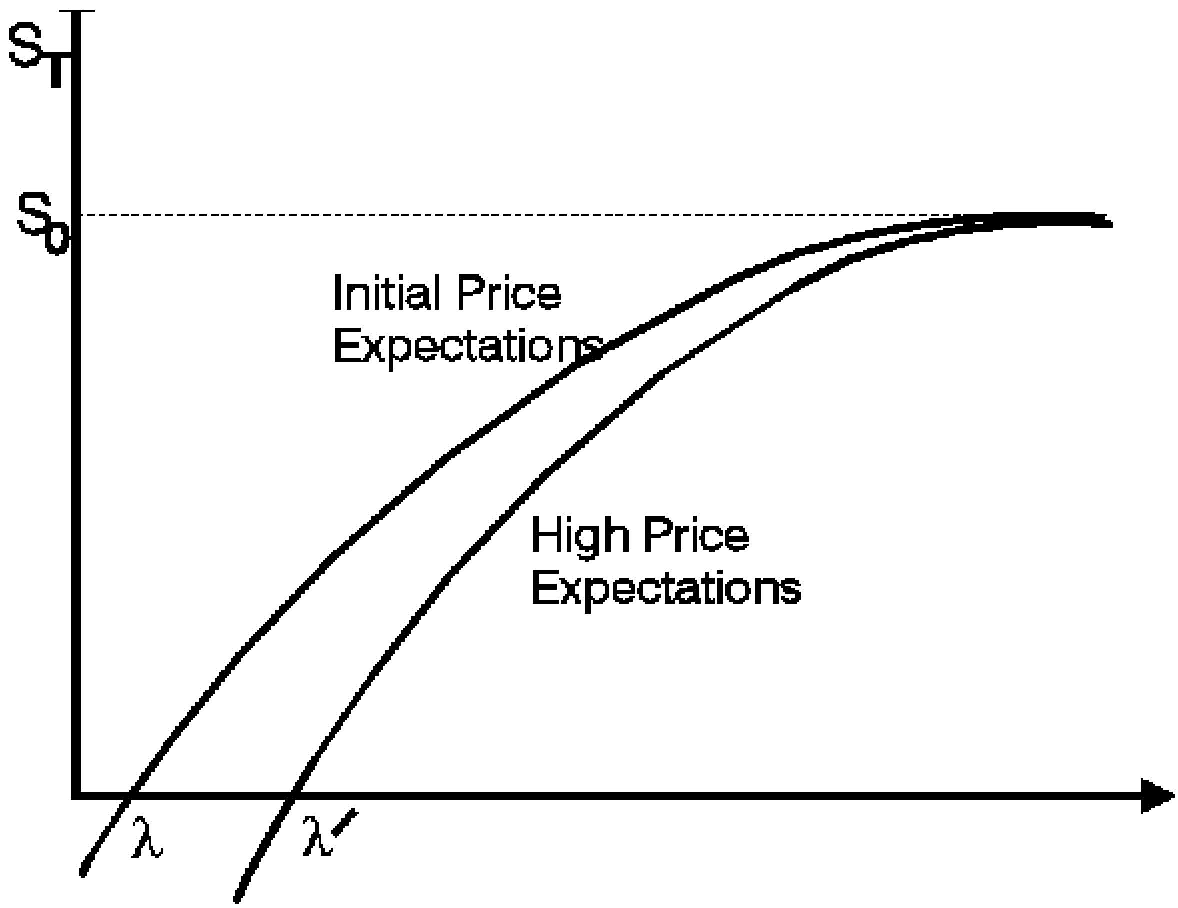

Figure 5
Impact of Expected Future Price Increases on λ

functions and thus can change prices facing an individual resource owner. Changes in the tax regime or in the international political situation can likewise influence prices. Here we examine the case in which current prices remain unchanged currently but in which beliefs about future prices do change. Such changes in expectations will lead to changes in current extraction patterns. Here we assume that those expectations turn out to be accurate, although the predicted alterations in current actions do not depend upon whether the changed beliefs are ultimately found to be correct or mistaken.

For this analysis, we assume that after some future time, τ, in the new situation all prices will be higher than they were in the old situation:

$$P_t' = P_t \quad \text{for } t \leq \tau; \qquad P_t' > P_t \quad \text{for } t > \tau$$

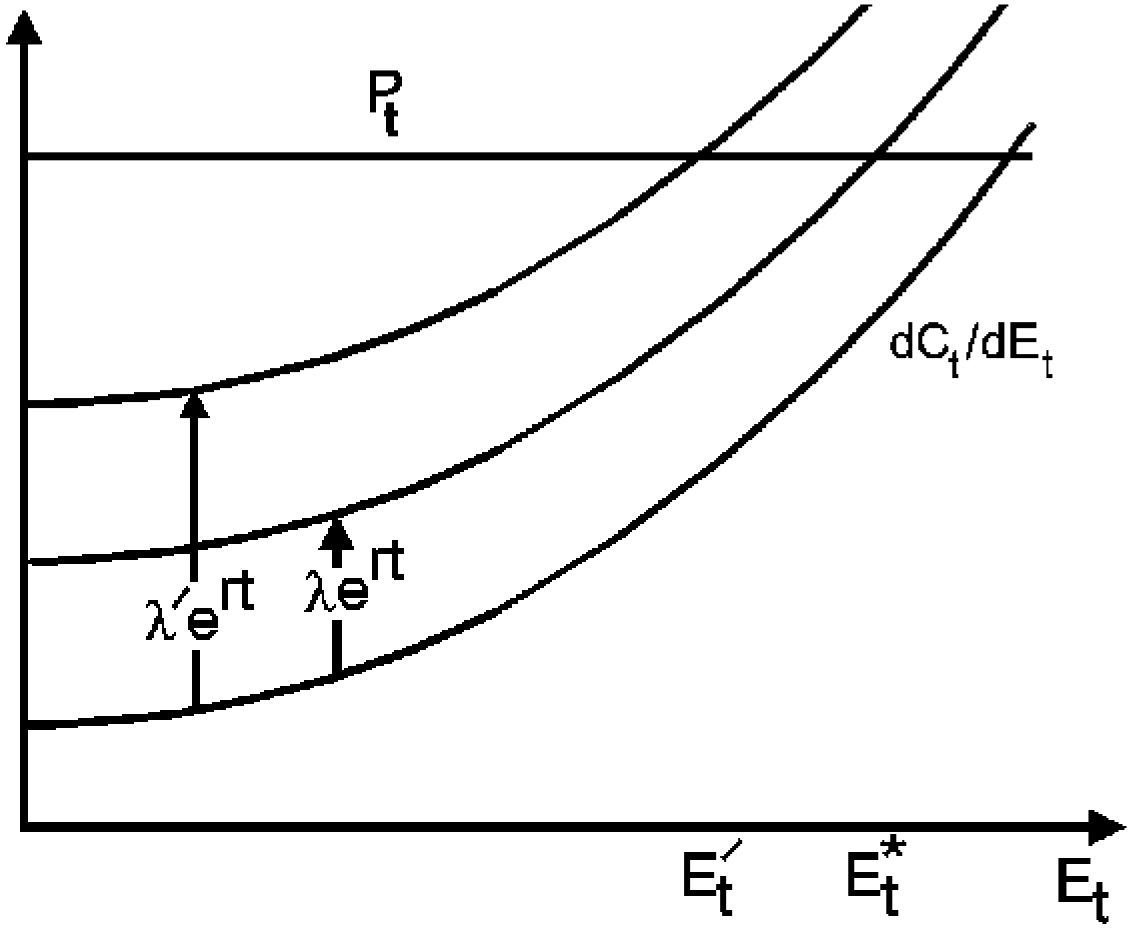

Figure 6
Impact of Increased λ on Early Year Extraction Rate

We can analyze this case in three steps. First we examine how changed price expectations would change S_T, for λ unchanged. Based on this impact, we examine changes in λ. Finally, based on changes in λ, we examine impacts on current extraction decisions.

Equation (15) shows that if λ were to remain unchanged, extraction would increase for all time greater than τ but would remain unchanged for all earlier time. If extraction increases, stock remaining at time T would decrease. But either final stock or opportunity cost must be equal to zero (Equation (16)); an increase in λ is needed to bring S_T down to zero. This set of changes is illustrated by the downward shift in the curve in Figure 5.

For times before τ, the change in price expectations increases opportunity cost but impacts no other elements of equation (16). For these early times optimal extraction must decline as a result of the changed expectations. This impact is illustrated in Figure 6.

For $t > \tau$, both opportunity cost and price increase. Therefore whether optimal extraction increases or decreases depends upon the sign of net changes in $P_t - \lambda e^{rt}$. However, λ must increase just enough that the reductions in extraction before τ must just equal (in total) the net increases in extraction after τ.

d. Impacts of excise taxes

Natural resource taxation is used a source of revenue by governments around the world. Such taxes have many forms, including *ad valorem* taxes and fixed magnitude excise taxes. Here we assume that a local taxing authority imposes an excise tax on the producer of a depletable resource.

We first assume that a tax of X per unit of extraction is rationally believed to be time invariant once it is imposed. This tax is assumed to have no impact on the pre-tax sales price so that the net price obtained by the resource owner for the extracted commodity is reduced by exactly the amount of the tax.

The tax would necessarily reduce the opportunity cost of extracting the resource. Thus by equation (15), extraction at time t will increase, decrease, or remain constant, depending on whether $X + \lambda' e^{rt}$ is smaller than, greater than, or equal to λe^{rt}, where λ' and λ are the present value opportunity costs after and before the tax, respectively.

Since X is time invariant, in early years $X + \lambda' e^{rt}$ will be larger than λe^{rt} and the extraction rate will decrease. (Remember that $\lambda' < \lambda$.) However, in later years the decline in the opportunity cost will be the dominant factor ($X + \lambda' e^{rt} < \lambda e^{rt}$) and extraction will increase[14].

The net result of a time invariant excise tax is to move extraction of depletable resources from the present, to the future and to promote greater conservation of natural resources.

We next assume that the excise tax is not time invariant. In particular, we assume that it is rationally expected to grow over time at just the interest rate, r, so that the present value of the tax is equal to X at each time. If the tax is not too large, it will have no effect on the pattern of extraction over time. The same analysis will show that if the tax grows at a rate faster than r, the tax will cause resource owners to extract more rapidly than they would absent the tax.

As in the previous excise tax example, this tax will lead to a reduction in the present value opportunity cost. At time t extraction will increase, decrease, or remain constant, depending on whether $X e^{rt} + \lambda' e^{rt}$ is smaller than, greater than, or equal to λe^{rt}. This relationship implies that extraction will increase, decrease, or remain constant if $[X + \lambda' - \lambda] e^{rt}$ is smaller than, greater than, or equal to zero. Although the magnitude of $[X + \lambda' - \lambda] e^{rt}$ changes over time, its sign does not, so that extraction must increase for all t, decrease for all t, or stay unchanged for all t. The first two possibilities would lead to a violation of equation (16); the third, which would occur if $[X + \lambda' - \lambda] = 0$, would not lead to a violation. Therefore, if the current value opportunity cost would decrease by an amount exactly equal to the excise tax, $(- [\lambda' - \lambda] = X)$, all equations would be satisfied. Therefore extraction would be the same with and without the excise tax.

For large enough X, the tax will reduce extraction at all times. If the tax rate were greater than the initial opportunity cost $(X > \lambda)$, then $- [\lambda' - \lambda] < X$. The decline in the opportunity cost would be smaller than the tax because λ' could not become negative. If the tax rate were larger than the pre-tax opportunity cost, then extraction would decline for each time and the resource would not be fully extracted within the time horizon.

A similar analysis for more rapidly growing taxes shows that the opportunity cost would decline as a result of the tax and would decline by more than the tax in earlier years. Because the tax would grow faster than the current value opportunity cost (more precisely, than the difference between the pre-tax and post-tax current value opportunity cost), in later years the relative magnitudes would be reversed. Therefore in early years, extraction would increase; in later years extraction would decline as a result of the rapidly growing excise tax.

e. The role of environmental externalities

In their chapter within this Handbook, Krautkraemer and Kolstadt discuss environmental externalities associated with depletable resource extraction and use and examine biases of market determined resource extraction patterns from the socially optimal rates. Among the externalities they examine are those whose costs depend upon the rate of extraction of depletable resources. Here we analyze an optimizing resource owner to examine the impact on extraction rates of motivating owners to internalize externalities.

Assume that the marginal environmental damage per unit of resource extraction is D_t and that the policy environment is changed so that while originally the resource deposit owner could ignore the environmental externality, now the owner now must bear the entire environmental cost. In the changed situation the net price received by the owner would be decreased by D_t.
Since internalization of the externality influences the net price in exactly the same manner as would a time dependent tax equal to D_t, results of the previous section can be used directly. If the present value of D_t declines over time, a internalization leads to less extraction in early years and more in later years. As a result of this intertemporal shift, the discounted present value of the environmental damages over the deposit life would be reduced. The individual resource owner who internalizes environmental consequences shifts those consequences, along with resource extraction, to a later time.

The converse is true if the present value of D_t grows. Then internalization of the externalities leads the resource owner to extract more rapidly than would otherwise be the case. This shift moves the externalities to an earlier time but also decreases their discounted present value.

f. The role of national security externalities

The final example is motivated by energy policy issues, although it could be applicable to imports of other strategic materials. Often raised in energy policy debates is the idea that energy security can be improved by extracting more resources domestically and importing fewer extracted commodities. For more background, see the chapter by Toman in this Handbook. Here we assume that there is a national benefit B_t per unit of additional resource extracted at time t. If all prices are otherwise correct, would the individually optimizing owner extract too rapidly or too slowly from a socially optimal perspective?

The result, is identical to those discussed above. If the present value of B_t declines over time, the socially optimal extraction rate will exceed the privately optimal rate. The optimizing owner will extract less in early years and more in later years than socially optimal. Conversely, if the present value of B_t is expected to grow, then social optimality would require a reduction in the extraction rate.

g. The role of the interest rate

A long standing concern in the depletable resources literature relates to biases in the interest rate. It seems generally agreed that if the interest rate used by the owners of depletable resources is higher than the social discount rate, resource owners will extract more rapidly than is socially optimal, and will therefore leave smaller stocks of depletable resources for the future (See, for example, Solow, 1974). Although the debate about the appropriate social rate of discount continues (See Lind for a discussion), generally interest rates used by corporations can be expected to exceed the socially optimal rates, if for no other reason, because of taxation of corporate profits. Thus the standard analysis concludes that high interest rates do provide an impetus for extracting depletable resources more rapidly than otherwise. Here we examine implications of high interest rates for extraction from a given deposit. We argue that the standard conclusion is correct if costs are unchanged by interest rates. However, if the cost function is increased by the higher interest rates, then high interest rates can well be expected to reduce extraction rates.

This analysis proceeds similarly to previous discussions for costs unchanged by the interest rate. Assume that in the new situation the interest rate (r') is higher than the interest rate in the old one (r) but that all prices and costs remain the same between the two cases.

Higher interest rates increase the current value opportunity cost for all future time if λ were to remain constant. Therefore, with λ unchanged, by equation (15), higher interest rates would lead to lower extraction rates at every time and would thereby decrease S_T. But for equation (16) to remain valid, λ would necessarily decrease: $\lambda' < \lambda$.

By equation (15), the net effects on the extraction path would depend upon the net changes in the current value opportunity cost, since no other factors would be altered. For early times, $\lambda' e^{r't} < \lambda e^{rt}$ and $E'_t > E_t$: the extraction rate would increase as a result of the higher interest rate. At some point the more rapidly growing exponential factor would lead to a reversal of the inequalities: $\lambda' e^{r't} > \lambda e^{rt}$. From that point on, extraction would be lower, reflecting the reduced size of the remaining resource stock.

For fixed cost functions, higher interest rates lead to more extraction in earlier years. This can be understood in terms of the capital theory discussion. The resource left in the ground is analogous to a capital investment, in that preserving the deposit involves forgoing current income in the expectation of earning greater future income. Higher interest rates imply that less investment is economically attractive and more of the resource is extracted in early years.

But there is a second impact of higher interest rates. For many depletable resources, capital costs are a large portion of the total and marginal costs. And the rental cost of capital increases with the interest rate. Thus high interest rates imply higher extraction costs. We have just shown that increases in unit cost (e.g. taxes or internalized externalities) reduce the depletion rate, as long as the magnitude of the cost increase declines in present value.

Increases in the interest rate therefore lead to two countervailing forces. By increasing the growth rate of the opportunity cost, such interest rate increases move extraction forward in time; by increasing the extraction cost, they move extraction backward in time. In principle either effect could dominate and one could have a non-monotonic relationship between interest rate and the extraction rate, as has been demonstrated by Jacobsen.

Which force dominates will depend upon the relative changes in the marginal cost function and in the opportunity cost. If the marginal cost function (evaluated at the initial extraction rate) is increased by more than the initial opportunity cost is decreased, increases in the interest rate will lead to reductions in the initial extraction rate, and vice versa. Thus if the initial opportunity cost is smaller than the increase in marginal cost, changes in marginal cost must always dominate; higher interest rates will imply lower initial extraction rates. This may well be the situation for U.S. oil and gas extraction. Only when the opportunity cost is a large fraction of price, as may well be the case for much Middle Eastern oil production, or when interest rates have very little impact on marginal costs, will higher interest rates imply more rapid extraction.

h. In summary

In summary, environmental, national security, or other externalities of depletable resource extraction can be expected to alter the extraction patterns over time. Similarly changes in expected prices or variations in the interest rate can impact these patterns. The directional impacts of interest rate variations will depend upon the magnitude of change in the cost function relative to change in the initial opportunity cost. The direction of impacts of the other variations will depend both on the sign of the externality or other variation and on the growth or decline of the present value of that change. For depletable resources, information about changing current conditions is not sufficient in itself to predict alterations in supply patterns. Rather changes in current conditions <u>relative</u> <u>to</u> changes in future circumstances must be examined in order to analyze changes in optimal current actions.

The need for a future information and the central role of the opportunity cost will remain relevant for more general models of depletable resource extraction. We turn now in our sequence of depletable resource models to the more general models in which the extraction costs may depend on the remaining stock of the resource.

C. Optimizing Models With Stock Effects

Extraction costs could depend upon remaining stock in several possible ways. When most of the original stock becomes depleted, total and marginal extraction costs will increase based upon physical difficulties of extracting remaining quantities. In addition, lower stock may increase costs independent of the extraction rate, as when reductions in remaining stock of water in an aquifer or intense mining leads to subsidence of overlying land. In addition, in earlier phases of resource extraction "learning by doing" could decrease costs as experience with a deposit is gained. If that experience is comes about through extracting the resource, then costs can be modeled as declining endogenously with reductions in remaining stock.

The general formulation has been presented as problem (8), which we will use in what follows. We will use the Kuhn-Tucker conditions to derive necessary conditions for optimal extraction paths and will discuss properties of the optimal extraction trajectories. Under our maintained assumption that the cost function is convex, the objective function for this problem is concave and the feasible set remains convex. Thus the first order necessary conditions for optimality will be sufficient conditions for optimality as well.

<u>1. Necessary Conditions for Optimality: Kuhn-Tucker Conditions</u>

$$Max\ \mathcal{L} = \sum_{t=1}^{T} [P_t E_t - C_t(E_t, S_{t-1})] e^{-rt} - \sum_{t=1}^{T} [S_t - S_{t-1} + E_t]\, \lambda_t + \mu\, S_T$$

Denoting the Lagrangian by $\mathcal{L}$ problem (8) can be converted to an unconstrained optimization: The mathematics can be simplified by substituting ϕ_t into the Lagrangian:

$$\phi_t = \lambda_t\, e^{rt} \tag{22}$$

The Lagrangian then becomes:

$$Max\ \mathcal{L} = \sum_{t=1}^{T} [P_t E_t - C_t(E_t, S_{t-1})] e^{-rt} - \sum_{t=1}^{T} [S_t - S_{t-1} + E_t]\ \phi_t \tag{23}$$

$$\partial\mathcal{L}/\partial E_t = [P_t - \partial C_t/\partial E_t] e^{-rt} - \phi_t\ e^{-rt} \quad \begin{cases} = 0 & \text{if } E_t > \\ \leq 0 & \text{if } E_t = \end{cases}$$

By the Kuhn-Tucker theorem, the first-order necessary conditions require that at the optimal point, the Lagrangian must be a maximum point with respect to each S_t and each E_t. Differentiating $\mathcal{L}$ with respect to each variable gives first order necessary conditions:

$$\partial\mathcal{L}/\partial S_{t-1} = -\partial C_t/\partial S_{t-1}\ e^{-rt} - \phi_{t-1}\ e^{-r(t-1)} + \phi_t\ e^{-rt} = 0 \qquad for\ t \leq T$$

$$\partial\mathcal{L}/\partial S_T = -\phi_T\ e^{-rT} + \mu = 0 \qquad for\ t = T$$

$$\mu\ S_T = 0$$

These equations provide the fundamental first order necessary conditions for optimality:

$$P_t \ \begin{matrix} = \\ \leq \end{matrix}\ \partial C_t/\partial E_t + \phi_t \quad \begin{cases} \text{if } E_t > 0 \\ \text{if } E_t = 0 \end{cases} \tag{24}$$

$$\phi_t = \phi_{t-1}\ e^{r} + \partial C_t/\partial S_{t-1} \qquad for\ t < T \tag{25}$$

$$\phi_T\ S_T = 0 \tag{26}$$

Equations (5), (24), (25), and (26) collectively define the time paths of the current value opportunity cost, extraction, and remaining stock. Equation (24) determines extraction, given ϕ_t. Equations (5) and (25) are difference equations governing the evolution of S_t and of ϕ_t, respectively. Equation (26) provides one boundary condition (for stock or opportunity cost) at the final time period, while the known value of S_0 provides the boundary condition for stock at the initial time. In principle these equations can be solved to determine the trajectories of each variable over all time.

Equation (24) is identical to equation (15). The optimal extraction rate is the one at which the marginal extraction cost plus the current value opportunity cost is equated to the price of the extracted commodity. The opportunity cost at any time is dependent on the prices and costs at all time, as without stock effects. ϕ_t is evaluated at the optimal values of the extraction rates, as in the absence of stock effects. The solution to equation (24) has been illustrated graphically in Figure 1.

The opportunity cost remains a central organizing concept even when stock effects are introduced. However, in contrast to models without stock effects, the present value shadow price is time dependent for models in which extraction cost depends on remaining stock. The role of the opportunity cost does not change here, although its dynamics does vary.

For ease of notation, it is valuable to define a one time period interest rate ρ, such that:

$$1 + \rho = e^{r} \tag{27}$$

Then equation (25) can be written as:

$$\phi_t = \phi_{t-1}\,(1+\rho) + \partial C_t/\partial S_{t-1} \qquad \text{for } t \le T \tag{25'}$$

Equation (25), which governs the dynamics of the current value opportunity cost, generalizes the relationships underlying the models without stock effects. If $\partial C_{t+1}/\partial S_t = 0$, equation (25) would become:

$$\phi_{t+1} = \phi_t\, e^{r} \qquad \text{for } t < T$$

The solution to this dynamic equation is:

$$\phi_t = \lambda\, e^{rt},$$

where λ is a constant. If $\partial C_t/\partial S_{t-1} = 0$, current value opportunity cost would grow at the interest rate and these equations would be identical to those for models without stock effects.

When $\partial C_t/\partial S_{t-1} < 0$, the normal case, the current value opportunity cost may grow or shrink. But if it grows, its growth rate will be lower than r so that the present value opportunity cost must shrink. If

$\partial C_t/\partial S_{t-1} > 0$ (for example, with learning by doing), then the current value opportunity cost growth rate is faster than r and the present value will increase over time.

