13. Term Structure of Interest Rates: Facts and Alternative Models

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Why study this in a monetary economics course?

While simple monetary models like these

$$\left.\begin{aligned} y &= -\beta r_{-1} + \lambda y_{-1} + \varepsilon \\ \pi &= \pi_{-1} + \alpha y_{-1} + \eta \\ r &= [(\lambda + \alpha q)/\beta]y + [q/\beta]\pi \end{aligned}\right\} \text{Ball (Keynesian)}$$

$$\left.\begin{aligned} i_t &= i_t^* + \phi_\pi(\pi_t - \overline{\pi}) + \phi_y(y_t - y_t^n) \\ \pi_t &= \kappa(y_t - y_t^n) + \beta E_t \pi_{t+1} \\ y_t &= E_t y_{t+1} - \sigma(i_t - E_t \pi_{t+1}) + g_t \end{aligned}\right\} \text{Woodford (New Keynesian)}$$

focus only on the short term interest rate (federal funds rate), the monetary transmission mechanism is more complex. Need to look at other interest rates (longer term, mortgage rates) And there is debate about QE at the zero bound.

Lecture Outline

- Review facts about the term structure
- Review models of the term structure.
  - "Pure expectations" model which has been used for policy evaluation for many years
  - No-arbitrage affine model, brings risk into account
  - Preferred habitat, market segmentation
    - An older theory, but recently revived
    - "runs counter to at least the past half century of mainstream frictionless finance theory."
      - Bauer and Rudebusch (2011)

U.S. Treasury Yield Curve: 1-16 May 2013

Date	1 Mo	3 Mo	6 Mo	1 Yr	2 Yr	3 Yr	5 Yr	7 Yr	10 Yr	20 Yr	30 Yr
05/01/13	0.03	0.06	0.08	0.11	0.20	0.30	0.65	1.07	1.66	2.44	2.83
05/02/13	0.02	0.05	0.08	0.11	0.20	0.30	0.65	1.07	1.66	2.44	2.82
05/03/13	0.02	0.05	0.08	0.11	0.22	0.34	0.73	1.17	1.78	2.58	2.96
05/06/13	0.01	0.04	0.08	0.11	0.22	0.34	0.74	1.19	1.80	2.60	2.98
05/07/13	0.01	0.04	0.08	0.10	0.22	0.35	0.75	1.21	1.82	2.62	3.00
05/08/13	0.01	0.04	0.08	0.11	0.22	0.35	0.75	1.20	1.81	2.61	2.99
05/09/13	0.02	0.04	0.08	0.11	0.22	0.35	0.75	1.20	1.81	2.60	3.01
05/10/13	0.02	0.04	0.08	0.11	0.26	0.38	0.82	1.28	1.90	2.70	3.10
05/13/13	0.02	0.05	0.08	0.13	0.24	0.40	0.83	1.30	1.92	2.73	3.13
05/14/13	0.01	0.05	0.09	0.12	0.26	0.41	0.85	1.33	1.96	2.77	3.17
05/15/13	0.01	0.04	0.09	0.12	0.26	0.40	0.84	1.32	1.94	2.76	3.16
05/16/13	0.00	0.03	0.