Equation (26), which provides a boundary condition for the difference equation, is identical to equation (16). Both show that the current value opportunity cost is equal to zero at time T unless the stock is totally depleted by the time horizon. Positive final stock may well be the norm, not the exception for depletable resources with stock effects. In those situations in which final opportunity cost is zero, the current value of opportunity cost must be shrinking, not growing, as the time horizon is reached.

Although these equations can be solved in principle, in practice their solution typically involves numerical simulations based upon explicit cost functions and price trajectories. Such solutions generally involve iterative procedures to guess the initial value of the opportunity cost, to calculate the adjustments of stock and opportunity cost to the time horizon, and to compare the final opportunity cost and stock to the conditions required by equation (26). If the final opportunity cost is too high, the initial opportunity cost can be decreased for the next iteration and conversely if the final opportunity cost is too low. In this way iterative procedures can find explicit solutions, given explicit cost functions and price trajectories.

For this chapter, however, we are more concerned with properties of the equations and solutions. Interpretation of the opportunity cost for depletable resources with stock effects will be the focus of the following section.

2. Interpretations of Opportunity Costs

When stock effects exist, the concept of opportunity cost is no longer dependent upon absolute limits to resource availability. An opportunity cost will exist even if the resource is not ultimately depleted. This conclusion is in sharp contrast to results from models without stock effects. In particular, the opportunity cost can be interpreted as the present value (discounted to time t) of future cost increases due to additional extraction at time t.

To examine this interpretation of opportunity cost, consider a resource deposit that will not ultimately be totally depleted. For such a resource $\phi_T = 0$, by equation (26), where T is the time horizon. For such a resource there is no absolute limit on availability; the owner could extract more of the resource without extracting less at another time.

To calculate the opportunity cost, we rearrange equation (25) and solve it recursively, beginning from t = T-1 and working backward. Rearranging gives:

$$\phi_t = \phi_{t+1}\, e^{-r} - \partial C_{t+1}/\partial S_t\, e^{-r} \tag{28}$$

The boundary condition $\phi_T = 0$ allows equation (28) to be solved, at least conceptually. The general solution to this recursive equation gives the current value opportunity cost and the present value opportunity cost as follows:

$$\phi_t \quad = \quad -\sum_{\tau=t+1}^{T} [\partial C_\tau/\partial S_{\tau-1}]\, e^{-r(\tau-t)} \tag{29}$$

$$\lambda_t = \phi_t\, e^{-rt} = -\sum_{\tau=t+1}^{T} [\partial C_\tau/\partial S_{\tau-1}]\, e^{-r\tau} \tag{30}$$

Equation (29) shows that the opportunity cost at time t is simply the present value, discounted to time t, of future incremental costs accruing as a result of extracting one more unit of the resource at time t. An additional unit extraction at time t requires no compensating extraction reduction. Yet it does lead to one unit stock reduction for all subsequent times. At time τ, that stock reduction causes a cost change of - $\partial C_t/\partial S_{t-1}$. The cost change is discounted by a factor $e^{-r(\tau-t)}$ to time t and by a factor $e^{-r\tau}$ to time zero. Summing terms over all subsequent time gives the opportunity cost as being equal to the present value of future costs stemming from additional current extraction, as shown in equations (29) and (30).

Equations (25) and (26) do provide for an opportunity cost component which derives from absolute limits to resource extraction if the deposit ultimately is totally depleted. This component rises at the interest rate, just as in models without stock effects. If the resource ultimately is totally depleted, the final value of the opportunity cost can be positive, say equal to ϕ_T. By equation (28), the opportunity cost at time t will include a term equal to $\phi_T\, e^{-r(T-t)}$. If we choose a constant λ equal to $\phi_T\, e^{-rT}$, this opportunity cost component would equal $\lambda\, e^{rt}$, the same form as obtained in models without stock effects[15].

Because (25) is a linear equation, opportunity cost can be determined as the sum of the two separately derived and conceptually separate components. Therefore the total opportunity cost will be the present value of incremental future costs [equation (29)] plus an exponentially rising term deriving from resource limits.

$$\phi_t = - \sum_{\tau = t+1}^{T} [\partial C_\tau / \partial S_{\tau-1}] \, e^{-r(\tau - t)} + \lambda \, e^{rt} \tag{31}$$

Equations (29) or (31) may well give opportunity costs which are small initially, which rise over time as the deposit is depleted, but then which ultimately decrease if the time horizon is ever approached. This pattern occurs if extraction costs are relatively insensitive to remaining stock when S_t is large but are very sensitive to S_t as the deposit approaches an economic shutdown. A limiting case would occur if $\partial C_t/\partial S_t = 0$ for large stock. Then during early years when stock is large, all non-zero terms on the right-hand side of equation (29) would remain from one year to the next, and the discount factor would be multiplied by e^r each year. During this time, opportunity cost would grow annually at the interest rate. But once $\partial C_t/\partial S_t = 0$ varied from zero, opportunity cost would stop growing at the interest rate because the number of non-zero terms in the summation would decrease from year to year while those that remained would grow at exactly the interest rate.

In summary, opportunity cost for depletable resource deposits stems from two conceptually separate phenomena. Pure depletion leads to an opportunity cost which rises at the rate of interest. But if resource deposits are not fully depleted, but rather shut down due to economic conditions, then this component is always zero. The second component stems directly from stock effects. If current extraction leads to future costs through reduced future stock levels, then the discounted present value of these additional costs is a component, and perhaps the only component of the opportunity cost.

3. Steady State Conditions

Without stock effects, steady state conditions were straight forward: extraction stops when stock is reduced to zero. From that time onward there would be no further extraction. For depletable resources with stock effects, extraction likewise ceases in steady state, but the deposit need not be totally depleted. We explore here the steady stock level at which extraction permanently ceases. To so do we make the further assumption that the external conditions -- price, cost functions, interest rate, are time invariant. Furthermore, we assume that the time horizon T is so far in the distant future that it is irrelevant for current decisions. Under these assumptions we can examine depletable resource steady state conditions.

Assumption: Time Invariance:
Prices, interest rate, and cost functions are independent of time. The time horizon is so long that its existence has virtually no impact on the optimal choices.

Under the time invariance assumption, equations (24) and (25) or (25') can define a steady state stock level at which opportunity cost remains constant, extraction stops, and stock remains constant. Steady state stock will depend upon prices, possibly the interest rate, and upon the cost function.

In steady state, opportunity cost remains constant from one time period to the next. Denoting steady state opportunity cost as ϕ: $\phi = \phi_t = \phi_{t+1}$. By equation (25'):

$$\phi = - \frac{\partial C/\partial S}{\rho} \tag{32}$$

where costs are evaluated at E=0, since in steady state all extraction ceases[16]. Steady state opportunity cost is the present value of a perpetual stream of incremental costs $-\partial C/\partial S$.

This interpretation can make the time horizon condition in the assumption clear. A stream of $-\partial C/\partial S$ for (T-t) years would have a discounted present value of $\phi\,[1 - e^{-r(T-t)}]$. However, for large enough values of (T-t), this discounted present value can be made arbitrarily close to ϕ. The time horizon is assumed to be distant enough that $e^{-r(T-t)}$ is virtually zero.

Combining equations (24) and (32) in steady state gives a condition defining steady state stock:

$$P = \partial C/\partial E\big|_{E=0} - \frac{\partial C/\partial S}{\rho} \tag{33}$$

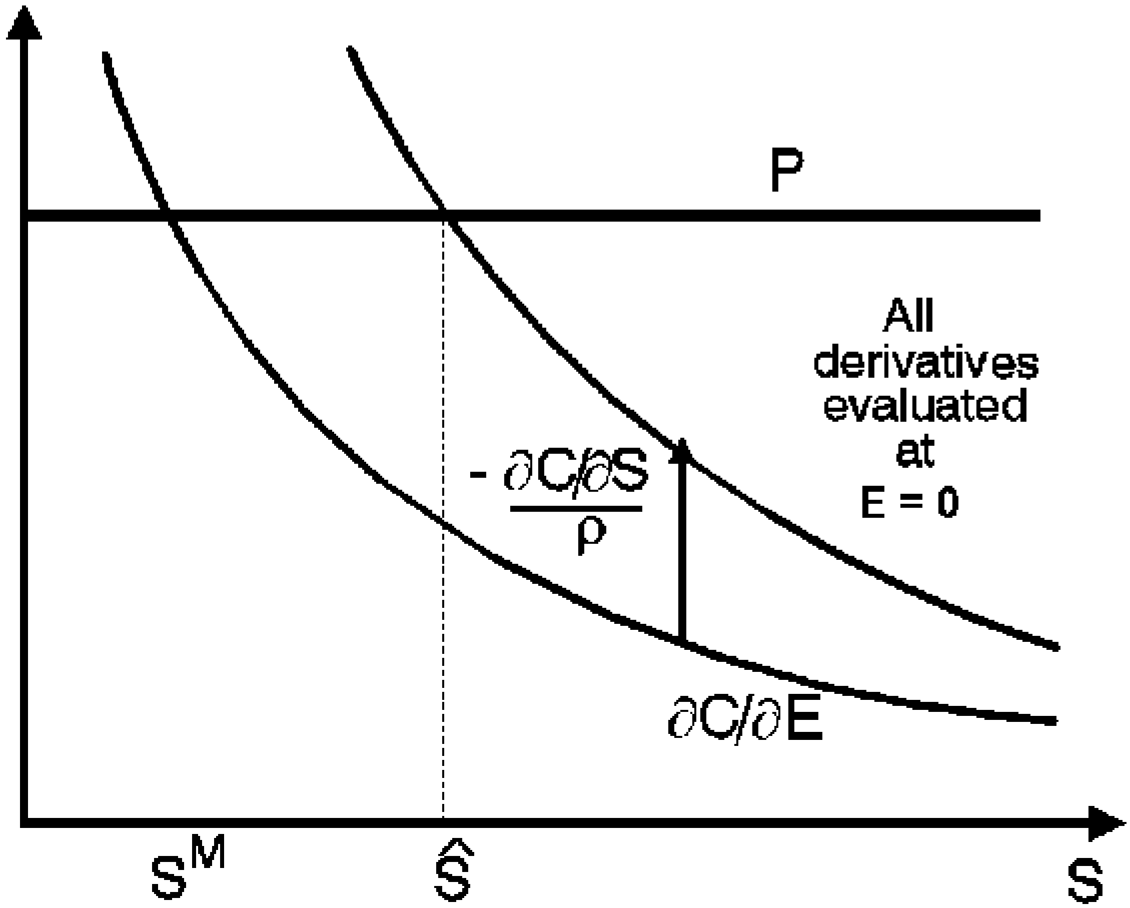

Figure 7
Steady State and Final Equilibrium Stocks

Each term on the right hand side of equation (33) is evaluated at E = 0. Since extraction rate is not a variable for that equation, steady state stock, $\hat{S}$, is the only variable, and thus $\hat{S}$ can be determined from equation (33). Figure 7 illustrates equation (33). The right hand side of equation (33) consists of two terms, both typically decreasing functions of stock. $\hat{S}$ is determined as that stock level at which price is just equal to the marginal extraction cost plus the steady state opportunity cost, both evaluated at zero extraction rate.

Equation (33) implies that the higher the price, the lower the steady state stock and the greater the quantity extracted before steady state is reached, as easily seen in Figure 7.

Figure 7 identifies S^M as that stock level for which marginal cost, evaluated at E=0, equals price. If stock exceed S^M when time reaches T, then extraction will occur at the time horizon. However, if total cost is a decreasing function of stock at zero extraction, and hence $\phi > 0$, then the steady state stock exceeds S^M. In this case, a deposit may converge to steady state before the time horizon but may then be reopened for extraction as time approaches the horizon. Extraction may terminate and then be reinitiated even with no change of price or cost function.

In many circumstances, if there is no extraction, there will be zero cost or some fixed shutdown cost, independent of the remaining stock. For example, the costs to plug and abandon an oil or natural gas well will typically not depend upon the amount of the resource remaining at the time of abandonment. In this situation $\phi = 0$ and the two curves in Figure 7 would be identical. Once extraction is halted, it will remain stopped forever, unless prices or costs were to change over time.

But there are many situations in which $\phi > 0$ because there remains a stock-dependent cost after all extraction ceases. These situations seem to be related to environmental consequences of resource extraction or use. For example, if a virgin forest is harvested, even after all harvesting has ceased, there will be long-term environmental consequences. The less forested area remaining, the higher the environmental cost. Similarly, the more coal is strip mined in a region, the higher the environmental costs or the one-time restoration costs. Subsidence of overlying land may occur as resources are extracted and the amount of subsidence may depend upon the total stock of the resource ultimately depleted. The greater the cumulative use of carbon-based resources, the greater the accumulation of CO_2 in the atmosphere even after use of such fuels stops. In such situations, $\partial C/\partial S < 0$ at E=0 and $\phi > 0$.

In summary, the total resource quantity ultimately extracted will be less than the initial stock if marginal extraction cost increases enough as stock declines toward zero. If price does not correspondingly rise, then economic shutdown will occur. Stock remaining at economic shutdown will depend on the marginal extraction cost (at E=0), the price of the extracted commodity, the magnitude of the stock dependent environmental or other costs, and the interest rate.

We turn now to analysis of trajectories toward the steady state.

4. Phase Diagrams for Dynamic Analysis

Under the time invariance assumption, phase diagrams provide convenient tools for comparative dynamic analysis of adjustment toward the steady state as well as for comparative static analysis of the steady state. Phase diagrams can also be used to gain insights into analysis of markets with time varying parameters.

A phase diagram is a graph in a space of S vs ϕ, divided into four regions, based upon the directions of movement of ϕ and S. Boundaries are defined by two loci: (1) the locus of points at which E is just reduced to zero and at which stock remains constant (the S-constant locus), and (2) the locus of points at which ϕ remains constant over time (the ϕ-constant locus).

The S-constant locus is defined by:

$$\phi = P - \partial C/\partial E|_{E=0} \tag{34}$$

Along or above the S-constant locus, S remains constant over time, while for lower values of ϕ, $E > 0$ and stock declines over time[17]. To the right of the ϕ-constant locus (higher values of S), opportunity cost increases over time, while to the left of that locus, opportunity cost decreases over time[18]. Figure 8 illustrates such a phase diagram for depletable resources with stock effects. The arrows indicate the direction of movement of opportunity cost and stock.

In Figure 8, the S-constant locus slopes upward as long as $\partial C/\partial E$ is a decreasing function of S, as normally be case absent learning by doing. The S-constant locus slopes downward for those stock levels at which $\partial C^2/\partial E\partial S < 0$ as may be the case with learning by doing.

The ϕ-constant locus is defined by:

$$\phi = -\frac{\partial C/\partial S}{\rho} \tag{35}$$

where $\partial C/\partial S$ is evaluated at the extraction rate consistent with the ϕ defined in equation (35). This locus is downward sloping, as has been drawn in Figure 8, if $\partial C^2/\partial E\partial S < 0$ (or not much greater than zero)[19]. This expression can be totally differentiated, recognizing that $\partial C/\partial S$ depends on S and E, which itself depends on S and ϕ, to calculate $\partial\phi \equiv \partial S$ as non-positive[20] as long as $\partial^2C/\partial S\,\partial E < \rho\,\partial^2C/\partial E^2$.

Steady state of the system can be illustrated as that portion of the ϕ-constant locus on which stock remains constant. The S-constant and the ϕ-constant loci intersect at $\hat{\phi}$, $\hat{S}$. Steady state of the system can occur at any portion of the ϕ-constant locus for which $S \leq \hat{S}$. At any such combinations of S and ϕ, extraction rate is zero, stock remains constant, as does opportunity cost. For the time horizon extremely far in the future (or infinite), the system either starts at one of these points (for $S_0 \leq \hat{S}$) or converges to $\hat{\phi}$, $\hat{S}$. For $S_0 \leq \hat{S}$, the resource is never extracted at all.

Figure 8 has been drawn with steady state opportunity cost positive, under the assumption that there exist environmental or other costs that lead to a positive incremental costs of reduced stock even after extraction ceases. Conversely, if $\partial C/\partial S = 0$ when $E = 0$, then the steady state would occur along the horizontal axis, with $\phi = 0$.

The difference equations determining changes of ϕ and S can be solved to give a unique optimal path of stock and opportunity cost from any initial stock to the final state. In the time invariant case, this unique optimal path must converge to the steady state. Under the time invariance assumption, the optimal extraction rate and opportunity cost given stock must be independent of time and must be independent of the initial stock. This implies that for every S there is a unique ϕ on its optimal extraction path, independent of the history of extraction. Such a mapping from S to ϕ defines a third locus of points in the phase diagram: the "convergent trajectory." The convergent trajectory provides a closed loop feedback control under the time invariance assumption in which: (1) ϕ is a function of S, (2) E is a

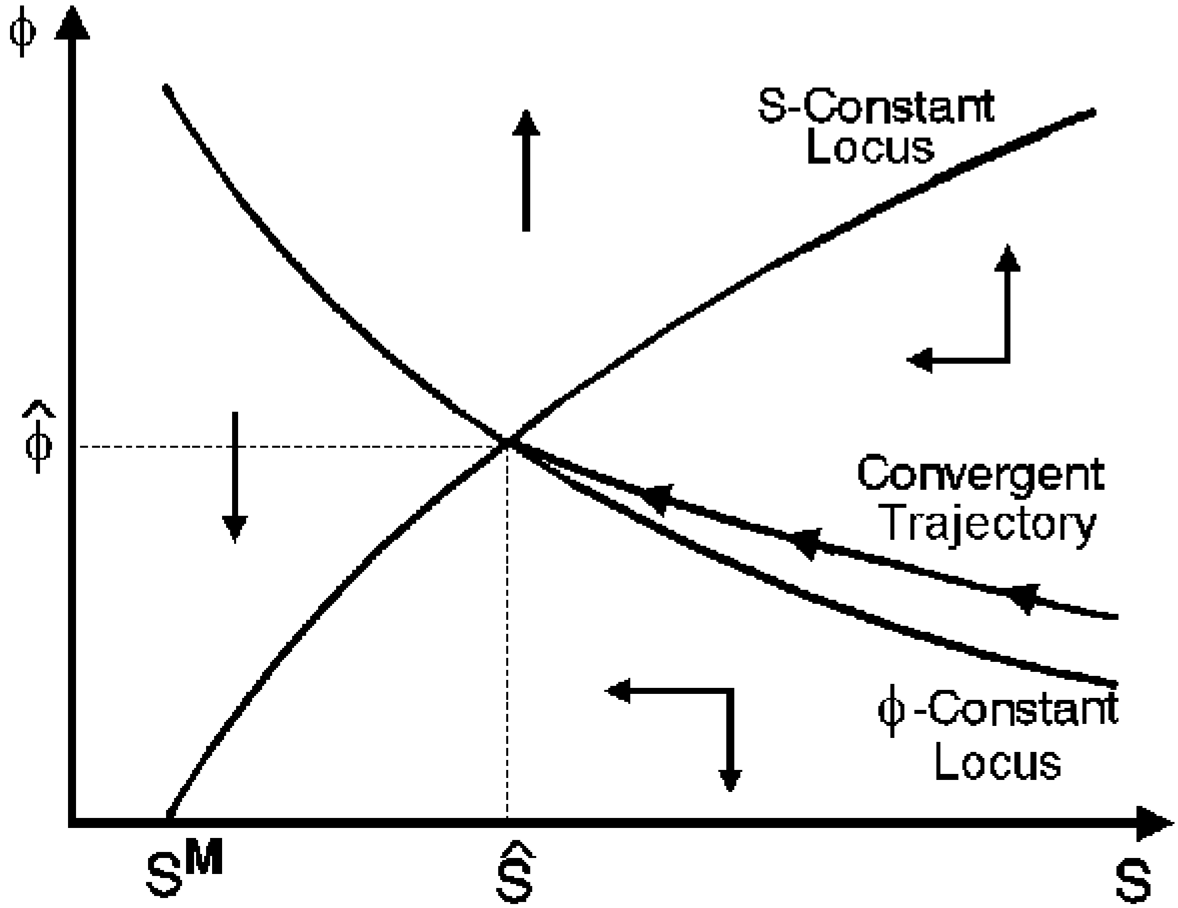

Figure 8
Phase Diagram with Convergent Locus

function of ϕ and S, and thus (3) E is a function of S. Figure 9 adds the convergent trajectory to the phase diagram of Figure 8. The convergent trajectory is indicated by the line with arrowheads indicating the direction of convergence toward the steady state.

For a finite time horizon problem, equation (26) determines conditions at the final time horizon. At time T, the final state of the system must be along either the horizontal or the vertical axis. Figure 8 has been drawn under the normal case in which the S-constant locus intersects the horizontal axis at some positive stock, S^M, the same S^M as in Figure 7. For a finite time horizon problem $S_T \geq S^M$ and $\phi_T = 0$ (unless the system starts with initial stock below S^M.) For the system to terminate on the horizontal axis, the entire path of opportunity cost must lie below the convergent trajectory. For any given stock, opportunity cost will be smaller, extraction will be larger, and stock will decline more rapidly than in the

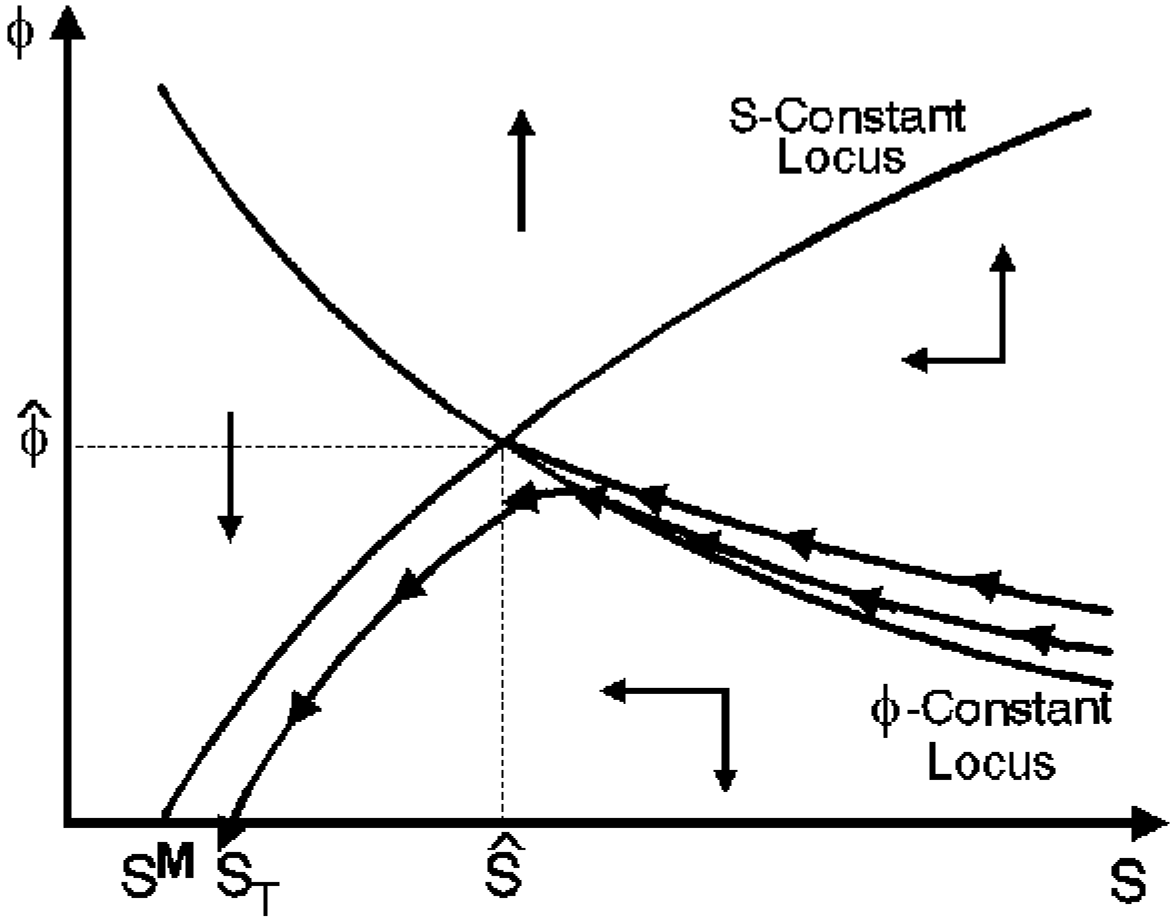

Figure 9
Finite Time vs Infinite Time Trajectories

convergent trajectory. This finite time horizon trajectory is illustrated in Figure 10. For very long time horizon, the actual path lies very close to the convergent trajectory and the system state remains very near the steady state for many time periods. Only as the time horizon is approached does the system trajectory diverge from the convergent trajectory.

The preceding discussion and graphs assumed that marginal cost and total cost were both decreasing functions of remaining stock at all stock levels. However, in early phases of development of a deposit, stock reductions may lead to reductions in costs rather than cost increases. In this case the S-constant locus may bend downward for high stock levels. In addition, as S increases in the region in which $\partial C^2/\partial E \partial S > 0$, the slope of the S-constant locus approaches minus infinity. This locus cuts through the S axis to give negative values of ϕ along the ϕ-constant locus. The locus never slopes upward.

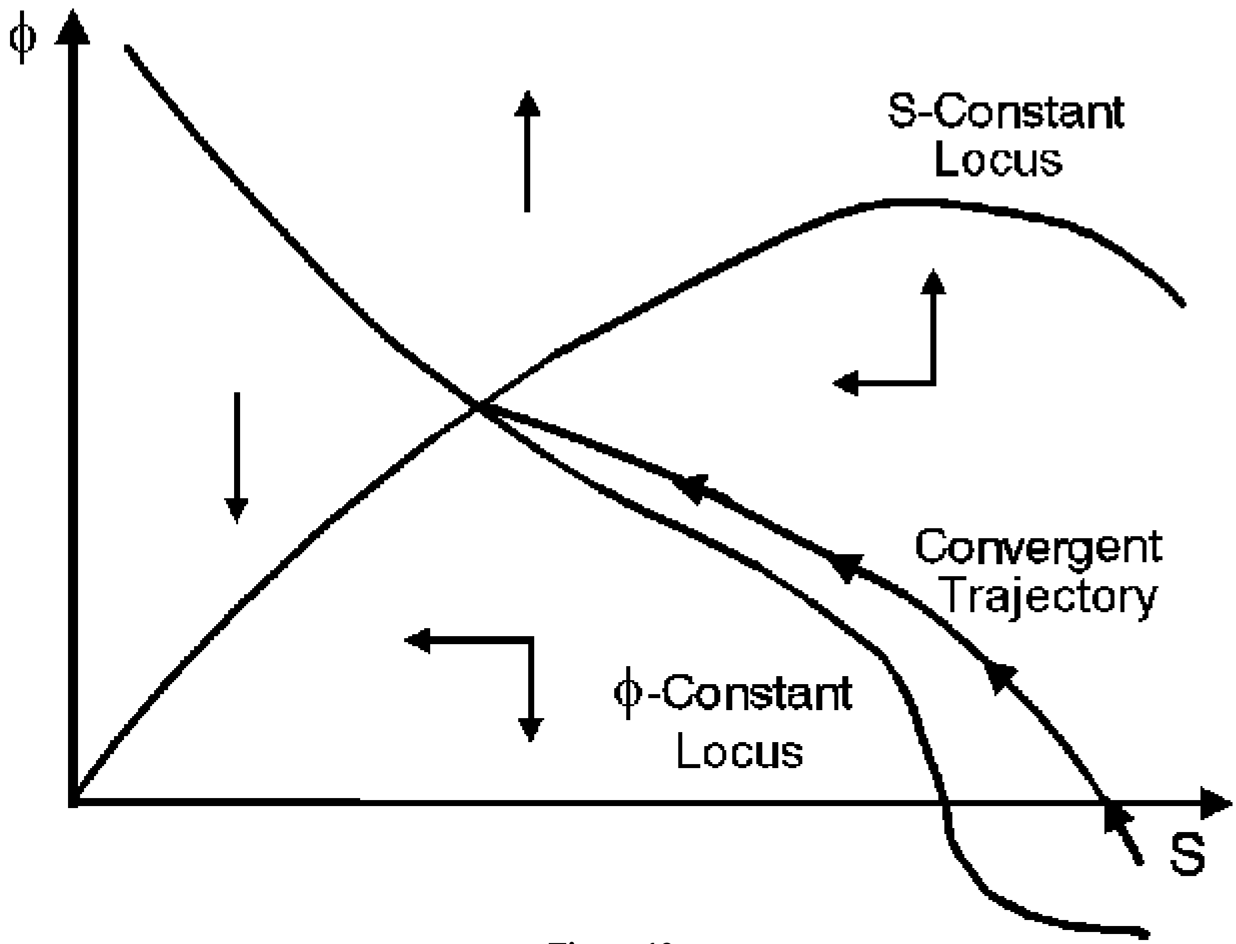

Figure 10
Phase Diagram with "Learning-by-Doing"

Rather ϕ gets negative enough and extraction rate gets positive enough that either $\partial^2C/\partial E^2$ increases so that $\partial^2C/\partial S\partial E < \rho \ \partial^2C/\partial E^2$ everywhere on the ϕ-constant locus or the locus does not exist for some values of S. A phase diagram for such "learning-by-doing" cost functions appears in Figure 11.

Figure 11 shows that the initial opportunity cost may be negative if learning-by doing is significant for high stock levels. In this case, the anticipation of future cost reductions would cause the firm to extract more rapidly than it would absent the incentive to reduce future costs.

Phase diagrams are particularly valuable for examining comparative dynamics of the optimal trajectories. It is such analyses to which we now move, utilizing phase diagrams whenever the system is time invariant or can be treated as such. Whenever the time invariance assumption is inappropriate, we will use equations (24) through (26) directly.

5. An Example

The analysis can be illustrated with a specific numerical example, in which extraction costs are quadratic in extraction rate and are inversely proportional to the remaining stock and that there is an environmental cost that is zero if stock equals the initial level but which increases as stock is depleted:

$$C_t(E_t, S_{t-1}) = \frac{K\,E_t^2}{2\,S_{t-1}} + M\left(\frac{1}{S_{t-1}} - \frac{1}{S_0}\right)$$

The constant K is positive and the constant M is non-negative[21]. The first term of the cost function is homogeneous of degree 1 and thus is only weakly convex but the second term makes the cost function strongly convex when M is positive. It can be easily seen that the second derivatives with respect to extraction rate and with respect to stock are both positive. The determinant of the Hessian matrix is always positive when M is positive and zero when M is zero. The second cross partial derivatives are negative. When M is zero, this is a limiting case of a weakly convex cost function and in that case the slopes of the S-constant locus and the ϕ-constant locus will be also limiting cases of the slopes discussed above.

Equation (24) can be solved to calculate the optimal extraction rate as a function of price and opportunity cost:

$$E_t = \{ (P_t - \phi_t) / K \} \, S_{t-1}$$

This equation allows us to calculate the S-constant locus easily. The S-constant locus is exactly equal to price, independent of the remaining stock. Stock remains constant if the opportunity cost is greater than or equal to price and stock declines for lower values of the opportunity cost.

Equation (25) can be rewritten based upon the optimal extraction rates:

$$\phi_t = \phi_{t-1}(1 + \rho) - \frac{(P_t - \phi_t)^2}{2\,K} - \frac{M}{S_{t-1}^{\;2}}$$

Note that for this example, the evolution of the opportunity cost depends on the interest rate, price, cost function, and remaining stock.

If we know the final time and the final value of the opportunity cost, these equations can always be solved (at least numerically) starting from the last time period and working backward in time to calculate the opportunity cost over time. This calculation can be made even if prices vary over time.

For this part of our analysis we will adopt the time invariance assumption and treat the terminal time as arbitrarily far in the future and assume that the price remains constant over time. Therefore we can drop the t subscript on price and focus attention on the convergent trajectory. At a later point we will return to this model in a time varying case.

The steady state opportunity cost can be calculated from the above equation by asking the value of ϕ that would make ϕ_t equal to ϕ_{t-1}. The resulting quadratic equation has a closed form solution:

$$\hat{\phi} = P - \rho K \left[\sqrt{1 + (2/\rho K)(P - M/\rho S^2)} - 1 \right]$$

The ϕ-constant locus is decreasing function of remaining stock and lies below the S-constant locus for $S > (M/\rho P)^{1/2}$. In steady-state, extraction rate will be zero and stock will equal to $(M/\rho P)^{1/2}$, but it would take infinite time to reach steady state. In steady state, $\phi = P$.

Optimal extraction along the convergent trajectory can be calculated:

$$E_t = \rho \left[\sqrt{1 + (2/\rho K)(P - M/\rho S^2)} - 1 \right] S_{t-1}$$

Along the convergent trajectory for very large stock or small value of M, both the stock and the extraction rate decline approximately geometrically, with extraction rate a virtually fixed proportion of remaining stock. However, for smaller values of stock and/or larger values of M, the fraction of the remaining stock extracted each period shrinks as S declines. The rate of extraction, and thus the rate of geometric decline is an increasing function of both price and interest rate and a decreasing function of the extraction cost. For a finite but long distant time horizon, opportunity cost is slightly below that calculated above and extraction rate is slightly above the rate calculated here.

We turn now to more general properties of solutions for more general cases.

6. Optimal Trajectories and Comparative Dynamics

The models can be used to examine characteristics of the optimal extraction path for a depletable resource with stock effects. We will present many of the same examples as examined previously to illustrate methods of analysis and conclusions that can be derived. In all the discussions that follow, we assume that the initial stock exceeds $\hat{S}$ so that some extraction will occur.

a. Extraction path under time invariant conditions

The phase diagrams of Figure 9 or 11 can be used to illustrate the optimal trajectory. In the optimal trajectory, if the ϕ-constant locus is downward sloping everywhere, the opportunity cost must begin in the region in which ϕ is increasing and S is decreasing over time and must converge to the crossing point of these two loci[22]. If the system began in any other region it could never converge to the steady state.

Characteristics of the time pattern of extraction are influenced by the derivative of marginal cost with respect to stock. In the absence of learning by doing or other phenomena which lead to positive values of $\partial^2C/\partial E\partial S$, extraction rate must decline over time; opportunity cost grows and as stock decreases, marginal cost for any extraction rate must increase. The basic pattern of extraction rates declining over time is similar to the pattern absent stock effects.

With learning by doing, two effects work in opposite directions: the rapid increase in the opportunity cost implies declining extraction but the decline in the marginal extraction cost as stock declines would have the opposite effect. Either effect could dominate and the extraction rate could increase or decrease over time.

b. Extraction path for prices varying: very long time horizon

When price varies over time, phase diagrams do not provide as decisive a conclusion as that outlined in the previous section, even when $\partial^2C/\partial E\partial S < 0$.

If price is increasing over time, extraction rate may grow or decline, depending upon the increase of price relative to the change in the opportunity cost. For rapidly increasing price, extraction rate will grow from one period to the next while for slowly increasing price, extraction rate will decline.

If price declines from t-1 to t, the S-constant locus will shift upward and the ϕ-constant locus downward from t-1 to t. Thus a decline in price could move the ϕ-constant locus from above the current value of ϕ to below the current value, allowing a convergence to the steady state. Thus an essential part of the previous section's argument is not valid if prices ever decline.

However, if we add an assumption that $-\partial C/\partial S$ is itself convex in extraction rate, maintaining assumptions that any time horizon is far in the future, that $\partial^2 C/\partial E\partial S < 0$ for all stock levels, and that the cost function is time invariant, we can establish that if $P_t \leq P_{t-t}$ then $E_t < E_{t-1}$. The maintained assumption that the cost function is convex does not imply that $-\partial C/\partial S$ is itself convex in E, unless additional structure is imposed on the cost function. For example, if the cost function is separable into two multiplicative or additive factors, one dependant only on S and the other only on E, then cost function convexity implies[23] that $-\partial C/\partial S$ is convex in E. With such additional assumptions declining price implies a declining extraction rate.

Assume then that $-\partial C/\partial S$ is convex in E, that $P_t \leq P_{t-1}$, but that $E_t \geq E_{t-1}$. We will show that this leads to a contradiction, thus implying that $E_t < E_{t-1}$. By equation (24) if $P_t \leq P_{t-1}$:

$$\phi_t - \phi_{t-1} \leq \partial C_{t-1}/\partial E_{t-1} - \partial C_t/\partial E_t$$

where the marginal costs on the right hand side are evaluated at S_{t-2} and S_{t-1} respectively. The right-hand side can be expanded, recognizing that $S_{t-2} - S_{t-1} = E_{t-1}$ and incorporating the dominance of extraction rate on marginal cost assumption:

$$\begin{aligned}\phi_t - \phi_{t-1} &\leq E_{t-1}\, \partial C^2/\partial S\partial E + [E_{t-1} - E_t]\, \partial C^2/\partial E^2 \\ &< E_{t-1}\, \partial C^2/\partial S\partial E - [E_{t-1} - E_t]\, \partial C^2/\partial S\partial E \\ &= E_t\, \partial C^2/\partial S\partial E\end{aligned}$$

The derivative on the right hand side is evaluated at stock between S_{t-2} and S_{t-1} and extraction rate between E_t and E_{t-1}.

A second limit can be placed on the change in the opportunity cost using equation (29) under the assumption that the time horizon is very far in the future:

$$\phi_t - \phi_{t-1} \geq \partial C_t/\partial S_{t-1} - \partial C_t/\partial S_{t-1}|_{E=0}$$

The first derivative on the right hand side is evaluated at S_{t-1} and E_t while the second term is evaluated at S_{t-1} and at E = 0. Under the assumption that $-\partial C/\partial S$ is convex in E, this inequality becomes:

$$\phi_t - \phi_{t-1} \geq \partial C_t/\partial S_{t-1} - \partial C_t/\partial S_{t-1}|_{E=0} \geq E_t\, \partial^2 C_t/\partial E_t \partial S_{t-1}$$

The two inequalities for the change in opportunity cost are inconsistent with one another. Thus the contradiction implies that whenever $P_t \leq P_{t-1}$, then $E_t < E_{t-1}$ for very long time horizon situations under the assumptions outlined here.

c. Extraction path as time approaches the horizon

Results from the previous section cannot be generalized as time approaches teh horizon. If the cost function is time invariant, $\partial C^2/\partial S \partial E < 0$, and $-\partial C/\partial S$ is convex in E, extraction rate can increase over time even while prices are declining. As time approaches the horizon, opportunity cost must decline and may decline faster than price. As stock declines, marginal cost of extraction from a given rate increases. If this marginal cost increase is not sufficient to compensate for the drop in opportunity cost, extraction will grow over time.

This possibility is illustrated with calculations based upon the numeric example presented above. We use the cost function of our example, with $S_0 = 10^5$ units, $M = 10^{10}$, $K = 10^3$, and $\rho = 0.1$. Price begins at $10 per unit and declines by $0.1 per unit each year over 21 years. Price and resulting

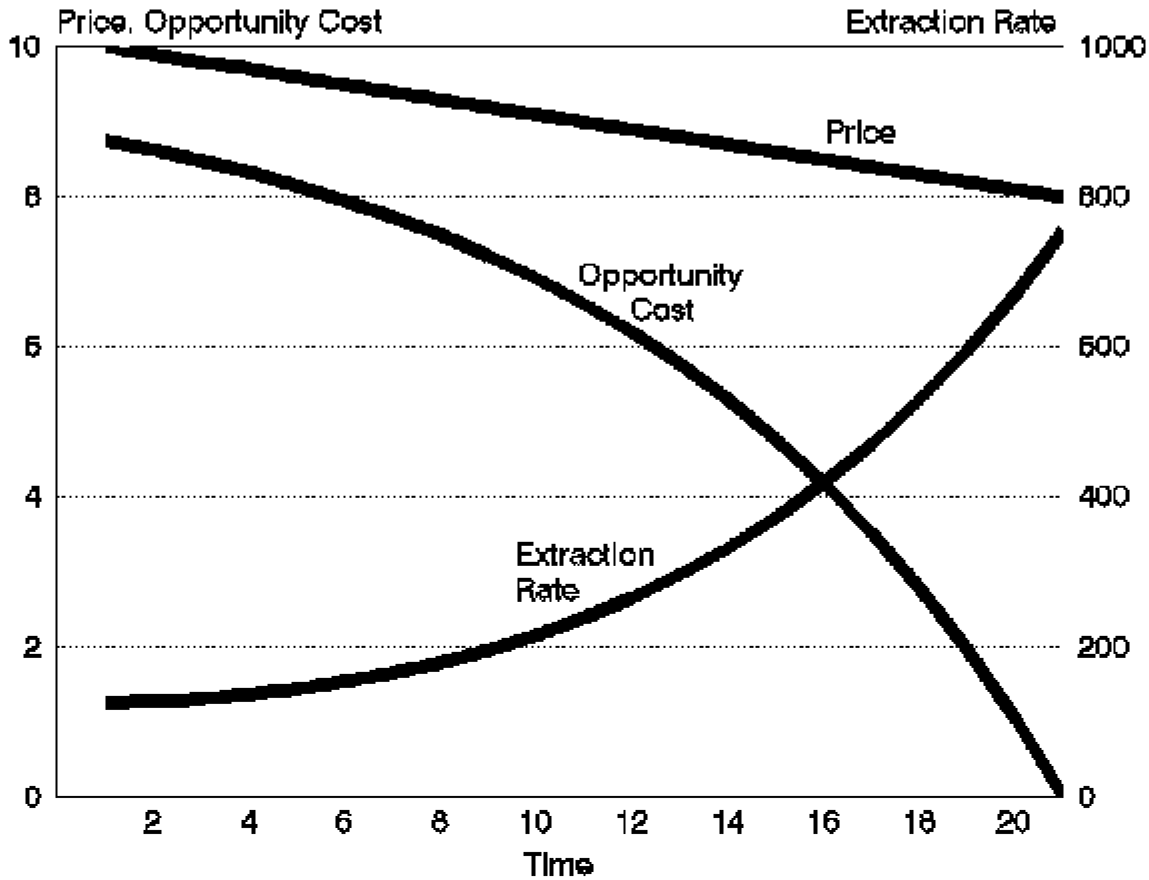

Figure 11Price, Opportunity Cost, and Extraction Rate Approaching Time Horizon

opportunity cost and extraction rate are plotted in Figure 12.

Figure 12 shows that for this particular example, extraction rate increases over time, starting at the lowest level and ending at its highest level, even in with price declining from one year to the next. Opportunity cost declines more rapidly than does the price, beginning at $8.75 per unit and ending in year 21 at $0. The rapid decline of the opportunity cost is the result of the large partial derivative of cost with respect to stock, even with low extraction rates.

For this example, had prices been constant at P and had the time horizon been sufficiently far distant, then extraction rate would have decreased over time, converging to zero as opportunity cost increased towards P. For the time horizon far in the future, ϕ_t converges to P, while as time approaches the horizon, the opportunity cost converges to zero.

Hence we see that it is quite possible for price movements and extraction rate movements to be inversely related when the system approaches a time horizon: declining prices may well be linked to increasing outputs, a situation which makes empirical research very difficult.

d. The role of price

Assume that in the first case price was always P and in the second, was P′, where P′ > P. Let δP = P′ - P. We examine here the comparative dynamics of the optimal extraction path.

First consider impacts on the steady state. We can totally differentiate equations (24) and (32) in order to determine impacts on the steady state stock and opportunity cost. The extraction rate is zero in steady state so that extraction rate is not treated as a variable in differentiating equations (24) and (32). We examine infinitesimal changes in P in order to determine the impacts on steady state stock and opportunity costs. For discrete changes, impacts can be integrated and properties of the infinitesimal change will be preserved .

Totally differentiating equations (24) and (32) gives:

$$\delta P = \partial C^2/\partial E \partial S \, \delta \hat{S} + \delta \hat{\phi}$$
$$\rho \, \delta \hat{\phi} = - \partial C^2/\partial S^2 \, \delta \hat{S}$$

Combining these equations gives the impacts on steady state stock and opportunity cost:

$$\delta \hat{S} = - \frac{\rho \, \delta P}{\partial C^2/\partial S^2 - \rho \, \partial C^2/\partial E \partial S}$$
$$\delta \hat{\phi} = \frac{\partial C^2/\partial S^2 \, \delta P}{\partial C^2/\partial S^2 - \rho \, \partial C^2/\partial E \partial S}$$

The numerators of the right-hand side expressions are both positive and the first term in the denominators is also positive. We expect no learning by doing in steady state and therefore the second term is also positive. Thus we can determine the signs of the changes and can bound the magnitude of

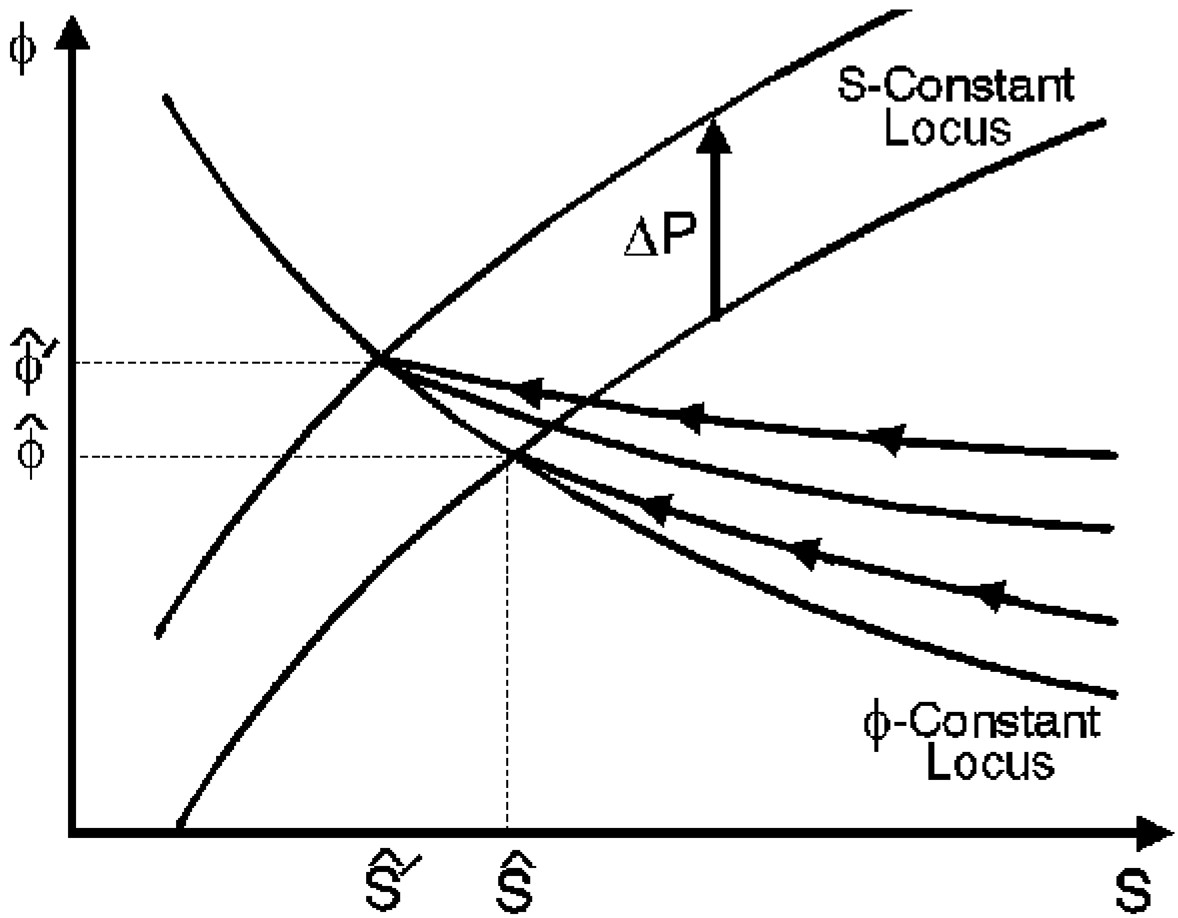

Figure 12
Phase Diagram with Increase in Price

the opportunity cost change. These equations show that the steady state opportunity cost must be an increasing function of price[24]. This increase must be less than the price change. The steady state stock, $\hat{S}$, must be smaller in the higher price case and a greater total amount of the resource must be extracted before steady state is reached.

These changes can also be examined through analysis of the changes in the phase diagram. Equation (24) implies that the S-constant locus would be shifted upward by the amount of the price increase. The form of equation (32) would be unchanged: for any fixed value of stock and extraction rate, steady state opportunity cost would be unchanged between the two cases. Differentiating equations (24) and (32) shows that ϕ-constant locus must rise by an amount smaller than δP as long as marginal cost is a decreasing function of remaining stock, a condition we expect at steady state. The changes in opportunity cost and in steady state stock are those indicated in the equations above. These shifts are shown in Figure 13.

We now examine the impacts of higher prices on the convergent trajectory. For those stock levels at which marginal cost is a decreasing function of remaining stock, higher price leads to a higher opportunity cost. However for levels at which marginal cost is an increasing function of remaining stock, we cannot assure that opportunity cost will decline with increases in price[25]. However, as long as the "dominance of extraction rate on marginal cost" assumption holds, the change in opportunity cost must always be smaller than δP. These variations in the opportunity cost imply that the optimal extraction for any given stock along the convergent trajectory is higher, the higher is the price of the extracted commodity.

These results can be established by examining infinitesimally small changes. Assume the converse: that for some S (at time t-1) the convergent trajectory is shifted upwards so that $\delta\phi_{t-1} \geq \delta P$. Totally differentiating equation (25) with respect to ϕ_t and E_t, for S fixed gives:

$$\delta\phi_t = \delta\phi_{t-1}(1+\rho) + \partial^2 C/\partial S\partial E \; \delta E_t$$

We can similarly differentiate equation (24) to obtain an expression for δE_t:

$$\delta E_t = \frac{\delta P - \delta\phi_t}{\partial^2 C/\partial E^2}$$

These two equations can be combined to eliminate δE_t:

$$[\delta\phi_t - \delta P]\,[1 + \frac{\partial^2 C/\partial S\partial E}{\partial^2 C/\partial E^2}] = [\delta\phi_{t-1} - \delta P]\,(1+\rho) + \rho \qquad (36)$$

By equation (36), if $\delta\phi_{t-1} \geq \delta P$, then $\delta\phi_t > \delta P$ since the "dominance of extraction rate on marginal cost" assumption (equation (12)) implies that the factor multiplying $\delta\phi_t - \delta P$ must be positive. In addition, since $\delta\phi_t > \delta P$, stock must decline less ($\delta S_t > 0$). Along the original convergent trajectory, stock is declining and opportunity cost is increasing. Therefore $\delta S_t > 0$ and $\delta\phi_t > \delta P$ implies that the new convergent trajectory would be shifted up beyond the initial trajectory by an amount greater than δP at stock equal to S_t. For the next period the same logic can be repeated. Hence if the new convergent trajectory is shifted above the initial one by an amount greater than δP at any stock level, then it must be shifted up by an amount greater than δP for all lower stock levels. However, in the steady state, the change in opportunity cost must be smaller than δP, a contradiction. Thus the trajectory cannot be shifted by greater than δP at any stock level.

Assume now that $\delta\phi_{t-1} \leq 0$ at some stock level. We can rearrange terms in equation (36):

$$\delta\phi_t \left[1 + \frac{\partial^2 C/\partial S\partial E}{\partial^2 C/\partial E^2}\right] = \delta\phi_{t-1}\,(1+\rho) \quad + \quad \left[\frac{\partial^2 C/\partial S\partial E}{\partial^2 C/\partial E^2}\right] \delta P \tag{37}$$

Equation (37) shows that if $\delta\phi_{t-1} \leq 0$ and $\partial^2 C/\partial S\partial E < 0$, then $\delta\phi_t < 0$, under the "dominance of extraction rate on marginal cost" assumption. In addition, since $\delta\phi_t < 0$ and $\delta P > 0$, stock must decline more ($\delta S_t < 0$). Along the original convergent trajectory, stock was declining and opportunity cost was increasing. Therefore $\delta S_t < 0$ and $\delta\phi_t < 0$ implies that the new convergent trajectory would be shifted down below the initial trajectory at stock equal to S_t. For the next period the same logic can be repeated if $\partial^2 C/\partial S\partial E < 0$. Hence if the new convergent trajectory is below the initial trajectory at any stock level, then it must be shifted below for all lower stock levels, under the assumption that $\partial^2 C/\partial S\partial E < 0$ for all lower stock levels. But since the convergent trajectory must shift upward at the steady state, it can not be so shifted at any such stock level unless $\partial^2 C/\partial S\partial E > 0$ for some lower stock level[26].

In summary, the net result of a constant increase in price for all time is an increase in the extraction rate from any stock level, to extract a greater total quantity, and to increase the steady state opportunity cost. Opportunity cost increases at all levels along the new convergent trajectory unless marginal cost is an increasing function of remaining stock over some range of stock.

e. The role of price expectations

As we did for resources without stock effects, we here again examine the case in which the prices do not change currently but in which beliefs about future prices do change. Again we assume that after some future time, τ, in the new situation all prices will be higher than they were in the old situation. We identify three cases:

Low Price Case:

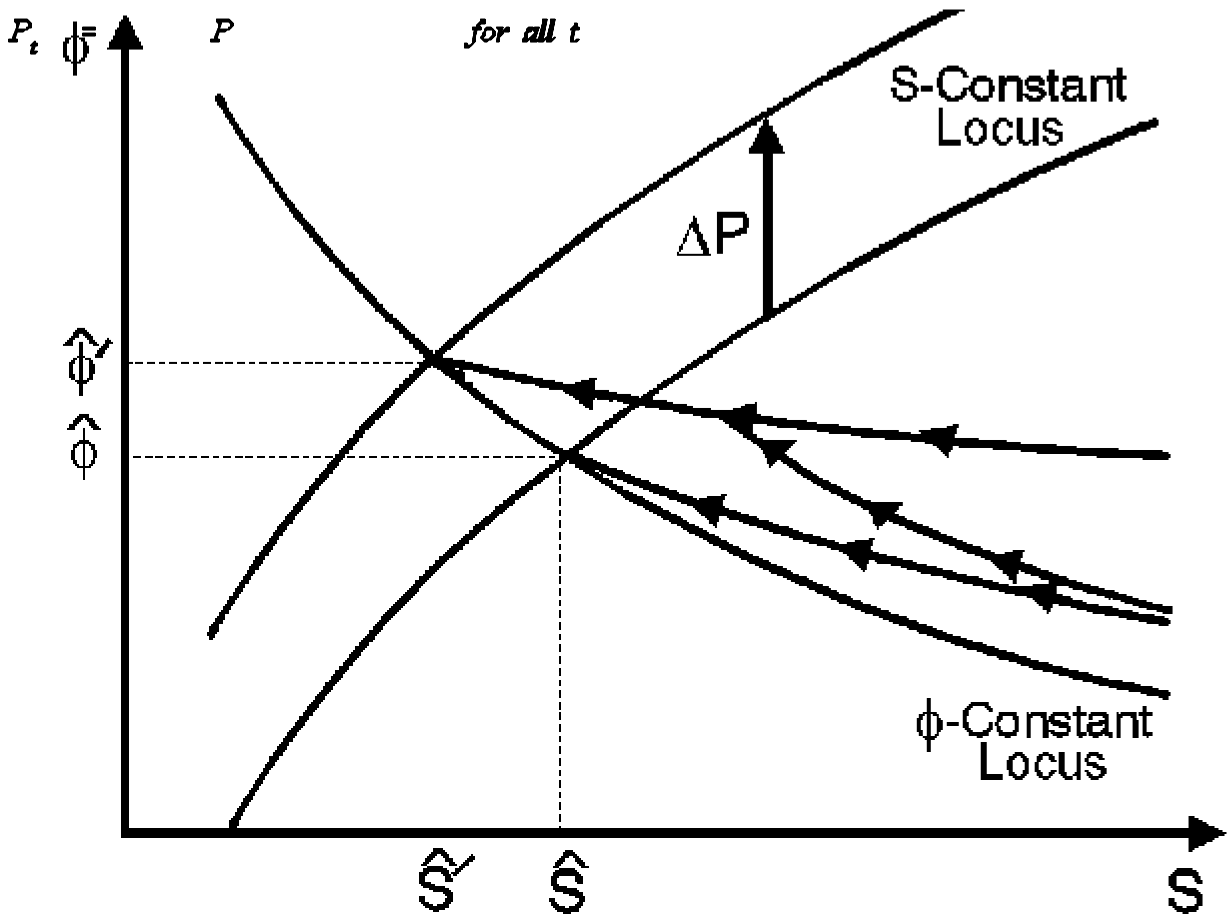

Figure 13
Optimal Trajectories for High, Low, and Changing Price Cases

High Price Case:

$$P_t \quad = \quad P' \; > \; P \qquad \textit{for all } t$$

Changing Price Case:

$$P_t' = \begin{cases} P & \textit{for } t < \tau \\ P' > P & \textit{for } t \geq \tau \end{cases}$$

Both the low and the high price cases satisfy the time invariance assumption and the changing price case maintains the time invariance assumption after time τ. The high and low price cases will provide upper and lower bounds for the opportunity cost (as a function of remaining stock) for the changing price case. We will show that at times before τ, the changing price case will exhibit lower extraction rates than will either of the other cases. Here again, expectations of changing prices will influence current behavior.

Figure 14 displays the phase diagram. In the low price case, steady state would be $\hat{\phi}$, $\hat{S}$ and the convergent trajectory is the lower of the three in Figure 14. In the high price case and in the changing price case steady state is $\hat{\phi}'$, $\hat{S}'$.

Before time τ the changing price case convergent trajectory is always strictly above the low price case convergent trajectory and is never as much as δP below the convergent trajectory in the high price case. This conclusion will be established in subsequent paragraphs. From time τ and after, the convergent trajectories in the high price case and in the changing price case are identical since all conditions are identical after time τ.

Equation (24) and the lower bounds on the opportunity cost imply that for times before τ, extraction will be lower in the changing price case than in either of the other cases. Price in the changing price case is the same as in the lower price case but the opportunity cost is strictly higher. Price in the high price case is δP larger than in the changing price case but the opportunity cost is increased by less than δP; hence extraction rate is higher in the high price case. The net result is that extraction is reduced in anticipation of the higher prices to come. Stock declines less rapidly in the changing price case than in either of the other two.

Equation (36) or (37) provides the basis for showing that the convergent trajectory for the changing price case can never be below that for the low price case. For times before τ, equation (36) or (37) show the evolution of $\delta\phi_t$, where $\delta\phi_t$ is based on the variation from the low price to the changing price case, recognizing that the variation in price before τ between these two cases is zero. These equations become:

$$\delta\phi_t \left[1 + \frac{\partial^2 C/\partial S \partial E}{\partial^2 C/\partial E^2}\right] = \delta\phi_{t-1} (1+\rho)$$

Under the "dominance of extraction rate on marginal cost" assumption (equation (12)), the bracketed factor on the left side of the equation is positive, as is $(1+\rho)$. Thus the sign of $\delta\phi_t$ is the same that of $\delta\phi_{t-1}$: if $\delta\phi_{t-1} \leq 0$, then $\delta\phi_t \leq 0$. But $\delta\phi_t \leq 0$ implies that $\delta E_t \geq 0$, so that $\delta S_t \leq 0$. Along the original convergent trajectory, stock was declining and opportunity cost was increasing. Therefore $\delta S_t \leq 0$ and $\delta\phi_t \leq 0$ imply that the changing price case convergent trajectory could not be above the low price trajectory at stock S_t either. The logic can be repeated for all lower stocks on the convergent trajectory until time τ. At time τ opportunity cost could not lie above on the low price convergent trajectory and therefore must lie below the high price convergent trajectory. But at time τ the convergent trajectory of the changing price case must be identical to the convergent trajectory for the high price case, a contradiction. This contradiction implies that the changing price case convergent trajectory must <u>always</u> lie above the low price case convergent trajectory.

Similarly, we can use the same approach to show that the high price case convergent trajectory can never be above the changing price case convergent trajectory by δP or more. In equation (36) let $\delta\phi_t$ now refer to the difference in opportunity cost between the high price case convergent trajectory and that for the changing price case. This equation implies that if at any time t-1, $\delta\phi_{t-1} \geq \delta P$, then $\delta\phi_t > \delta P$. This implies that $\delta E_t < 0$ and $\delta S_t > 0$. The combination of $\delta\phi_t > \delta P$ and $\delta S_t > 0$ implies that at S_t the high price convergent trajectory will also exceed the changing price convergent trajectory by more than δP. The logic applies for all future times and corresponding stock levels. But we know that at time τ the convergent trajectory of the changing price case must be identical to the convergent trajectory for the high price case, a contradiction. Hence the opportunity cost on the convergent trajectory for the high price case can never exceed the equivalent opportunity cost for the changing price case by δP or more.

The results are similar to those obtained for models without stock effects. For times before τ, the change in expectations leads to an increase in opportunity cost but impacts no other elements of equation (24). Thus for these early times, optimal extraction for the given level of stock must decline as a result of the changed expectations about future prices. The effect would be very small for times long before the anticipated price increase, since the opportunity cost would be raised only slightly. However, as the date of price increase becomes imminent, the effect would get larger and extraction rates would be significantly reduced.

f. The role of the price trajectory

This section relaxes the time invariance assumption, allowing prices and costs to vary over time, and expands the questions addressed in the previous two sections. We assume an increase in price both now and in the future and examine the impact of such changes on extraction from a given stock. We will focus attention only on those price changes whose present value remains constant or declines over time: $\delta P_t \leq (1+\rho)\, \delta P_{t-1}$. This restriction is in contrast to the previous analysis in which the expectation of a very rapid price increase motivated extraction reductions in anticipation of the increase.

Under these fairly general conditions, the optimizing trajectory for the high price case will lead to greater extraction from a given stock than would occur with the lower price case. These results will not necessarily hold if the present value of price rises over time.

Equations (24) and (25) can be differentiated totally to give the relationships between changes in the variables. Differentiating equations (24) and (25) plus the depletability constraint of problem (8) gives:

$$\delta P_t = \partial^2 C/\partial E^2 \; \delta E_t + \partial^2 C/\partial E \partial S \; \delta S_{t-1} + \delta \phi_t$$

$$\delta \phi_t = e^r \; \delta \phi_{t-1} + \partial^2 C/\partial S^2 \; \delta S_{t-1} + \partial^2 C/\partial E \partial S \; \delta E_t$$

$$\delta S_t = \delta S_{t-1} - \delta E_t$$

These equations can be solved by eliminating δE_t so as to show the evolution of $\delta \phi_t - \delta P_t$ and of δS_t over time:

$$\begin{aligned} [\delta \phi_t - \delta P_t]\; [\, \partial^2 C/\partial E^2 + \partial^2 C/\partial E \partial S \,] = {} & \partial^2 C/\partial E^2 \; [e^r \; \delta \phi_{t-1} - \delta P_t] \\ & + [\, \partial^2 C/\partial E^2 \; \partial^2 C/\partial S^2 - (\partial^2 C/\partial E \end{aligned} \tag{38}$$

$$\delta S_t = \frac{[\, \partial^2 C/\partial E^2 + \partial^2 C/\partial E \partial S \,]}{\partial^2 C/\partial E^2} \; \delta S_{t-1} + \frac{[\delta \phi_t - \delta P_t]}{\partial^2 C/\partial E^2} \tag{39}$$

Equations (38) and (39) hold for any trajectory of price changes. However, if we include the restriction on the rate of price growth -- $\delta P_t \leq (1+\rho)\, \delta P_{t-1}$ -- equation (38) becomes:

$$[\delta\phi_t - \delta P_t]\,[\partial^2 C/\partial E^2 + \partial^2 C/\partial E \partial S] \geq$$
$$\partial^2 C/\partial E^2\,(1+\rho)\,[\delta\phi_{t-1} - \delta P_{t-1}]$$
$$+\,[\partial^2 C/\partial E^2\,\partial^2 C/\partial S^2 - (\partial^2 C/\partial E \partial S)^2]\,\delta S_{t-1}$$

Under the "dominance of extraction rate on marginal cost" assumption, the factor multiplying $[\delta\phi_t - \delta P_t]$ is positive. Cost function convexity implies the factors multiplying $[\delta\phi_{t-1} - \delta P_{t-1}]$ and δS_t must be non-negative. In equation (39), the factor multiplying δS_{t-1} is positive and smaller than 1 and that multiplying $\delta\phi_t - \delta P_t$ is positive. Thus equations (38) and (39) imply:

$$\textit{If} \quad \delta S_{t-1} \geq 0 \quad \textit{and} \quad \delta\phi_{t-1} - \delta P_{t-1} > 0 \quad \textit{then}$$
$$\delta S_t > 0 \quad \textit{and} \quad \delta\phi_t - \delta P_t > 0$$

These inequalities imply that if at <u>any</u> time in the high price situation (1) stock is increased or unchanged and (2) the opportunity cost increase exceeds the price increase, then stock would always afterward be increased and the change in opportunity cost would always exceed the price change. At the initial time S_0 is unchanged by definition. Therefore by equation (24), both $[\delta\phi_1 - \delta P_1]$ and δS_1 must be positive or both must be negative. Therefore if $\delta\phi_1 > \delta P_1$, then $\delta S_1 > 0$, $\delta\phi_t > \delta P_t$, and $\delta S_t > 0$ for all subsequent time.

But if at the time horizon (no matter how far distant) stock and opportunity cost were both to increase as a result of the price increase, equation (26) could not hold: both opportunity cost and stock would be positive. This contradiction shows that $\delta\phi_1 \leq \delta P_1$. Furthermore, if the deposit were not ultimately to be fully depleted, then the final opportunity cost would be zero: it cannot increase, as would be required if $\delta\phi_1 = \delta P_1$. Therefore $\delta\phi_1$ is strictly smaller than δP_1 and the initial extraction rate must be higher in the higher price case[27].

As a result of the price increase, from any stock level, the firm would extract at a higher rate and would, over time, have smaller stocks. Smaller stocks would reduce optimal extraction rate from that which would be optimal had past prices been lower and had more stock been left for future use. Higher prices would therefore move extraction toward the present from the future and would increase the total amount of the resource ultimately extracted.

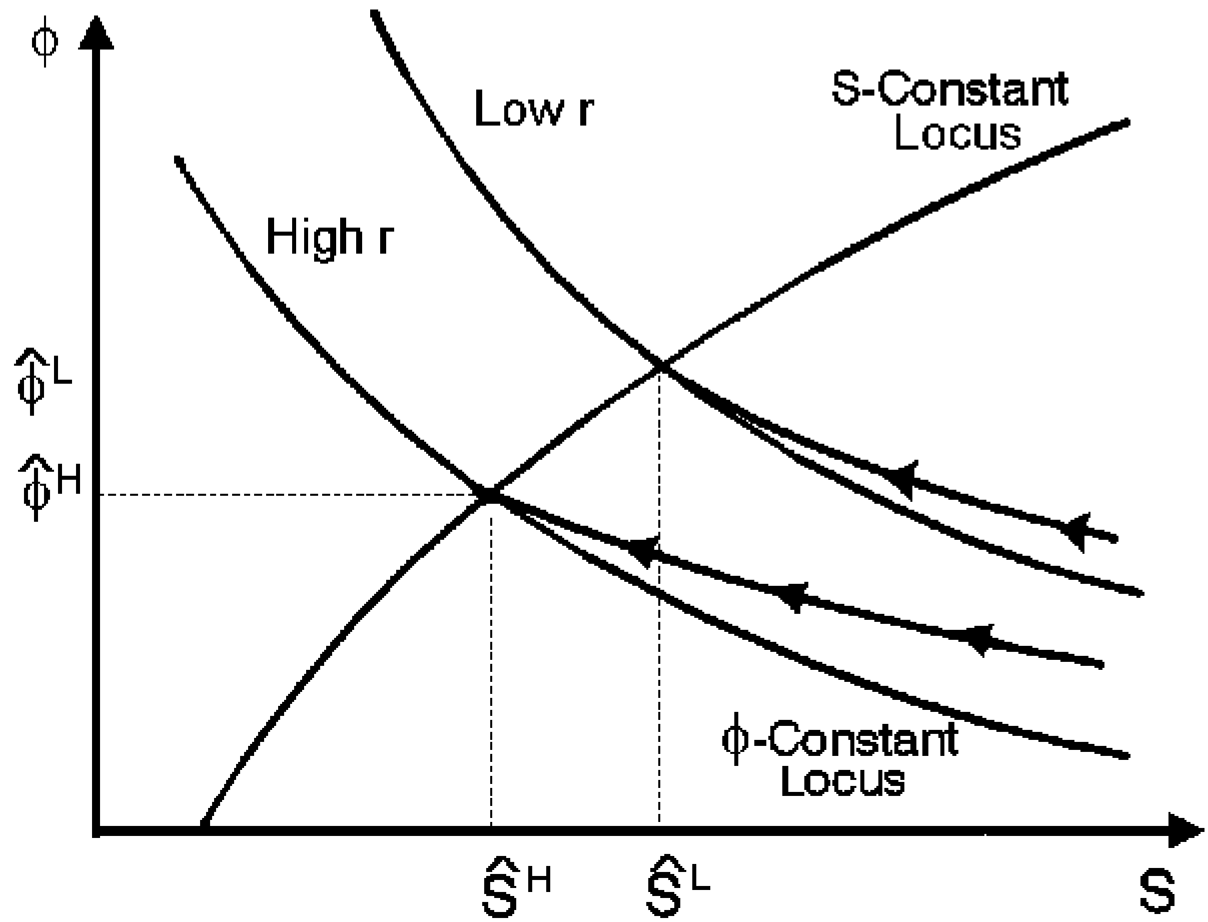

Figure 14
Impacts of Interest Rates on the Optimal Trajectory

g. The role of the interest rate

Under the time invariance assumption, the model implies that the higher the interest rate, the more rapid the extraction from a given stock, again under the assumption that interest rates have no impact on extraction costs. A demonstration follows. It must be remembered, however, that interest rate increases may also increase marginal extraction costs and that such cost increases can imply that the higher the interest rate, the less rapid the extraction from a given stock.

Equation (24) remains unchanged by the interest rate difference; only equation (25) is altered. This variation shifts the ϕ-constant trajectory downward in the high interest rate case and therefore leads to lower steady state opportunity cost and lower steady state stock (unless the steady state opportunity cost is zero and then there is no change in these steady state variables). These differences are illustrated in the phase diagram of Figure 15.

Mathematically, we can examine changes in the steady state by totally differentiating equations (24) and (32) to give impacts on steady state stock and opportunity cost:

$$\delta\hat{S} = - \frac{\hat{\phi}\ \delta\rho}{\partial C^2/\partial S^2 - \rho\ \partial C^2/\partial E \partial S} \leq 0$$

$$\delta\hat{\phi} = \frac{\hat{\phi}\ \partial C^2/\partial S \partial E\ \ \delta P}{\partial C^2/\partial S^2 - \rho\ \partial C^2/\partial E \partial S} \leq 0$$

We turn now to the examination of the convergent trajectory, as illustrated in Figure 15 and show that the convergent trajectory with high interest rates is always lower than the trajectory with low interest rates. The basic idea is that by equation (25), for a given stock and opportunity cost, with low interest rates, opportunity cost will grow less rapidly than with higher interest rates. Growing less rapidly, but ending at a higher level, implies that the opportunity cost must always be higher in the low interest rate case.

Equations (24) and (25) plus the depletability constraint of problem (8) can be differentiated totally and combined to give the relationships between changes in the variables from the initial situation. We use the same procedure as in the previous section to show:

$$\delta\phi_t\ [\ \partial^2 C/\partial E^2 + \partial^2 C/\partial E \partial S\ ] = \partial^2 C/\partial E^2\ (1+\rho)\ \delta\phi_{t-1} + [\ \partial^2 C/\partial E^2\ \partial^2 C/\partial S^2 \quad (40)$$

$$\delta S_t = \frac{[\ \partial^2 C/\partial E^2 + \partial^2 C/\partial E \partial S\ ]}{\partial^2 C/\partial E^2}\ \delta S_{t-1} + \frac{1}{\partial^2 C/\partial E^2}\ \delta(\quad (41)$$

Each of the factors multiplying $\delta\phi_t$, $\delta\phi_{t-1}$, and δS_{t-1} must be positive. If ϕ_{t-1} is positive (which will always be the case absent learning by doing) the factor multiplying $\delta\rho$ is also strictly positive. This leads to the condition:

$$\textit{For } \phi_{t-1} \geq 0, \ \textit{if } \delta S_{t-1} \geq 0 \ \textit{ and } \ \delta\phi_{t-1} \geq 0 \ \textit{ then}$$
$$\delta S_t > 0 \ \textit{ and } \ \delta\phi_t > 0$$

These inequalities imply that if at any time in the high interest rate situation both stock and opportunity cost are increased or unchanged, then stock and opportunity cost would always afterward be increased. S_0 is unchanged by definition. Therefore by equation (24), both $\delta\phi_1$ and δS_1 must be positive or both must be negative. Therefore if $\delta\phi_1 > 0$, then $\delta S_1 > 0$, and hence $\delta\phi_t > 0$, and $\delta S_t > 0$ for all subsequent time.

But if at the final time (no matter how far distant) both stock and opportunity cost were both to be increased as a result of the price increase, then equation (26) could not hold: both opportunity cost and stock would be positive. This contradiction shows that $\delta\phi_1$ is strictly negative and the initial extraction rate must be higher in the higher interest rate case[28].

The net result of the interest rate increase then is that from any stock level, the firm would extract that stock at a higher rate and would, as time goes on, be left with smaller stocks. Over time, these smaller stocks would reduce the optimal extraction rate from that which would be optimal had the interest been lower and had more stock been left for future use. Higher interest rates would therefore move extraction toward the present from the future and would increase the total amount of the resource ultimately extracted. This result is similar to that obtained in the absence of stock effects.

h. The role of externalities

As discussed above, some environmental or national security externalities associated with depletable resource extraction or use imply costs which depend upon the rate of depletable resource extraction. Since externalities imply a divergence between the price and the social value of the output, then the results derived in previous sections for price changes can be used directly to analyze such externalities.

Other externalities, however, may be related to the stock of the resource extracted to date. Subsidence of land, the amount of carbon dioxide accumulation in the atmosphere, and some local environmental impacts are more related to the total quantity of the resource extracted to date. Here we assume there are some environmental damages whose costs increase with the amount of the resource extracted (and hence decrease with the amount of the resource remaining). We will adopt the assumption of time invariance in order to illustrate the results but the general conclusions can be obtained for a more general case.

This formulation implies that $\partial C/\partial S$ is more negative when the externalities are internalized. Such impacts can be analyzed by use of equations (24) and (25), in which the only change is that for a given value of ϕ and S, the right hand side of equation (25) is reduced when externalities are incorporated.

Figure 15 can be used to illustrate this situation as well as that of changed interest rates, modifying the "Low r" label to "Internalize Externalities" and the "High r" label to "Ignore Externalities". The ϕ-constant locus is increased when the stock related externality is internalized and therefore in steady state the opportunity cost and the final stock are both increased. Less of the resource is extracted cumulatively over time.

The convergent trajectory is higher for all stock levels when externalities are internalized, as illustrated in Figure 15. This result can be established by assuming the opposite, that for some stock at some time the opportunity cost is less when externalities are incorporated. Then the opportunity cost will always be less, contradicting the conclusion that in steady state the opportunity cost will be increased. The formal mathematical equations would be analogous to equations (40) and (41), except that the externality (the change in $\partial C/\partial S$) would appear in equation (40) in place of $\phi_{t-1}\,\delta\rho$.

Higher opportunity costs imply lower extraction rates for a given stock level. The net result of internalizing stock related externalities is a reduction in the depletable resource extraction rate and a decrease in the amount ultimately extracted. This result is different from that obtained for flow-related externalities. When the externality depends on extraction rate rather than on stock, then the direction of bias depends on whether the present value of the marginal external cost grows or declines. If flow-related externalities are small now but are expected to be large at a later time, then an incorporation of external costs into decisionmaking would lead to <u>more</u> extraction in early years, avoiding those times in which the marginal externality cost were large. However, for stock related externalities this is not true. Internalization of <u>any</u> stock related externalities (for which marginal external costs are positive) will result less rapid extraction of the depletable resource with smaller quantities ultimately extracted.

We turn now from analysis of individual deposits to analysis of depletable resources in a market environment.

III. EXTRACTION WITH PRICES DETERMINED ENDOGENOUSLY

The theory presented so far is directed toward the profit maximizing extraction trajectories of a single resource or a group of resources operating in a competitive market. In particular, each decisionmaker correctly believes that his or her actions alone have no effect on the overall market price. In the theory presented in the previous sections we did not address the issue of the price determination and in particular, the impact of resource extraction patterns on the prevailing prices.

In this section, we address joint determination of prices and quantities in markets. We first address purely competitive markets and then turn to monopolistic markets. More complex market structures are not addressed here but are examined in the chapters by Newbery and Karp and by Teece and Sunding.

A. Competitive Equilibrium

The concept of competitive equilibrium for depletable resource markets is fundamentally identical to the competitive equilibrium concept for conventional commodities. Each firm and consumer is so small that its input and output decisions can have no significant impact on the prices prevailing in the market. Market clearing occurs when the quantity supplied equals the quantity demanded in each market.

In addition, however, markets are explicitly linked over time in that firms can substitute extraction among time periods. It is not meaningful to address a static concept of competitive equilibrium which fails to recognize the essential intertemporal linkage through depletable resource supply. Thus competitive equilibrium for a depletable resource requires market clearing in <u>each</u> time period.

Definition: <u>Competitive Equilibrium</u>. Competitive equilibrium is said to occur if each agent is so small as to have no individual impact on prevailing prices and if for each time period the quantity of the commodity extracted by all firms together is equal to the quantity of the commodity demanded.

We define E^i_t as the quantity of the commodity extracted by the i^{th} firm at time t. Then the market supply at time t, Q_t, is equal to the summation of extraction quantities over all firms:

$$Q_t = \sum_i E^i_t \qquad (42)$$

This market supply will be a function of all current and future commodity prices, current stocks remaining in each resource deposit, cost functions for each deposit, and the interest rate. The impacts of these factors on the market supply function derives directly from the impacts discussed above for individual deposits.

The market demand for the commodity will be represented by a conventional static demand function. Although demand can be analyzed more fully, as shown in the chapter authored by Jorgenson and Wilcoxen and that authored by Slade, Kolstadt, and Weiner, we will adopt this fairly simple static formulation in the remainder of this chapter. We will denote $Q^D(P_t,t)$ as the demand function at time t. We will assume that $Q^D(P_t,t)$ is a continuous decreasing function of price.

Competitive equilibrium is characterized by a sequence of prices over all time such that supply equals demand at each t:

$$Q_t = \sum_i E^i_t = Q^D(P_t,t) \qquad \text{for all } t \tag{43}$$

In most of our analysis it will be more convenient to use the inverse demand function $P(Q_t,t)$ to determine market clearing price as a function of the market quantity supplied:

$$P_t = P(Q_t,t) \tag{44}$$

The inverse demand function is defined so that $Q^D(P(Q_t,t),t) = Q_t$ for all values of Q_t. If market demand is finite for $P_t = 0$, the inverse demand function is defined to be zero for all market quantities greater than or equal to the market demand at zero price[29].

In competitive equilibrium, then, equations (15) and (16) or (24) through (26) must hold for each firm, and equations (42) and (44) must link the firms to one another through a market.

Depletability of the resource immediately implies than in a competitive equilibrium there is an upper limit on the cumulative quantity of the resource demanded over all time. Summing equation (43) over time gives:

$$\sum_{t=1}^{T} Q^D(P_t,t) = \sum_{t=1}^{T} \sum_i E^i_t = \sum_i \sum_{t=1}^{T} E^i_t \leq \sum_i S^i_0 \blacktriangle S \tag{45}$$

The right hand side of equation (45), S^A, is the initial aggregate resource base. Depletability of the resource implies that the cumulative demand can be no larger than the initial aggregate resource base. For T arbitrarily large, the cumulative demand remains bounded above by the natural endowment of the resource. At some time as the resource is depleted, prices must increase enough to drive demand to zero or demand functions must shift so as to reduce or eliminate demand without large price increases. For an infinite time horizon, price must ultimately increase enough for demand to converge to zero.

There may exist some price which reduces demand to zero. In that case, there will always exist some price trajectory for which inequality (45) holds. We will refer to the price which just reduces demand to zero as the "choke price", and denote it as P^C_t. Note that the choke price may itself be a function of time.

The existence of a choke price can be related either to the demand function for the extracted commodity or to the supply function for a perfect substitute. On the demand side of the market, consumers may be willing to forgo all use of a particular commodity or class of commodities if price rises enough. Dasgupta discusses such possibilities in his chapter within this volume. Commodities can be described as "essential" or "non-essential", depending upon whether demand can be driven to zero at finite prices. If a commodity is "non-essential", it will have a finite choke price.

On the supply side, there may be technologies which allow large or unlimited quantities of a perfect or virtually perfect substitute to be produced at some price. Such "backstop technologies", as they have been dubbed by Nordhaus, can assure the existence of a choke price, with choke price equal to the average or marginal cost of the backstop technology. Even essential commodities may have a choke price if a backstop technology exists.

If no choke price exists, inequality (45) could still be satisfied, even for arbitrarily long time horizons, as long as demand converges towards zero for high enough prices. For this chapter, however, we will assume that a choke price always exists, since such an assumption allows us to avoid mathematical difficulties. We will not differentiate between the two reasons for the existence of a choke price, but will simply assume that the commodity is non-essential.

Assumption: Non-essential, static, continuous demand. Market demand in each time will be a continuous function of price at that time and will be independent of price at other times. A finite choke price will exist at each time.

1. Existence of Competitive Equilibrium

The first question is whether a competitive equilibrium exists at all. Under our assumptions that cost functions are convex and that a choke price exists, we can establish existence of a competitive equilibrium.

We start with markets for which all cost functions are strictly convex. With strict convexity, each price trajectory leads to a unique optimal extraction path for each deposit. This extraction path is a continuous function of the price trajectory. For this situation we can use the Brouwer fixed point theorem to establish the existence of a competitive equilibrium.

A fixed point of a function f is a point x such that x = f(x). The simplest fixed point theorem, due to Brouwer, is applicable to functions which map an m-dimensional real number, denoted by x, back to m-dimensional real numbers. In particular, if X is a subset of the space of m-dimensional real numbers, $\boldsymbol{R}^m$, the Brouwer fixed point theorem provides conditions under which one can assure that a fixed point exists for a function which maps X into itself[30].

Brouwer Fixed Point Theorem: If X is a compact convex subset of $\boldsymbol{R}^m$ and f is a continuous function mapping X into X, then there exists a fixed point of f.

To establish existence of a competitive equilibrium, we will define f to be a mapping from market quantity trajectories back to market quantity trajectories. A market quantity trajectory is an m-dimensional vector, with m corresponding to the number of time periods before the time horizon. Let X be the set of market quantity trajectories such that $0 \leq Q_t \leq S^A$ for each t, limiting the possible market clearing quantity trajectories to a set for which market quantity in any time period is no greater than the aggregate total resource base, S^A. Then X is a compact convex subset of $\boldsymbol{R}^m$. For every market quantity trajectory in X, the inverse demand functions, $P_t(Q_t,t)$, as defined in equation (44), map that quantity trajectory to a unique trajectory of prices. For every price trajectory, there is a unique optimal extraction trajectory for each deposit (under the strict convexity assumption). Those individual extraction trajectories define a trajectory of market supply, Q_t, as indicated by equation (42), thus completing the mapping. The physical constraints facing each firm assure that the trajectory of market supply is a member of set X. The demand function is a continuous downward sloping curve, and thus the inverse demand function is a continuous function; the mapping from the market quantity trajectory to the price trajectory is a continuous function. Strict convexity of cost functions implies that the mapping from the price trajectory back to the market supply trajectory is also a continuous function. Thus the mapping from the market quantity trajectory to the price trajectory back to a market quantity trajectory is a continuous mapping from set X into set X.

All of the premises of the Brouwer fixed point theorem are met. Thus its conclusion holds: there exists a fixed point of the mapping we have constructed. That fixed point is a market demand quantity trajectory which leads to a market price trajectory, which in turn leads back to the same market quantity trajectory. Alternatively, it can be envisioned as a price trajectory which leads to a market supply and a market demand trajectory which are identical to one another. A competitive equilibrium exists.

The role of the choke price should be noted in particular. With finite choke prices, it is possible to define a price trajectory for any market quantity trajectory in X, since for each possible market supply trajectory, the inverse demand functions imply that $0 \leq P_t \leq P^C{}_t$ for each t. If choke prices do not exist, then price trajectories do not exist for some market quantity trajectories in X and thus the mapping needed for the Brouwer fixed point theorem cannot be defined.

To illustrate possible non-existence of a competitive equilibrium, assume that the commodity were essential and that market demand always exceeded the quantity Q^M for all finite prices until the time horizon of T. Then cumulative market demand must exceed Q^M T for all finite prices. If $Q^A < Q^M T$, then no competitive equilibrium will exist. Prices would be driven to infinity and markets would still not clear.

We now turn to markets for which cost functions for some deposits are weakly, but not strictly, convex. For example, Hotelling costs are in this class. In this situation market supply for a given price trajectory is not unique. Rather a range of extraction rates may be consistent with profit maximizing for each firm. Thus we can define only a correspondence -- a set valued mapping -- from the price trajectory to the market supply trajectory.

For our analysis we will follow the same approach as above, using inverse demand functions to provide a mapping from the same set X of possible market quantity trajectories. Optimizing choices of firms will define a correspondence back to a set of market quantity trajectories. The existence of a fixed point will be established. The fixed point will describe a trajectory of market clearing prices and resulting trajectory of market demands. This demand trajectory will equal one of the (possibly many) optimal supply trajectories for the market clearing price trajectory.

The Kakutani fixed point theorem provides the necessary mathematical tool to establish the existence of a competitive equilibrium under the weak convexity assumption[31]:

<u>Kakutani fixed point theorem</u>. Let X be a compact convex subset of $\boldsymbol{R}^m$, and let γ be a closed correspondence from X into subsets of X. If $\gamma(x)$ is a convex set for every $x \in X$, then there is a fixed point.

To establish existence of a competitive equilibrium, again let X be the set of market quantity trajectories such that $0 \leq Q_t \leq S^A$ for each t. X is a compact convex subset of $\boldsymbol{R}^m$. For every market quantity trajectory in X, the inverse demand functions, $P_t(Q_t,t)$, as defined in equation (44), map to a unique trajectory of prices. Optimizing responses of firms to the price trajectory complete the mapping back to market quantities. The physical constraints facing the firm assure that all trajectories of market supply belong to set X.

The set of optimal extraction trajectories for each deposit must be convex. To see this, consider a convex combination of two optimal trajectories, with the first trajectory given a weight of β and the second given a weight of $1-\beta$, where $0 < \beta < 1$. The convex combination would remain feasible since it would have no more cumulative extraction than would the maximum of the two optimal trajectories. The present value of the revenue from the convex combination would be a weighted average of the present values of the two revenues, with weights corresponding to β and $1-\beta$. The present value of the costs would be no greater than the weighted average of the two optimal present values of costs (with weights of β and $1-\beta$), under the assumption that the cost function is convex. Therefore the present value of profit from the convex combination could be no smaller than the weighted average of present values of profits for the two optimal trajectories. But if the two trajectories are optimal, they must give the same present value of profit and no feasible trajectory could give strictly greater present value of profits. Thus the convex combination must give identical present value of profits as either of these two optimal trajectories: the set of individual optimal trajectories is convex.

The market demand is a summation of individual demands and thus the set of market supply trajectories is the set summation of optimal supply trajectories from individual firms. Because the set summation of convex sets is itself a convex set, the set of market supply quantity trajectories must be convex for every price trajectory and thus for every initial market demand quantity trajectory.

We need only to show that the correspondence so constructed is closed. However, every upper hemi-continuous mapping must be closed, so we need only show that the mapping is upper hemi-continuous. Finally, to establish the upper hemi-continuity of the correspondence we must invoke the theory of the maximum[32]:

<u>Theorem of the Maximum:</u> Let f(x,a) be the objective function of the constrained maximization problem: Maximize f(x,a) such that x is in G(a). Assume that f(x,a) is a continuous function with a compact range and that the constraint set G(a) is a nonempty, compact-valued continuous correspondence of a. Then x(a) is an upper-hemicontinuous correspondence.

For our problem, the constraint set for each firm is fixed, nonempty, and compact. The objective function is continuous in both prices and in extraction trajectories. The market demand function is a continuous downward sloping curve, and thus the inverse demand function is a continuous function of market demand. Therefore the objective function is continuous in the market quantity trajectory as well. For each firm, the feasible set of extraction quantities is a compact set. Thus the set of optimal extraction trajectories must be a compact set for any given price trajectory. The set of market supply trajectories is the set summation of individual firm extraction trajectories. Compactness of the individual sets implies compactness of the set summation. Hence the conditions of the theory of the maximum are met and thus the correspondence we have defined is upper hemicontinuous and hence closed.

All premises of the Kakutani fixed point theorem are met. Thus its conclusion holds: there exists a fixed point of the mapping we have constructed. Thus a competitive equilibrium exists even with weak convexity of the cost functions.

Given the existence of an equilibrium, we can now turn to characteristics of the price and quantity trajectories in competitive equilibrium. The nature of these trajectories will depend on several factors:

- The direction and rate of change of the demand function over time;
- The rate of technological change in resource extraction; and
- The nature of the cost function.

These relationships will be examined in what follows. We start with the classical Hotelling cost assumption, progress through non-Hotelling models without stock effects and end with models which include stock effects.

2. Hotelling Cost Models

For this section we assume that the Hotelling cost assumption is satisfied. There may be many deposits, each characterized by different costs and initial stocks.

Competitive equilibrium price and quantity trajectories can be solved comparatively simply under Hotelling assumptions. By equation (21), extraction can be positive for resource deposit i only during that time interval in which price is growing along a Hotelling path:

$$P_t = c^i + \lambda^i e^{rt} \qquad (46)$$

Here c^i represents the marginal cost for resource i and λ^i, the present value opportunity cost associated with resource stock i. The opportunity cost, λ^i, is equal to the maximum (over time) of the present value of $P_t - c^i$. Whenever price is below $c^i + \lambda^i e^{rt}$, there would be no extraction from the i^{th} deposit. If $\lambda^i > 0$, then deposit i must ultimately be totally depleted.

We can explore properties of competitive equilibrium, assuming that all firms satisfy the Hotelling cost assumption. We will assume that any time horizon is far enough in the future that extraction ceases because of resource depletion, not because of a time horizon. We start first with the simplest case: Hotelling cost models with no technological progress. The demand function, however, will be allowed to depend on time. Once results are presented, we will examine the differences in results if there is technological progress.

a. Hotelling Cost Models With No Technological Change

We assume that there is no technical progress so that extraction costs for any given deposit are independent of time. Demand functions may be time varying. Several results will emerge from the analysis. (1) Whenever demand is positive, price will always be growing and present value of price always declining. (2) If the demand function is time invariant or shrinking over time, then market supply and demand will decline over time. (3) If several resource deposits have different costs, then resources will be extracted strictly in order of increasing cost, except that at the transition time between two deposits both could be extracted simultaneously. (4) Higher cost resources will have lower opportunity costs. (5) Each resource with an extraction cost below the maximum market price at will ultimately be totally depleted. (6) Once started, extraction from a deposit will not stop until it is totally depleted, unless there is sufficient stock of the resource for it to be a "backstop technology." We will now demonstrate these propositions.

Whenever demand is positive, price will always be growing and present value of price always declining. By equation (46), for extraction to occur from deposit i, price must be changing along the following path where λ^i is strictly positive for any resource which will ultimately be totally depleted:

$$P_t = c^i + \lambda^i e^{rt}$$

Along this path, price is growing but the present value of price, $P_t\ e^{-rt}$, is declining. If price were declining or growing so rapidly that $P_t\ e^{-rt}$ were increasing, the equation above could not hold for any deposit and market supply would equal zero. That would not be a competitive equilibrium if there were positive demand.

If the demand function is time invariant or shrinking over time, then market supply and demand will decline over time because price must always be rising. However, if the demand function is increasing rapidly enough, then market clearing quantities can grow over time. For example if, for some time, the inverse demand function increases at the interest rate, so that $P(Q_t,t) = P(Q_t)\ e^{rt}$ for a time interval, then $P_t\ e^{-rt}$ declining implies that market clearing supply and demand must be growing.

The order of competitive extraction will be strictly from low to high cost. To demonstrate, assume the converse, that deposit i is more costly than deposit j -- $c^i > c^j$ -- but that some quantity of resource i is extracted before some quantity of deposit j is extracted -- $E^i_t > 0$ and $E^j_\tau > 0$ for two times t, and τ, for $\tau > t$.

By equation (46) both the following inequalities must hold:

$$P_t = c^i + \lambda^i e^{rt} \geq c^j + \lambda^j e^{rt}$$
$$P_\tau = c^j + \lambda^j e^{r\tau} \geq c^i + \lambda^i e^{r\tau}$$

The inequalities can be rewritten as follows:

$$[\lambda^j - \lambda^i]\, e^{rt} \leq c^i - c^j \leq [\lambda^j - \lambda^i]\, e^{r\tau}$$

By these inequalities, $\lambda^i - \lambda^j > 0$, and thus these inequalities also imply that $e^{rt} \leq e^{r\tau}$, a contradiction, because $\tau > t$ and r is positive. It is thus established that in a competitive equilibrium with time invariant

Hotelling costs, deposits must be extracted strictly in order of cost, with low cost deposits used up before higher cost deposits are initiated.

At the moment of transition between two deposits, both would typically be extracted simultaneously[33]. Therefore at such times of transition, equation (46) must hold for both resources. If τ is the transition time from the j^{th} to the i^{th} deposit, then:

$$\lambda^j - \lambda^i = [\, c^i - c^j \,]\, e^{-r\tau} \qquad (47)$$

Equation (47) shows that higher cost deposits will have lower opportunity costs.

In the Hotelling case competitive equilibrium, price exceeds the unit cost of the deposit being extracted by an amount equal to the opportunity cost of that resource. Opportunity cost increases at the rate of interest. When the next resource is being extracted, its opportunity cost becomes the relevant one for determining price and the price increases based on growth of that lower opportunity cost.

Ultimately, all deposits will be fully depleted if their costs are strictly below the maximum value that the market price reaches over time, P^M. If a deposit were not fully extracted, then its opportunity cost would be zero. But if firm i's opportunity cost were zero and $P^M > c^i$, it would extract its entire stock. P^M must be at least as high as the maximum value of the choke price at the time of final shutdown or afterwards. Otherwise there would be a positive demand for the resource. Therefore, all deposits whose costs are below any value of the choke price at or after final shutdown will ultimately be fully depleted.

Since 1) resources will be extracted strictly in order of increasing cost, and 2) a resource will ultimately be fully depleted if price ever exceeds its cost, then once started, extraction from a deposit will not stop until it is totally depleted. The exception is if the resource is the last being extracted, market price at maximum just equals its cost, and its stock is sufficiently large to satisfy all demand for all time that choke price remains above its extraction cost. If this condition holds, then the resource satisfies all characteristics of a "backstop technology."

Market quantity can either increase or decrease over time, depending upon changes in the demand function. For demand functions that do not grow rapidly, market clearing quantities will decline over time. However, for rapidly growing demand functions, quantities can grow.

Based on these characteristics, in what follows we develop equations to allow explicit calculation of market clearing prices and quantities for resources with Hotelling costs.

At the time of transition from the j^{th} to the i^{th} deposit, all deposits with cost lower than c^j must be depleted, while at the end of the preceding time some of the j^{th} deposit will remain. Cumulative demand to time τ must be at least as great as the total initial stock of resources having cost j or smaller but cumulative demand up to time τ-1 must be smaller than the total initial stock of resources having cost smaller than or equal to c^j:

$$\sum_{t=1}^{\tau-1} Q^D(P_t,t) < \sum_{\{k: c^k \le c^j\}} S^k_0 \le \sum_{t=1}^{\tau} Q^D(P_t,t) \tag{48}$$

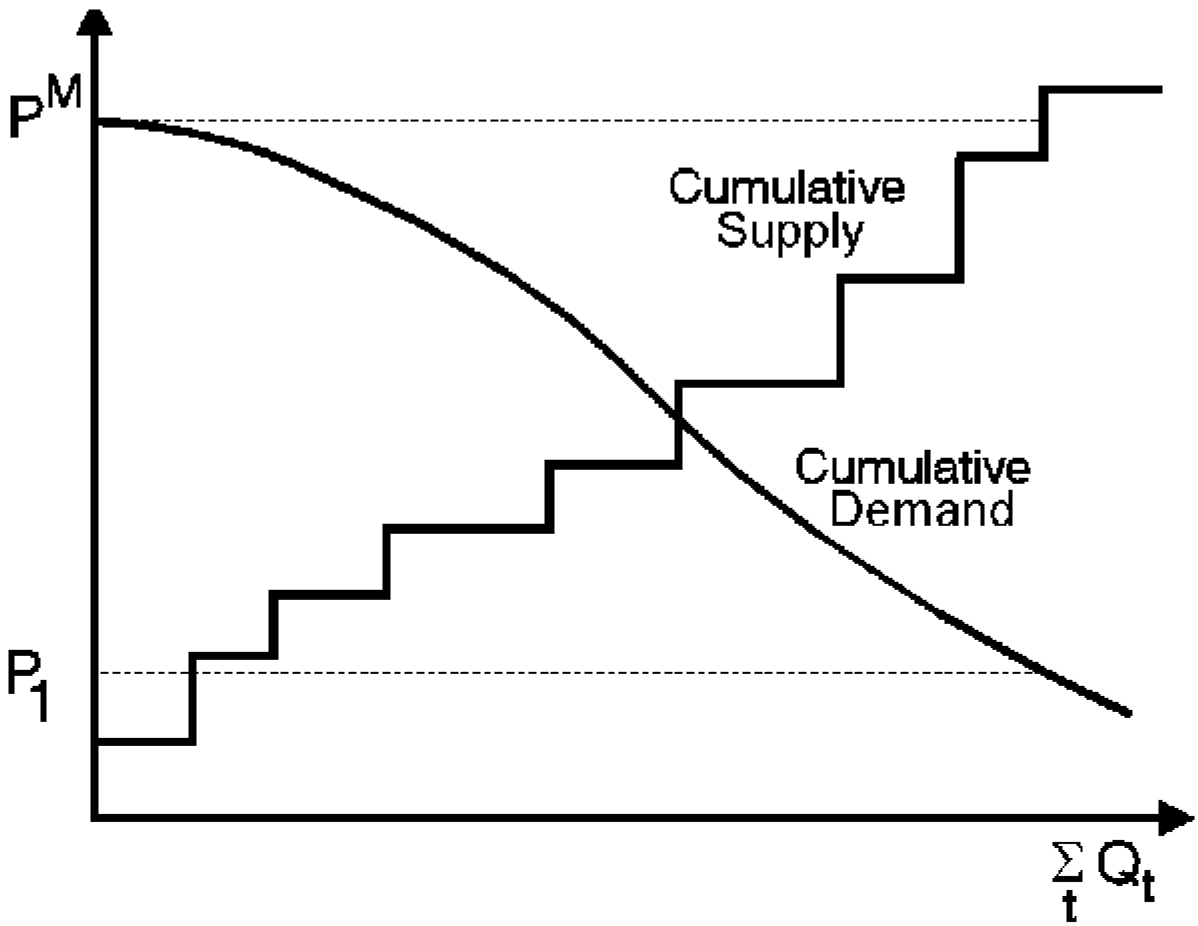

Figure 15
Determination of the Initial Price: Hotelling Costs

At the final time, the right hand inequality of (48) may be an inequality or a strict equality, depending upon whether the maximum price ever obtained is equal to or greater than the unit cost of the most expensive resource extracted:

$$\sum_{t=1}^{T} Q^D(P_t,t) \quad \begin{matrix} = \\ \le \end{matrix} \quad \sum_{\{k:c^k \le P^M\}} S^k_0 \quad \begin{cases} \text{if } Max\ \{c^k : c^k \le P^M \\ \text{if } Max\ \{c^k : c^k \le 1 \end{cases} \qquad (49)$$

Here P^M denotes the maximum value reached by P_t over all time.

The competitive equilibrium price path can be determined uniquely by equations (46) through (49). If P_1 is known, equation (46) determines price until the lowest cost deposit is totally depleted and determines opportunity cost for that deposit. Once the deposit is depleted, equation (47) determines opportunity cost for the next lowest cost deposit. Inequality (48) determines the transition timing. Equation (46) determines price until the next transition, and so on. The final time of positive extraction is determined as the last time before price equals or exceed the choke price (unless extraction occurs up to a final time horizon).

Finally, equation (49) must also hold in equilibrium. Unless choke price is declining over time, P^M would typically equal choke price at final shutdown of extraction. With a discrete set of extraction costs (as might be typical in a quantitative model), the right hand side of equation (49) is a step function, with cumulative extraction increasing in steps as P^M increases. The right hand side will depend on initial price, P_1, to the extent that P_1 influences time of final shutdown and choke price depends on final shutdown time. The left hand side of this equation represents cumulative demand until final shutdown and is a decreasing, continuous function of P_1. If P_1 is too high, cumulative demand is reduced, and resources for which $c^i < P^M$ will remain after all extraction ceases. If P_1 is too low, the cumulative demand will exceed the total stock of resources with costs no greater than P^M. Neither case would be a competitive equilibrium. A unique P_1 exists which allows (49) to be satisfied.

Determination of P_1 through equation (49) is diagrammed in Figure 16 for a time invariant demand function. Cumulative demand is a decreasing function of P_1. With time invariant demand, cumulative demand is zero for $P_1 = P^M$ because P^M just equals the choke price. Cumulative supply is an increasing step function of P^M. This diagram illustrates choke price determining which resources will be depleted and thus determining the cumulative supply. Cumulative supply determines cumulative demand, which in turn determines P_1.

b. Hotelling Cost Models with Technological Progress

With technological progress in resource extraction, costs decline over time. In this case each of the equations (46) through (49) must continue to hold, but with each c^i declining over time. Resources may not be extracted strictly in order of increasing costs. If the low costs decline rapidly enough relative to the high cost resources, then competitive firms could extract low cost resources after higher cost deposits.

In addition, equilibrium prices need not necessarily grow. By equation (46), if c^i shrinks more than $\lambda^i e^{rt}$ grows, then price will decline over time. For example, assume that each cost function decreases at the same exponential rate. In that case, low cost resources will still be extracted before high cost resources. Higher cost resources will still have lower opportunity costs. Right after transition from a lower cost to a higher cost resource the price growth rate must decline and could become negative. While that resource is being extracted, the opportunity cost absolute growth increases and unit cost absolute decline will become smaller. Thus price could again begin growing after some time. The process could then repeat as progressively poorer grades of resources are extracted. With technological progress, equilibrium price may either increase or decrease over time and could alternate between periods of price growth and periods of price decline. Market clearing quantities could similarly alternative between growth and decline.

<u>3. Non-Hotelling Models Without Stock Effects</u>

We now turn to more complex models in which marginal extraction costs from individual deposits are increasing functions of extraction rate, but remain independent of remaining stock. In competitive equilibrium of such a model, the quantity supplied for each resource as a function of the price trajectory is determined by equations (15) and (16). Market demand as a function of price is determined by the demand function and equation (43) links the market together, equating market supply to market demand.

Characteristics of the equilibrium will depend upon properties of the cost functions and of the demand function. We will first explore models in which all cost functions are time invariant but in which the demand function may vary over time. We will then relax the assumption of time invariant demand functions and admit technical progress.

We assume now that there is no technical progress so that extraction costs for any given deposit are independent of time. Several results will emerge from the analysis. (1) If the inverse demand function is time invariant or growing, equilibrium price will always be growing. If the present value of the inverse demand function is never growing, then the present value of equilibrium price will always be declining. (2) If the inverse demand function is time invariant or declining over time, then market supply and demand will decline over time. If the present value of the inverse demand function is growing, then market supply and demand will increase over time. (3) If several deposits have different costs but identical initial stocks, these deposits will generally be extracted in order of increasing cost, but typically there will be deposits of several different costs being extracted simultaneously. (4) Higher cost deposits will have lower opportunity costs for initial stock fixed and larger deposits will have lower opportunity costs for cost functions fixed. (5) Each resource will ultimately be totally depleted if the market price remains above its minimum marginal extraction cost for long enough.

These propositions are far more limited than those obtained for Hotelling cost functions. Thus insights from Hotelling models cannot necessarily be generalized to non-Hotelling models. We will demonstrate these propositions in turn.

The time pattern of increases or decrease in the market clearing price trajectories and quantity trajectories will depend upon the rate at which the inverse demand function shifts. We will identify three ranges: (a) $P(Q_t,t)$ declining over time, (b) $P(Q_t,t)$ increasing over time but $P(Q_t,t)\ e^{-rt}$ decreasing over time, and (c) $P(Q_t,t)\ e^{-rt}$ increasing over time. These three ranges lead to the two sets of propositions above about price changes and quantity changes. In the middle range we can place upper and lower bounds on price changes but no bounds on quantity changes. In the upper range we can put lower bounds on price and quantity changes while in the lower range we can put upper bounds on both rates of change.

If the inverse demand function is time invariant or growing, equilibrium price will always be growing. By equation (15), if the equilibrium price were constant or declining, extraction rate from each firm would decline over time and the market supply would decline. Declining supply (and hence demand) would imply increasing prices, a contradiction.

If the present value of the inverse demand function is never growing, then the present value of equilibrium price will always be declining. By equation (15), if $P_t\ e^{-rt}$ were constant or increasing from one time period to the next, extraction would be increasing for each deposit and thus market supply would be growing over time. Growing market supply (and hence market demand) implies that $P_t\ e^{-rt}$ would be declining over time, a contradiction.

If the inverse demand function is time invariant or declining over time, then market supply and demand will decline over time. If quantities were increasing, then prices would be declining. But by equation (15), declining prices imply declining extraction rates from all deposits and hence declining market supply, a contradiction.

If the present value of the inverse demand function grows, then market supply and demand will increase over time. If quantities were decreasing, the present value of price would be increasing. But by equation (15), increasing present value of prices implies increasing extraction rates from all deposits and hence increasing market supply, a contradiction.

In summary, unless demand functions are shrinking over time, prices must rise until the choke price is reached. The rate of price growth will depend upon movements of the demand function, characteristics of individual cost functions, the interest rate, and the sizes of the various deposits. The overall characteristics for the three ranges of inverse demand function change are displayed in Table 2.

Rate of Change: Inverse Demand Function	Rate of Change: Market Clearing Prices	Rate of Change: Market Clearing Quantities
Declining	Present value declining	Declining
Growing, Present value declining	Increasing, Present value declining	
Present value growing	Increasing	Growing

Table 2
Time Patterns of Market Clearing Price and Quantity Trajectories

Assume that there were two deposits in a market and that one had a lower marginal cost than the other for each rate of extraction. The first will be referred to as the "low cost" deposit and the second the "high cost" deposit. We assume that they are both ultimately totally extracted. In market clearing, if both initial stocks are the same size, then the high cost deposit will have a lower opportunity cost than will the low cost deposit. (If they were ultimately not totally depleted, then they would have zero opportunity costs.) To see this, assume that the opportunity cost of the low cost deposit were at least as low as that of the high cost deposit. Then by equation (15), the low cost deposit would have a higher extraction rate at all times[34]. But because both have the same initial stock, this would be impossible.

If two deposits have the same marginal cost function, but different initial stocks, the larger deposit will have a lower opportunity cost than the smaller. If the reverse were true, then by equation (15), the smaller deposit would have higher extraction over all time, which again would be physically impossible, unless the deposits were ultimately not totally depleted.

Deposits will generally be extracted in order of increasing cost, but typically there will be deposits of several different costs being extracted simultaneously, even though they may each have different cost functions. This pattern corresponds far more closely to that found in reality than the does the pattern forced by Hotelling cost functions, in that many different resource grades could be extracted simultaneously under this model.

More precisely, two propositions can be established to formalize the sense in which deposits are generally extracted in order of increasing cost. Extraction from the lower cost deposit must either be initiated first, terminated first, or both. When the lower cost deposit reaches its peak extraction rate, the extraction rate of the higher cost deposit will still be increasing.

Let i represent the high cost deposit and j the low cost deposit. Then $\lambda^j > \lambda^i$. Assume that extraction from deposit i is positive and extraction from deposit j is still zero at time τ. Then, because both deposits face the same price, equation (15) implies:

$$dC^i_\tau/dE^i_\tau + \lambda^i e^{r\tau} < C^j_\tau/dE^j_\tau + \lambda^j e^{r\tau}$$

where both marginal costs are evaluated at zero extraction rate. Because $\lambda^j > \lambda^i$, the right hand side the above inequality grows over time more than does the left hand side. Thus the inequality must remain valid for all future time. Consider that time at which the extraction rate of deposit j (the low cost deposit) just declines to zero. Then this inequality implies that the extraction rate of the high cost resource must still be positive, in fact, must be higher than it was when extraction was first initiated for the low cost resource. Hence if the high cost extraction is initiated before the low cost extraction, then extraction must stop for the low cost resource first.

We can now demonstrate that when the lower cost deposit reaches its peak extraction rate, the extraction rate of the higher cost deposit will still be increasing. If both are being extracted simultaneously, then the following equation must hold, where the marginal costs are now evaluated at the positive extraction rates:

$$dC^i_t/dE^i_t + \lambda^i e^{rt} = C^j_t/dE^j_t + \lambda^j e^{rt}$$

When the extraction rate from deposit j just reaches its maximum, extraction rate remains constant from that period to the next. Thus the marginal extraction cost of resource j just remains constant. However the opportunity costs continue to grow, with $\lambda^j e^{rt}$ increasing by more than does $\lambda^i e^{rt}$ (since $\lambda^j > \lambda^i$).

Thus the marginal extraction cost of resource i must continue to increase from one time period to the next for the equation above to hold. Therefore the i^{th} extraction rate must still be increasing; the extraction rate from the i^{th} resource will not yet be at its peak[35].

For models assuming Hotelling costs, we concluded that each deposit will ultimately be totally depleted if the market price ever gets above its marginal extraction cost. That conclusion is not valid for non-Hotelling models. We can only conclude that each resource will ultimately be totally depleted if the market price remains above its minimum marginal extraction cost for long enough. If it were not totally depleted, then the opportunity cost would be zero and extraction rate would be positive as long as market price exceeded the minimum marginal extraction cost. With enough time the deposit would be totally depleted.

Once technical progress is introduced, fewer results can be established. In particular, if extraction costs decline over time, then prices tend to increase less rapidly or may decrease even though they would have increased absent technical progress. In Table 2, all results predicting declines in price or present value of price or those predicting growth in quantities can still be established. Those predicting increases in price or present value of price or those predicting declines in quantities cannot.

Other results must be weakened or eliminated as well. Since models with Hotelling costs are extreme limiting cases of this class of models, the wide range of possibilities for Hotelling solutions implies an even wider range for these models.

Additional characteristics of the competitive equilibrium, either with or without technical progress, are dependent upon the specific demand function and cost functions. Typically numerical simulation is required to solve such models. The equations presented within this chapter provide a complete basis for such numerical simulations, once market demand functions, costs functions, and initial stocks are specified.

4. Models With Stock Effects

A more complex case is one in which marginal extraction cost from any particular deposit is an increasing function of the extraction rate and a decreasing function of the remaining stock. In competitive equilibrium of such a model, the quantity supplied for each resource as a function of the price trajectory is determined by equations (24) through (26). Market demand as a function of price is determined by the demand function and equation (43) links the market together, equating market supply to market demand.

We assume now that there is no technical progress so that extraction costs for any given deposit are independent of time, although dependent on remaining stock. Unless otherwise indicated, we will assume that the time horizon is so far in the future that its existence has no significant impacts on the extraction patterns. In addition, we will assume that total cost and marginal cost are a decreasing function of the amount of the stock remaining.

A few results will emerge from the analysis: If the inverse demand function is time invariant or growing, equilibrium price will always be growing except when market supply and demand are zero[36]. A corollary follows: If the inverse demand function is time invariant market supply and demand will decline over time. Under the further assumption that $-\partial C/\partial S$ is convex in E, if the inverse demand function is time invariant or declining over time, market supply and demand will decline over time.

Note that these propositions are far more limited than those obtained for models without stock effects. Thus some of the conclusions based upon models without stock effects cannot be generalized to models with stock effects. We will demonstrate the propositions in turn.

If the inverse demand function is time invariant or growing, equilibrium price will always be growing (except when market supply and demand are zero) if the time horizon is so far in the future that its existence has no significant impacts on the extraction patterns. There exists <u>some</u> time τ after which[37] equilibrium price will not be lower than P_τ, since the system must at some time reach a choke price, which is itself constant or growing.

Equations (24) through (26) imply that if τ exists after which prices will not be lower than P_τ, then $P_{\tau-1} < P_\tau$. To demonstrate, assume that $P_{\tau-1} \geq P_\tau$. As a consequence, market supply and demand must be non-decreasing from time τ-1 to τ and thus so must be output from some deposit, say deposit i. By equation (24) , non-decreasing output with non-increasing price occurs only if $\phi^i_\tau < \phi^i_{\tau-1}$. Hence ϕ^i_τ must be below the ϕ-constant locus. Because future price will never be lower than P_τ, the ϕ-constant locus will not decline from its position at time τ; thus ϕ^i_τ will always stay below the ϕ-constant locus and can never converge to a steady state. Rather opportunity cost must forever decrease and must become and stay negative, violating condition (26). This contradiction implies that $P_{\tau-1} < P_\tau$. Hence it follows that price will never again be lower than $P_{\tau-1}$. The same logic then shows that price will never again be lower than $P_{\tau-2}$, and so on. Thus price must always be rising if market supply and demand are positive.

From this conclusion it follows directly that if the inverse demand function is time invariant then market supply and demand will decline over time, since in this situation prices will always be increasing whenever demand is positive.

Under the further assumption that $-\partial C/\partial S$ is convex in E, if the inverse demand function is time invariant or declining over time, market supply and demand will decline over time. If market supply and demand were increasing or constant from time τ-1 to τ then $P_{\tau-1} \geq P_{\tau}$ and $E^i_{\tau-1} \leq E^i_{\tau}$ for at least one deposit, i. But we have shown in a previous section of this chapter that under these assumptions, whenever price is declining, extraction must also be declining for each individual deposit. This contradicts the supposition that market supply in and demand were increasing or constant.

The results change dramatically if time were approaching a final horizon. Then it would be possible for the inverse demand function to either shrink or grow over time, yet have competitive equilibrium prices decline and output increase over time. We have seen from the example illustrated in Figure 12 that for an individual deposit, it is possible for both prices and output to decline over time as the final horizon is reached. If all deposits were similar to that individual deposit, then this would be true for the overall market supply. One could easily construct an inverse demand function which was either growing or shrinking and which gave the quantity path of Figure 12 when the price path of Figure 12 occurred.

5. Models with New Discoveries

In the work presented so far, locations of all resource deposits are known at the beginning of time and firms can extract from these deposits at any time. However, one important feature of most depletable resource systems is the process of discovery of new deposits. Discovery implies that these deposits were not available for extraction in the time prior to discovery, in contrast to the assumption made so far. The assumption of exogenous discovery can be readily incorporated into the theory, although endogenous discovery requires a careful discussion of uncertainty and information, a discussion beyond the scope of this chapter.

Assume that the various deposits will be discovered at various dates, denoted by $T_D{}^i$. Then for each deposit we need only add the additional constraint that $E^i_t = 0$ for $t < T_D{}^i$. If that constraint is added, the optimal extraction trajectory for each firm follows the same optimality conditions for all time after discovery.

Under this model, discovery of additional deposits has no effect on optimal trajectories of previously discovered deposits, except to the extent such discoveries change current prices or expectations of future prices. And if all participants in the market have rational expectations about future discoveries, the actual event of a new discovery need have no significant impact on any prices and thus would have no effect on optimal extraction of any other deposits[38].

Competitive equilibrium still would be characterized by supply and demand equality for all times. The properties of this competitive equilibrium can differ as a result of the addition of new resource deposits over time. Late discovery implies that more of the resource stock remains at any given time than would have been the case had the deposit been discovered earlier. Thus early year prices would be increased and extraction rates decreased and later year prices would be decreased and extraction rates increased if such discovery constraints were incorporated into the model. Therefore, even without technical progress, market clearing prices need not rise over time, except in Hotelling models. Prices could fall over a long period of time if new deposits were constantly being discovered. All deposits, once discovered, would be extracted at high rates initially and their extraction rates would decline over time. However, if sufficient numbers of new deposits were being discovered, this process could dominate the decline from existing deposits so that supply would be increasing over time, satisfying a growing demand.

Empirically, these observations are important, in that we have observed depletable resource prices declining over significant periods of U.S. and world history as new deposits become available. This price decline does not suggest the invalidity of depletable resource theory, since once new discoveries are incorporated into the competitive models, we can generate patterns of declining prices along with decreasing extraction from existing deposits and growing total market quantities. But such incorporation of new discoveries does make empirical research in depletable resource theory even more challenging.

We turn now from competitive market models to examine an alternative market structure.

B. Depletable Resource Monopoly

In the analysis so far, we have maintained the assumption that all firms and consumers are price takers in markets for commodities extracted from depletable resource stocks. However, as discussed more fully in the chapters by Teece and Sunding and by Newbery and Karp, many commodities extracted from natural resources are sold in non-perfectly competitive markets. The diamond cartel, the Organization of Petroleum Exporting Countries (OPEC), and the copper cartel are examples of organizations who have market power to influence prices at which their outputs are sold.

In this section we examine models which assume the opposite extreme from perfectly competitive markets, models of markets dominated by a single extractor of the depletable resource. Such monopolies, taking into account the impact of their output choices on market clearing prices, can generally be expected to operate differently from perfectly competitive firms. More complex structures are covered in the two chapters of this Handbook cited above. We will maintain the monopoly assumption throughout this section, unless otherwise indicated.

Assumption: <u>Monopoly.</u>
A single firm controls all deposits of the depletable resource. That firm chooses extraction patterns so as to maximize the discounted present value of its profit. There are many consumers of the commodity, all of which are price takers.

We will assume that a monopoly faces the same conditions as do competitive firms, except that it owns all of deposits, incurs all costs, and collects all revenues from extraction of the resource. The monopoly controls N resource deposits, numbered from 1 to N.

As in a competitive model, if markets clear, price is determined by the inverse demand function from equation (44). Revenue obtained from selling E^i_t units from resource deposit i depends upon the commodity price, which in turn depends on the total quantity sold from all deposits. Total monopoly revenue will be denoted as $R_t(E^1_t, E^2_t, \ldots, E^N_t)$:

$$R_t(E^1_t, E^2_t, \ldots, E^N_t) \;=\; \sum_i R^i_t(E^1_t, E^2_t, \ldots, E^N_t) \;=\; \sum_i E^i_t\, P_t(\qquad (50)$$

Optimization of each deposit must account for the impact of extraction from that individual deposit on prices facing all extraction.

The cost facing the monopolist is the sum of costs over all resource deposits. Underlying cost functions could be as simple as Hotelling cost functions or could be more complicated, with each marginal cost depending on the extraction rate and remaining stock in that deposit. We will go directly to the more general cost functions, recognizing that the simpler models are all special cases of the more general cost functions.

The problem facing the monopoly is then:

$$Max\ \Pi = \sum_{i} \sum_{t=1}^{T} [R^{i}_{t}(E^{1}_{t}, E^{2}_{t}, \ldots, E^{N}_{t}) - C_{t}(E^{i}_{t}, S^{i}_{t-1})]\, e^{-rt}$$

$$\text{Under:} \quad S^{i}_{t} = S^{i}_{t-1} - E^{i}_{t}, \quad E^{i}_{t} \geq 0, \quad S^{i}_{T} \geq 0 \tag{51}$$

1. Necessary Conditions for Optimality

This monopoly optimization can be solved using the same methods applied above. The Lagrangian can be written:

$$Max\ \mathfrak{L} = \sum_{i} \left\{ \sum_{t=1}^{T} \{ [R^{i}_{t}(E^{1}_{t}, E^{2}_{t}, \ldots, E^{N}_{t}) - C_{t}(E^{i}_{t}, S^{i}_{t-1})]\, e^{-rt} - [S^{i}_{t} - S^{i}_{t-1} + E^{i}_{t}]\, \phi^{i}_{t}\, e^{-rt_{t}} \} + \mu^{i} S^{i}_{T} \right\}$$

First-order necessary conditions require that at the optimal point, the Lagrangian must be a stationary point with respect to each S^{i}_{t} and E^{i}_{t}. Differentiating $\mathfrak{L}$ with respect to each variable and combining equations:

$$\partial R_{t}/\partial E^{i}_{t} \begin{matrix} = \\ \leq \end{matrix} \partial C^{i}_{t}/\partial E^{i}_{t} + \phi^{i}_{t} \qquad \begin{cases} \text{if } E^{i}_{t} > 0 \\ \text{if } E^{i}_{t} = 0 \end{cases} \tag{52}$$

$$\phi^{i}_{t} = \phi^{i}_{t-1}\, e^{r} + \partial C^{i}_{t}/\partial S^{i}_{t-1} \qquad \text{for } t < T \tag{53}$$

$$\phi^{i}_{T}\, S^{i}_{T} = 0 \tag{54}$$

Equation (50) can be differentiated to relate marginal revenue to market clearing price:

$$R'_t \triangleq \partial R_t/\partial E^i_t = P_t(Q_t) + Q_t\, \partial P_t/\partial Q_t = P_t(Q_t)\,[1 - 1/\epsilon_t($$ (55)

where R'_t is marginal revenue at time t, Q_t is the market quantity of the resource, and $\epsilon_t(Q_t)$ is the elasticity of demand, defined to be a positive number. R'_t is a function of Q_t and depends on the shape of the inverse demand function.

2. Characterizing Monopoly vs. Competitive Equilibrium Solutions

The conditions for monopoly optimality are very similar to the conditions describing the competitive equilibrium. Equations (52) through (55) are identical to equations (24) through (26) plus (42) and (44), which define optimal paths for competitive deposits, with one crucial exception. In equation (52) marginal revenue appears in place of the price. Marginal revenue is itself simply price scaled down by a factor dependent on elasticity of demand.

In a competitive industry each deposit is small enough that its extraction will not influence price significantly *from the perspective of the individual deposit owner or manager*. The same must be true if these deposits are organized monopolistically. Each deposit manager faces a price and a marginal revenue which he or she cannot influence significantly *from the perspective of the individual deposit manager*. But insignificant impacts at the deposit level can be important when applied to the entire enterprise. Therefore, the manager in a monopolistic industry must look to marginal revenue; while the owner or manager in a competitive industry must look to price. In both markets, optimality conditions for individual resource deposits can be solved separately, using but one variable to link the various commodity outputs[39]: marginal revenue for a monopoly and price for a competive firm. All differences in market clearing trajectories stem from the difference between marginal revenue and price.

Equation (55) shows that for any given demand function there is a mapping that relates price to marginal revenue. For "well behaved" demand functions marginal revenue is a monotonically increasing function of price. In what follows we assume that monotonic relationship to hold. In addition, if a choke price exists, then at that choke price marginal revenue and price are equal, although for all other prices marginal revenue is strictly smaller than price. The monotonic relationship implies that once characteristics of the marginal revenue trajectory are determined, these characteristics can be translated to the price trajectory.

In previous sections, we have analyzed characteristics of the price trajectory for a competitive equilibrium. For a monopoly, each of these characteristics become precisely the equivalent characteristics of the optimal marginal revenue trajectory. For given demand functions, the marginal revenue trajectory can be translated directly into a price trajectory. Thus the analysis of a competitive industry can be translated completely to analysis of monopoly solutions. We now turn to several examples of that principle.

a. Hotelling Cost Models with No Technology Changes

We again assume that the Hotelling cost assumption is satisfied. Deposits are characterized by different costs and initial stocks; extraction costs are independent of time. Any time horizon is far enough in the future that extraction ceases due to resource depletion and not to a time horizon. Inverse demand functions may be increasing, decreasing, or constant.

Several results can be translated immediately from the competitive equilibrium analysis: (1) Whenever demand is positive, marginal revenue will always be growing and present value of marginal revenue always declining. (2) If the marginal revenue function is time invariant or shrinking over time, then market supply and demand will decline over time. (3) If several resource deposits have different costs, then resources will be extracted strictly in order of increasing cost, except that at the transition point between two deposits both could be extracted simultaneously. (4) Higher cost resources will have lower opportunity costs. (5) Each resource with an extraction cost below the maximum marginal revenue will ultimately be totally depleted. (6) Once started, extraction from a deposit will not stop until it is totally depleted, unless there is sufficient stock of the resource for it to be a "backstop technology."

We need not demonstrate these propositions because they are established in exactly the same manner as the parallel propositions for a competitive equilibrium.

Characteristics of the optimal price trajectory can be ascertained from the marginal revenue trajectory for particular demand functions. As examples, we examine two demand functions: the constant elasticity and the linear functions.

b. Hotelling Cost Models: Constant Elasticity Demand Functions

The constant elasticity demand function has the form:

$$Q^D_t = A\, P_t^{-\eta} \qquad \textit{with } \eta > 1 \tag{56}$$

where A and η are positive constants. Elasticity of demand must exceed 1.0 or the monopoly could obtain unlimited profit by selling a vanishingly small total output and no optimal would exist. Note that for the constant elasticity demand function no choke price exists.

While the i^{th} resource is being extracted the first order necessary conditions for the monopoly solution become:

$$R'_t = P_t \left[\frac{\epsilon - 1}{\epsilon}\right] = c^i + \lambda^i e^{rt}$$

This equation can be rearranged to give the price equation applicable while the i^{th} resource is being extracted:

$$P_t = c^i \left[\frac{\epsilon}{\epsilon - 1}\right] + \lambda^i \left[\frac{\epsilon}{\epsilon - 1}\right] e^{rt} \tag{57}$$

Equation (57) can be compared with equation (46) which shows the competitive price trajectory. The cost term on the right hand side of equation (57) is multiplied by a factor greater than 1: $[\epsilon/(\epsilon-1)]$. The second term is simply a new opportunity cost which increases at the interest rate, just like the second term on the right hand side of equation (46).

The optimal price and quantity trajectories for a monopoly facing a constant elasticity demand function would be identical to equilibrium trajectories in an equivalent competitive market with extraction costs scaled up by a factor of $[\epsilon/(\epsilon-1)]$.

Equation (57) can be used to illustrate the case, discussed by Stiglitz, in which a monopoly extraction path is identical to that of an equivalent competitive industry. Stiglitz assumed a constant elasticity demand function with $\eta > 1$ and $c^i = 0$ for all deposits. For a monopoly, equation (57) becomes:

$$P_t = \lambda^i \left[\frac{\epsilon}{\epsilon - 1}\right] e^{rt} = k^i e^{rt}$$

where k^i is a positive constant selected so that cumulative demand equals the initial stock. For a competitive industry, equation (46) becomes:

$$P_t = \lambda^i e^{rt}$$

where λ^i is a positive constant selected so that cumulative demand just equals initial stock.

The two price trajectories must be identical. They both rise at the interest rate and they both begin at a level which just causes the resource to be totally depleted over all time.

This special case, although not representative of many depletable resources, illustrates an important point. If there are sufficient incentives to assure ultimate total depletion of resources, a monopolist cannot manipulate market prices by altering ultimate cumulative extraction. The only option is to shift the time pattern of the given total extracted quantity. The monopolist can increase its profit only if there are intertemporal differences in the discounted present value of the "wedge" between price and marginal revenue. This "wedge" will be formalized as a "market imperfection function" and discussed more fully in a later section of this chapter.

In the constant elasticity case, the "wedge" between price and marginal revenue is equal to the P_t/η. With zero extraction cost, both price and marginal revenue rise at the interest rate, as does this "wedge". Thus the present value of this "wedge" is independent of time. There are no intertemporal differences to motivate the monopolist and thus the monopolist chooses an extraction path identical to that of a competitive industry.

For positive extraction costs equation (57) implies that the initial monopoly price will exceed the competitive equilibrium price. Monopoly price will rise less rapidly than the competitive price so that in later years the monopoly would have a larger stock remaining and would charge a price lower than would occur were the market perfectly competitive. If the initial monopoly price did not exceed the competitive equilibrium price, by equation (57), the initial λ^i in the competitive equilibrium would be larger than λ^i $[\epsilon/(\epsilon\text{-}1)]$ for the monopoly and the monopoly price would grow more slowly than the competitive equilibrium price. If the monopoly price starts lower and grows slower, it would always be below the competitive price and more of the resource would be extracted. However, in the competitve industry all would be extracted, so extracting more would be infeasible. This contradiction implies that the initial monopoly price will exceed the initial competitive equilibrium price.

In this case, the monopoly would shift extraction from the present to the future, charging higher than competitive prices at early times and lower prices at later times.

c. Hotelling Cost Models: Linear Demand Functions

The linear inverse demand function has the form:

$$P_t = P^C - B\,Q^D{}_t \tag{59}$$

where B is a positive constant and P^C is the choke price. Calculating marginal revenue gives the necessary condition for the price path while resource deposit i is being extracted:

$$P_t = \frac{(c^i + P^C)}{2} + \frac{\lambda^i}{2} e^{rt} \tag{60}$$

Equation (60) can be compared with equations (46) and (57).   In a monopoly with a linear cost function, the cost term on the right of equation (60) is the average of actual cost and choke price. Since the only deposits which are extracted are those for which unit cost is smaller than the choke price, this average is larger than the actual unit cost. The second term is simply a new opportunity cost which increases at the interest rate.

The optimal price and quantity trajectories for a monopoly facing a linear demand function are identical to the equilibrium trajectories in a competitive market with all extraction costs increased to the average of actual cost and choke price.

A similar analysis to that above shows that this monopoly will initially charge a price higher than the competitive equilibrium price and will thereby extract the resource more slowly than would be the case in competitive equilibrium. Prices will rise less rapidly for the monopoly. If the time horizon is long enough, extraction will cease when the choke price is reached. The monopoly facing a linear demand function will reach this choke price strictly later than would the competitive industry[40] and will extract the resource over a longer period.

d. Non-Hotelling Models Without Stock Effects

Here again, we can use all of the results obtained for competitive markets, translating price to marginal revenue. Several conclusions follow for models in which all cost functions are time invariant but in which the demand function may vary over time: (1) If the inverse demand function, and hence the marginal revenue function, is time invariant or growing, marginal revenue will always be growing. If the present value of the marginal revenue function is never growing, then the present value of marginal revenue will always be declining. (2) If the marginal revenue function is time invariant or declining over time, then market supply and demand will decline over time. If the present value of the marginal revenue function is growing, then market supply and demand will increase over time. (3) If several deposits have different costs but identical initial stocks, these deposits will generally be extracted in order of increasing cost, but typically there will be deposits of several different costs being extracted simultaneously. (4) Higher cost deposits will have lower opportunity costs for initial stock fixed and larger deposits will have lower opportunity costs for cost functions fixed, unless they are ultimately not totally depleted. (5) Each resource will ultimately be totally depleted if market price remains above its minimum marginal extraction cost for long enough.

e. Non-Hotelling Models with Stock Effects

We again assume that extraction costs for any given deposit are independent of time and that the time horizon is so far in the future that its existence has no significant impacts on the extraction patterns. Total cost and marginal cost are assumed to be decreasing functions of the amount of the stock remaining.

A few results will emerge emerge immediately based on the parallel of the monopoly solution to the competitive equilibrium: If the marginal revenue function is time invariant or growing, marginal revenue will always be growing except when market supply and demand are zero[41]. A corollary follows: If the marginal revenue function is time invariant, market supply and demand will decline over time. Under the further assumption that $-\partial C/\partial S$ is convex in E, if the marginal revenue function is time invariant or declining over time, market supply and demand will decline over time.

C. Comparative Dynamics and Intertemporal Bias

We saw above that the monopoly optimum and the competitive equilibrium solution were described by the same equations, with but one difference. In a competitive industry each deposit owner faces price derived from an inverse demand function while in a monopolistic industry, each deposit manager faces marginal revenue derived from a marginal revenue function. These differ by a time varying "wedge" equal to $Q_t\ \partial P_t/\partial Q_t$. We will refer to this "wedge" as a "market imperfection function" and denote it as $g(Q_t,t)$. Thus the market imperfection function for the monopoly problem is just:

$$g(Q_t,t) \quad = \quad Q_t\ \partial P_t/\partial Q_t \quad < \quad 0$$

The market imperfection function depends on market conditions and may well vary in its magnitude and sign over time, but is the same for all deposits participating in the market.

Here we analyze how properties of the market imperfection function translate to changes in extraction and price trajectories.

Market imperfection functions are applicable to additional situations. Any postulated alteration in the demand function can be referred to as a "market imperfection function". For example, an excise tax on resource extraction in competitive markets reduces the inverse demand function by the amount of the tax. Regulations on use of the extracted commodity, failure to internalize externalities common to all the firms, expectations of the development of a backstop technology, or exogenous shifts in the extraction of a close substitute product: all could shift the inverse demand function. The shift may well vary over time but it is felt the same for all deposits. If the inverse demand function shifts from $P_t(Q_t)$ to $P'_t(Q_t)$, the market imperfection function will equal the difference between these inverse demand functions:

$$g(Q_t,t) \;=\; P'_t(Q_t) - P_t(Q_t)$$

The market imperfection function allows analysis of market quantity trajectories as if the inverse demand function shifts in a competitive market. Whenever there are no stock effects, properties of the market imperfection function -- whether its sign is positive or negative and whether its present value grows or shrinks -- are sufficient for determining whether the resource will be extracted more rapidly or more slowly as a result of the change.

With market imperfection functions, the new market solution becomes:

$$P_t + g(Q_t,t) \;\begin{matrix} = \\ \leq \end{matrix}\; dC^i_t/dE^i_t + \lambda^i e^{rt} \quad \begin{cases} \text{if } E_t > 0 \\ \text{if } E_t = 0 \end{cases} \qquad (64)$$

$$\lambda^i S^i_T = 0 \qquad (65)$$

$$Q_t \;=\; \sum_i E^i_t \qquad (66)$$

$$P_t \;=\; P_t(Q_t) \qquad (67)$$

If $g(Q_t,t)$ is identically zero, equations (64) through (67) represent the unchanged competitive equilibrium. Analysis of changing conditions reduces to analysis of changing solutions to equations (64) through (67) as the function $g(Q_t,t)$ changes. In conducting our analysis we assume that there are no stock effects and that $P_t(Q_t) + g(Q_t,t)$ is a decreasing function of Q_t. We can formally define the market imperfection function:

Definition: Market Imperfection Function. If one market situation can be represented by equations (64) through (67) with $g(Q_t,t) \equiv 0$ and another by the same equations but with $g(Q_t,t) \neq 0$, then $g(Q_t,t)$ is referred to as a market imperfection function.

1. The Impact of Market Impact Functions

In important special cases we can analyze impacts of market imperfection functions on extraction trajectories. For situations in which all deposits will ultimately be fully depleted, if present value of the market imperfection function either 1) decreases below its initial level for all future time, 2) increases above its initial level for all future time, or 3) equals its initial level for all time, we can determine the direction of changes in the first period market rate of extraction, or more generally, in the extraction rate standardized for remaining resource stocks. If the initial extraction rate is increased, stock will decline more rapidly and at some later time less of the resource will be extracted. Therefore the absolute change in extraction will be predicted only for the initial time.[42] The direction of change will depend upon whether the market imperfection function is positive or negative. If some resource deposits are not ultimately fully depleted, then results will be available for constant or declining present value of the market imperfection function but not for increasing present value.

For positive values of the market imperfection function, Table 3 summarizes impacts on the initial rate of extraction. For negative values, impacts on extraction would be opposite those indicated here. Proof of this result appears in the Appendix.

Present Value of Market Imperfection Function Over Time	Are all Deposits Ultimately Fully Depleted?	
	Yes	No
$g(Q_t,t)\, e^{-r(t-1)} < g(Q_1,1)$ for all $t > 1$	$\Delta Q_1 > 0$	$\Delta Q_1 > 0$
$g(Q_t,t)\, e^{-r(t-1)} = g(Q_1,1)$ for all $t > 1$	$\Delta Q_0 = 0$	$\Delta Q_1 > 0$
$g(Q_t,t)\, e^{-r(t-1)} > g(Q_1,1)$ for all $t > 1$	$\Delta Q_1 < 0$	Indeterminate

Table 3
Impacts of Market Imperfection Function on Initial Extraction Rates:
Positive Values of Market Imperfection Function

Table 3 indicates that if the present value of the market imperfection function is positive initially and decreases below its initial level for all future time, then the initial extraction rate will increase as a result of the changed conditions, independently of whether all deposits are ultimately fully depleted. Results will be reversed for a negative market imperfection function. If the present value of the market imperfection function remains constant over time, then the entire extraction trajectory will be unchanged if all deposits are ultimately totally depleted but the intial extraction rate will increase if some deposits are not ultimately totally depleted. Finally, if the market imperfection function is positive and its present value decreases below its initial level for all future time, then the initial extraction rate would be decreased if all resources will ultimately be fully depleted but its sign would be indeterminate if some deposits would not ultimately be fully depleted.

2. Application: Intertemporal Bias Under Monopoly

The theory of market imperfection functions allows a generalization of the results obtained previously for intertemporal bias of monopolies. We previously have shown that a monopoly (absent technical progress) facing either a linear or a constant elasticity demand function would reduce supply and raise price relative to performance of a competitive industry with the same remaining stocks.

These results can be generalized using the theory of market imperfection functions.
The market imperfection function for a monopoly is $g(Q_t,t) = Q_t\, \partial P_t/\partial Q_t < 0$. For the constant elasticity demand function in equation (56), $g(Q_t,t) = - P_t/\epsilon$. If the present value of price were rationally expected to decline, present value of the market imperfection function would always be smaller than its initial value. By Table 3, such a monopoly would extract its resources less rapidly than would a competitive industry. Similarly, if price were rationally expected to grow at the interest rate so that expected value of price were expected to remain constant, present value of the market imperfection function would remain constant over time. By Table 3, if all deposits would ultimately be fully depleted, such a monopoly would extract its resources at exactly the same rate as would a competitive industry.

In a competitive market with Hotelling costs, present value of price will decline unless extraction is costless or extraction costs are expected to increase rapidly enough. Absent these two circumstances, the monopoly would always extract less rapidly than would a competitive industry. Similarly, for a non-Hotelling costs without stock effects, present value of price must decline unless the present value inverse demand function were increasing or cost functions were increasing rapidly enough. Absent these circumstances, the monopoly facing a constant elasticity demand function would extract more slowly and charge higher prices than would a competitive industry.

For competitive markets with linear demand functions, the market imperfection function equals the difference between current price and the choke price: $g(Q_t,t) = P_t(Q_t) - P^C$, by equation (59). The market imperfection function is negative and its magnitude is increasing with market quantity. Therefore, for time invariant demand functions, unless Q_t is increasing rapidly, the present value of the market imperfection function must be declining in absolute value. By Table 3, a monopoly would extract more slowly than would a competitive industry and would charge higher prices. If present value of the inverse demand function is growing over time, present value of the market imperfection function would be declining in absolute value whenever market quantity is stationary or declining. Again, the monopoly would reduce extraction rates and charge higher prices than would a competitive industry.

In summary, monopolies typically can be expected to extract less rapidly and to charge higher prices, relative to stocks, than a competitive industry, although this need not be the case if demand functions are growing rapidly enough. Nor need it be the case for all shapes of demand functions. But market imperfection functions can be used in many cases to derive theoretical conditions which determine monopoly directions of intertemporal bias.

3. Application: Expected Future Demand Function Changes

Here we use the theory of market imperfection functions to examine market implications of a change in regulatory or economic conditions rationally expected to decrease demand functions in the future but to have no effect on current demand functions. For example, in the mid 1970's, U.S. efficiency standards were set for future vintages of automobiles. Those standards were expected to reduce demand functions for petroleum starting five to ten years after their passage but to have no effect on demand functions before that time.

Under these assumptions the market imperfection function was zero during the mad 1970's. The expected future demand reduction implied that $g(Q_t,t) < 0$ for all $t > \tau$, where τ is the date at which the standards would first become effective. Table 3 suggests that if ultimate full depletion were expected, the market response would be an initial <u>increase</u> in extraction and reduction in initial price. Anticipation of future demand function changes would lead to market responses before the exogneous changes in fact occured. After τ, extraction could be increased or decreased, depending on the particular form of the demand function shift. However, at some future times, extraction would be reduced compensating for the initial increase. However, if ultimate full depletion is not to be expected, the immediate changes could not be predicted directly from Table 3. Furthermore, Table 3 has not been proved to apply to situations in which stock effects are important.

IV. IN CONCLUSION

This chapter has presented a deterministic theory of depletable resource economics, both for individual deposits and for market equilibrium including extraction from a group of deposits. For both situations we have examined a sequence of models, beginning from the most common Hotelling models, progressing through non-Hotelling models without stock effects, and finally ending with non-Hotelling models with stock effects.

As we have gone through the sequence, results became scarcer and scarcer. Under Hotelling assumption we could quantify the price path by a limited set of parameters of the problem and could examine comparative dynamics in detail. With non-Hotelling costs absent stock effects, we could characterize directions of price and/or quantity changes in the overall market and could derive general theorems describing the comparative dynamics. But once stock effects were introduced, even the direction of price and quantity change impacts of changing conditions could not be established in general. The problem became even more complex when the normal process of new deposit discovery was added to the models and results were still fewer.

Since results from the simpler models may not remain valid for more complex models, without further work we cannot be confident that insights from the simpler models will remain valid for the more complex. Serious empirical work, such as that discussed in the Epple and Londregan chapter within this volume, is necessary to specify appropriate cost functions, demand conditions, and market structures. More sophisticated theoretical models are needed to address the many phenomena ignored by the models discussed in this chapter. While much work has been completed and some of that work is described in subsequent chapters of this Handbook, much needed work remains. Our hope is that this Handbook will help to motivate renewed theoretical and empirical attention to issues of depletable resource economics.

V. APPENDIX: PROOFS

A. Marginal Cost for a Discrete Time Cost Function (Equation (10))

As E_t varies, instantaneous extraction rates, $\varepsilon(\gamma)$, must vary so that their sum remains equal to E_t. Their variation leads to the change in cost:

$$\partial C_t/\partial E_t = \int_t^{t+L} [\partial g/\partial \varepsilon \ \partial \varepsilon/\partial E_t - \partial g/\partial S \int_t^{t+\gamma} \partial \varepsilon/\partial E_t \, d\theta] \, d\gamma$$

We can change the order of integration for the second term under the integral above and switch the designation of the differentials $d\gamma$ and $d\theta$ to rewrite the above equation as:

$$\partial C_t/\partial E_t = \int_t^{t+L} [\partial g/\partial \varepsilon \ \partial \varepsilon/\partial E_t] \, d\gamma - \int_t^{t+L} \partial \varepsilon/\partial E_t \int_\theta^{t+L} [\partial g/\partial S \, d\theta] \, d\gamma$$

Combining the terms under one integral gives:

$$\partial C_t/\partial E_t = \int_t^{t+L} \{ \partial \varepsilon/\partial E_t [\partial g/\partial \varepsilon - \int_{t+\gamma}^{t+L} \partial g/\partial S \, d\theta] \} \, d\gamma$$

The derivative $\partial C_t/\partial E_t$ would seem to depend on the resulting changes in the $\varepsilon(\gamma)$ and the impacts of those changes on cost. However, because the cost function is the result of a minimization problem, it is not necessary to know how each $\varepsilon(\gamma)$ changes in response to changes in E_t. First order conditions for optimality within the time interval can be expressed as follows, where Ψ is a constant, independent of time within the interval:

$$\partial g/\partial \varepsilon - \int_{t+\gamma}^{t+L} \partial g/\partial S \, d\theta = \Psi$$

Inserting this relationship into the equation above gives:

$$\partial C_t/\partial E_t = \psi \int_t^{t+L} \partial \varepsilon/\partial E_t \, d\gamma = \psi = \partial g/\partial \varepsilon - \int_t^{t+L} \partial g/\partial S \, d\theta$$

where the functions on the right hand side of the equation are all evaluated at $\gamma = t$. Equation (10) is derived.

B. Intertemporal Bias Result

In establishing the results of Table 3, we begin with an analysis of how the opportunity costs change in response to the change in $g(Q_t,t)$. Basic result can be stated as Lemma 1:

Lemma 1: The maximum and minimum values (over time) of the market imperfection function place limits on changes in present value opportunity cost for the i^{th} deposit:

$$\text{if } \max_t g(Q_t,t) > 0 \text{ or } \sum_{t=0}^{T} E'^i_t = S_0, \text{ then } \Delta\lambda^i \le \max_t \{ g(Q_t,t)\, e^{-rt} \}$$

$$\text{if } \min_t g(Q_t,t) < 0 \text{ or } \sum_{t=0}^{T} E^i_t = S_0 \text{ then } \Delta\lambda^i \ge \min_t \{ g(Q_t,t)\, e^{-rt} \}$$

where E^i_t and E'^i_t represent extraction absent and with the market imperfection function, respectively. The maxima and minima are defined over that time that deposit i is being extracted. If $g(Q_\tau,\tau)\, e^{-r\tau}$ reaches its maximum (or minimum) value at only one time, then $\Delta\lambda^i < \max\{g(Q_t,t)\ e^{-rt}\}$ (or $\Delta\lambda^i > \min\{g(Q_t,t)\ e^{-rt}\}$.

Assume the converse of the first part of Lemma 1: for that deposit with the largest value of $\Delta\lambda^j$ (here denoted as deposit i), that either $\Delta\lambda^i > \max\{g(Q_t,t)\ e^{-rt}\}$ or that
$\Delta\lambda^i = \max\{g(Q_t,t)\ e^{-rt}\} > \{g(Q_\tau,\tau)\, e^{-r\tau}\}$ for all other $\tau \neq t$ and for which $E'^i_t > 0$. If $\max g(Q_t,t) > 0$ then $\Delta\lambda^i > 0$ and $\lambda^i > 0$: the deposit will ultimately be totally depleted. If $\max g(Q_t,t) \le 0$, then by the premise of the Lemma, deposit i will be totally depleted. In either case, for deposit i, $\Delta E^i_\tau \ge 0$ at some time τ. Then at τ, for all j:
$0 \le \Delta\{P_\tau + g(Q_\tau,\tau) - \lambda^i e^{r\tau}\} \le \Delta\{P_\tau + g(Q_\tau,\tau) - \lambda^j e^{r\tau}\}$. This inequality implies that $\Delta E^j_\tau \ge 0$ for all j. But then $\Delta Q_\tau \ge 0$ and $\Delta P_\tau \le 0$. But since $g(Q_\tau,\tau) < \Delta\lambda^i e^{r\tau}$,
$\Delta\{P_\tau + g(Q_\tau,\tau) - \lambda^i e^{r\tau}\} < 0$, a contradiction. Thus $\Delta\lambda^i \le \max\{g(Q_t,t)\ e^{-rt}\}$ and $\Delta\lambda^i < \max\{g(Q_t,t)\, e^{-rt}\}$ if $\max\{g(Q_t,t)\, e^{-rt}\} > g(Q_\tau,\tau)\, e^{-r\tau}$ for all τ that deposit i is being extracted. The second part of Lemma 1 is established in the same manner.

Lemma 1 leads us directly to the main result linking market imperfection functions to intertemporal bias.

Theorem 1:
If $g(Q_1,1) > 0$ and all deposits would be fully depleted in the original situation, then

- **if $g(Q_t,t)\, e^{-r(t-1)} < g(Q_1,1)$ for all $t > 1$, then $\Delta Q_1 > 0$;**
- **if $g(Q_t,t)\, e^{-r(t-1)} = g(Q_1,1)$ for all t, then $\Delta Q_1 = 0$;**
- **if $g(Q_t,t)\, e^{-r(t-1)} > g(Q_1,1)$ for all $t > 1$, then $\Delta Q_1 < 0$.**

If $g(Q_1,1) > 0$ and some deposits would not ultimatly be fully depleted in the original situation, then

if $g(Q_t,t)\, e^{-r(t-1)} < g(Q_1,1)$ for all t>1, then $\Delta Q_1 > 0$;

if $g(Q_t,t)\, e^{-r(t-1)} = g(Q_1,1)$ for all t, then $\Delta Q_1 \geq 0$.

If $g(Q_1,1) < 0$, all the inequalities above are reversed.

Assume that $g(Q_1,1) > 0$. Assume first that $g(Q_t,t)\, e^{-rt} = g(Q_1,1)$ for all t and that all deposits ultimately are fully depleted in the original case. By Lemma 1, $\max \{g(Q_t,t)\, e^{-rt}\} \leq \Delta\lambda^i \leq \max \{g(Q_t,t)\, e^{-rt}\}$ for all i. Thus $\max \{ g(Q_t,t)\, e^{-rt} \} = \Delta\lambda^i = \max \{g(Q_t,t)\, e^{-rt} \} = g(Q_t,t)\, e^{-rt}$ for all t. Then for all i and all t: $\Delta\{P_t + g(Q_t,t) - \lambda^i e^{rt}\} = \Delta P_t$. Therefore if $\Delta P_t > 0$, then $\Delta E^i_t > 0$, $\Delta Q_t > 0$, and $\Delta P_t < 0$, a contradiction. A similar contradiction is reached if it is postulated that $\Delta P_t < 0$. Therefore $\Delta P_t = 0$, $\Delta E^i_t = 0$, and $\Delta Q_t = 0$ for all i and all t if all deposits would be fully depleted. If some deposits would not be fully depleted, then
$\Delta\lambda^i < \max \{g(Q_t,t)\, e^{-rt}\} = g(Q_1,1)$. Therefore $\Delta Q_1 \geq 0$ in this situation.

Assume next that $g(Q_t,t)\, e^{-rt} < g(Q_1,1)$ for all $t > 1$. Then by Lemma 1, $\Delta\lambda^i < g(Q_1,1)$ for all i, independent of whether all deposits would be fully depleted. Assume the converse of the theorem, that $\Delta Q_1 \leq 0$. Then $\Delta P_1 \geq 0$. These results imply that $\Delta\{P_1 + g(Q_1,1) - \lambda^i\} > 0$ for all i, thus that $\Delta E^i_1 > 0$ for all i, and that $\Delta Q_1 > 0$, a contradiction. Thus $\Delta Q_1 > 0$.

Assume finally that $g(Q_t,t)\, e^{-rt} > g(Q_1,1)$ for all $t > 1$. Then by Lemma 1, $\Delta\lambda^i > g(Q_1,1)$ for all i if all deposits would be fully depleted. Assume the converse, that $\Delta Q_1 \geq 0$. Then $\Delta P_1 \leq 0$. These results imply that $\Delta\{P_1 + g(Q_1,1) - \lambda^i\} < 0$ for all i and thus that $\Delta E^i_1 < 0$ for all i and that $\Delta Q_1 < 0$, a contradiction. Thus $\Delta Q_1 < 0$ if all deposits would be fully depleted in the initial case.

A similar demonstration can be used when $g(Q_1,1) < 0$.

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VII. ENDNOTES

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2. There can be both consumptive and non-consumptive uses of many resources. Unless otherwise indicated, by "use", we refer to consumptive use.

3. British Thermal Unit, a measure of the energy content of energy commodities.

4. These examples suggest that depletable resources can be viewed a limiting case of renewable resources, a case in which the renewal rate has been reduced to zero. This interpretation will not be used here, but it can link the theory of depletable resources and that of renewable resources.

5. We use finite, but arbitrarily long time periods to avoid the mathematical complications associated with an infinite time horizon model.

6. This could be a minimization of the discounted present value of $g(\varepsilon(\gamma),S(\gamma))$. However it is envisioned that L is short enough that discounting within a time period is irrelevant. If a discounted present value is used, none of the subsequent conclusions are modified. Under the formulation as stated here, the necessary condition for optimization is:

$$\partial g/\partial \varepsilon - \int_{t+\gamma}^{t+L} \partial g/\partial S \, d\theta = \Psi$$

where Ψ is a constant independent of γ and each of the partial derivatives is evaluated at the time $t+\gamma$ within the time interval. Theory for solving this optimization is identical to that for solving the overall optimization problems discussed in this paper. This equation leads directly to equation (10) in the body of the chapter.

7. Because $C_t(E_t,S_{t-1})$ is defined based on the result of an optimization, the impact on cost for a small variation in the $\varepsilon(\gamma)$ will be zero as long as the total of all the $\varepsilon(\gamma)$ is unchanged.

8. The inequality of equation (12) might not apply to the underlying continuous time model, yet it must apply to the discrete time model. To illustrate, assume that the inequality was reversed in a discrete time model. Consider two cases in which the P_t and opportunity costs were unchanged between the two cases (see the discussion of first order necessary conditions for optimality at a later point in this chapter.) In the first, the initial stock was higher by δS_{t-1}. As a result the optimal extraction rate in the first case is increased by δE_t, where $\delta E_t = -\delta S_{t-1} [\partial^2 C_t/\partial S_{t-1}\partial E_t] / [\partial^2 C_t/\partial E_t^2] > \delta S_{t-1}$. Consider the change in stock after time period t: $\delta S_t = \delta S_{t-1} - \delta E_t < 0$. Higher stock at the beginning of a time period would lead to lower stock at the end of the period!

This could not happen in a continuous time model even if the inequality were likewise reversed. Assume that $[\partial^2 g/\partial S\partial\varepsilon] < -[\partial^2 g/\partial\varepsilon^2]$ and that the initial stock were higher by δS_{t-1}. Instantaneous extraction rate would increase by $\delta\varepsilon = -\delta S_{t-1} [\partial^2 g/\partial S\partial\varepsilon] / [\partial^2 g/\partial\varepsilon^2] > \delta S_{t-1}$. Therefore the rate of stock decline over time would increase by an amount larger than δS_{t-1}. The stock increase δS would decline over time. But in so doing, it would reduce the magnitude of $\delta\varepsilon$ and hence of the rate of decline of δS. It would never be possible for δS to become negative because as δS converged to zero, so would $\delta\varepsilon$. Thus if $\delta S_{t-1} > 0$, then $\delta S_t > 0$, even if the inequality were reversed.

Hence if the discrete time model is consistent with an underlying continuous time model, then it must be true that $[\partial^2 C_t/\partial S_{t-1}\partial E_t] > -[\partial^2 C_t/\partial E_t^2]$, even if $[\partial^2 g/\partial S\partial\varepsilon] < -[\partial^2 g/\partial\varepsilon^2]$.

9. This ordering by extraction cost will be discussed more fully in a later chapter section.

10. Convexity of the cost function and linearity of the revenue function together imply that the objective function is weakly concave. The constraints define a convex set. The set of optimal points for a concave function, constrained to a convex set, must always be convex. Hence for this problem, while the optimal point may not be unique, the set of optimal points will be convex.

11. **A clear explanation of the Kuhn-Tucker theorem appears in Varian.**

12. If binding non-negativity constraints on the extraction rates were explicitly included in the test, the same conclusion would hold as long as at least one extraction rate were positive. The derivative with respect to E_t would equal 1 for the t^{th} depletability constraint and 0 for all other depletability constraints. If the non-negativity constraint were not binding at one time, say, time t, then the gradient of the non-negativity constraint at time t would not be included in the set of gradients tested for linear independence. Hence there would be but one single gradient that included a non-zero derivative for E_t. This gradient could not be obtained as a linear combination of the other gradients.

13. Second order conditions for optimality will be simply the convexity conditions on the cost function. If these conditions were not valid, larger changes would allow an increase in profit from the optimal, a contradiction from the concept of optimality.

14. Here it is assumed that the paths of tax plus opportunity cost will cross. If they did not, say because tax plus opportunity cost in the new situation was always lower than in the initial situation, then equation (16) would be violated. In this case extraction would be increased at every time period and final stock would decrease. But if the deposit were ultimately totally depleted in the original situation, then the final stock could not decrease. If the deposit were not ultimately totally depleted in the original situation, then the original opportunity cost would be zero and could not decrease further. In this situation tax plus opportunity cost must increase in moving to the new situation.

15. The basic theory of difference equations shows that the solutions to equations such as (28) will in general be the sum of a homogeneous solution and a particular solution. The homogeneous solution must grow at the interest rate and can always be written as $\lambda\, e^{rt}$. The particular solution is just equation (29).

16. Note that the steady state opportunity cost is independent of the length of a time interval (L), since both the numerator and the denominator are roughly proportional to L.

17. This relationship can be seen from equation (24). If E = 0 for one value of ϕ, then if ϕ increases, E remains equal to zero. If ϕ decreases below the minimum level which gives E=0, then extraction rate will become positive.

18. This result can be shown from equation (25). From the ϕ constant locus, if S increases, the first term on the right hand side of equation (25) remains unchanged. The second term increases (or remains constant) under the convexity assumption, as will be shown in what follows. If S increases by δS, where δS is infinitesimally small, then the change in $\partial C/\partial S$ will be: $\delta[\partial C/\partial S] = \partial^2 C/\partial S^2\ \delta S + \partial^2 C/\partial S\partial E\ \delta E$. By equation (24) the change in E will be $\delta E = -[\partial^2 C/\partial S\partial E] / [\partial^2 C/\partial E^2]\ \delta S$.

Combining these two equations gives:
$\delta[\partial C/\partial S] = \partial^2C/\partial S^2 \ \delta S - [\partial^2C/\partial S\partial E]^2 / [\partial^2C/\partial E^2] \ \delta S$
$= \{ 1/ [\partial^2C/\partial E^2]\} \{ [\partial^2C/\partial S^2] \ [\partial^2C/\partial E^2] - [\partial^2C/\partial S\partial E]^2 \} \geq 0$ for $\delta S > 0$. Thus the right hand side of equation (25) increases or remains constant when S increases. Therefore ϕ must increase in time, or in the limit, be constant over time, to the right of the ϕ-constant locus.

19. Differentiating $\rho \phi = -\partial C/\partial S$ gives the following for infinitesimally small changes in ϕ, S, and E: $\rho \delta\phi = -\partial^2C/\partial S^2 \delta S - \partial^2C/\partial S\partial E \delta E$. Differentiating $P = \partial C/\partial E + \phi$ gives:
$\partial^2C/\partial E\partial S \ \delta S + \partial^2C/\partial E^2 \ \delta E + \delta\phi = 0$. Combining these two expressions:
$\rho \delta\phi = -\partial^2C/\partial S^2 \delta S + \partial^2C/\partial S\partial E [\partial^2C/\partial E\partial S \ \delta S + \delta\phi] / [\partial^2C/\partial E^2]$, or
$\{\rho \ \partial^2C/\partial E^2 - \partial^2C/\partial S\partial E \} \delta\phi = - \{ [\partial^2C/\partial S^2] \ [\partial^2C/\partial E^2] - [\partial^2C/\partial S\partial E]^2 \} \ \delta S$.
Convexity of the cost function implies that the expression in brackets on the right hand side of this expression is non-negative. The first term in brackets on the left hand side of this expression is always positive and the second term is positive as long as $\partial^2C/\partial S\partial E < 0$. This establishes the result that ϕ is a decreasing function of S as long as $\partial^2C/\partial S\partial E < \rho \ \partial^2C/\partial E^2$.

20. Notice that this condition will be independent of the length of the interval in the discrete model. In particular, as L increases, ρ would increase in proportion to L, as would $\partial^2C/\partial S\partial E$. $\partial^2C/\partial E^2$ would be roughly independent of L.

21. Additional limits are placed on these parameters by the requirements of equation (11). In using this equation, we choose parameters so that equation (11) is valid for all values of the variables in the optimal solution.

22. If the initial stock was below the steady state level, however, this pattern would be impossible and therefore no extraction at all would occur. However this possibility has been ruled out by assumption.

23. I would like to thank Robert Patrick for for pointing out this result about multiplicatively separable functions.

24. Note that the steady state stock and opportunity cost are independent of L. Both the numerators and the denominators will tend to be proportional to the length of the time period.

25. Equation (29) suggests this counter-intuitive result for stock levels from which $\partial^2C/\partial S\partial E > 0$. Higher prices lead to more extraction in early years when $\partial^2C/\partial S\partial E > 0$ and less extraction in later years when $\partial^2C/\partial S\partial E < 0$. Thus $\partial C/\partial S$ <u>increases</u> in both early and later years. Thus the discounted present value of $-\partial C/\partial S$ must decrease as a result of the price increase. Note that if $\partial^2C/\partial S\partial E < 0$ for all years, $\partial C/\partial S$ would <u>decrease</u> in early years and increase in later years. In that case the discounted present value of $-\partial C/\partial S$ could be expected to increase as a result of the price increase.

26. We could reasonably assume that if marginal cost is decreasing in S for one stock level it is decreasing in S for all lower levels. Convexity implies that if $\partial C/\partial S$ is negative for some level, it must be negative for all lower levels if E is held constant at those levels. However, that does not strictly imply that if marginal cost is decreasing in S for one stock level it is decreasing in S for all lower levels. Thus the awkward statement in the text.

27. Alternatively, in an infinite time problem which converged to a steady state, the increase in the steady state opportunity cost must be smaller than the increase in the price in the steady state as shown in the body of the chapter. But this would be impossible if the initial opportunity cost increase equalled or exceeded the initial price increase.

28. Alternatively, in an infinite time problem which converged to a steady state, the increase in the steady state opportunity cost must be negative, as shown in the body of the chapter. But this would be impossible if the initial opportunity cost increase were non-negative.

29. This clarification of the concept will be necessary for establishing the existence of a competitive equilibrium.

30. A discussion of these fixed point concepts as well as the basic mathematical concepts embedded in the fixed point theorems appears in the Green and Heller chapter within the *Handbook of Mathematical Economics*.

31. This statement is quoted directly from the Green and Heller chapter of the *Handbook of Mathematical Economics*. A correspondence γ is closed if and only if its graph is a closed set. The Green and Heller chapter also provides a discussion of upper hemicontinuity of a correspondence. Essentially, a correspondence $\gamma(a)$ is upper hemicontinuous if it does not blow up discontinuously in any small neighborhood of any point a.

32. See for example, Varian, for more discussion. The statement given here is a restatement of the theorem as cited by Varian.

33. The statement assumes that the firms producing resources of a given cost do not have exactly enough stock left during the last period of production from that resource grade to meet total market demand precisely. If they did, then there would be a small range of possible price variation at that time. The shorter the time intervals in the model, the smaller the range of possible variation. In the limit of a continuous time model, there would be no possibilities for price variation at the time of transition.

34. It should be remembered that both resources face the same price.

35. This discussion assumes that time intervals are so small that at the maximum extraction rate there is virtually no change in extraction rate from one time to the next. For larger time intervals, the more precise statement would be: if the low cost deposit increases in extraction rate from one time period to the next, then the higher cost deposit must also increase in extraction rates between these two time periods.

36. Here it is important that we assumed that the time horizon is so far in the future that its existence has no significant impacts on the extraction patterns. As the system approaches a time horizon, we have seen that extraction can rise over time with constant prices. For the competitive equilibrium, that implies that prices can be falling as the horizon is approached.

37. Price could decline as the time horizon is approached. However that time is assumed to be so far in the future as to be irrelevant for the current choices.

38. Note that depletable resource models based upon an aggregate cost function for all existing resources, rather than the disaggregated concept used here, could well lead implicitly to predictions of changes in the extraction from existing resources, based upon the discovery of new deposits.

39. This fact is perhaps important for decentralization of decision making within a monopoly or a cartel. The central controller needs only to communicate one variable to managers of the deposits, the derivative $Q_t\ \partial P_t/\partial Q_t$, or equivalently, P_t/ϵ_t. The central controller must motivate or cause the manager of each deposit to maximize its own profit, with relevant price adjusted downward by P_t/ϵ_t.

40. This proposition can be proved by assuming the converse and examining the difference in opportunity costs backward, starting from the time which the choke price is reached.

41. Here it is important that we assumed that the time horizon is so far in the future that its existence has no significant impacts on the extraction patterns. As the system approaches a time horizon, we have seen that extraction can rise over time with constant prices. For the competitive equilibrium, that implies that prices can be falling as the horizon is approached.

42. More precisely, the prediction is only for the first period when the remaining stock varies with the market imperfection function. However, as will be seen, when the discounted present value of the market imperfection function remains constant over time and the entire resource will be fully depleted, there is no initial impact on extraction. Thus the stock will be unchanged and there will never be an impact on extraction rates.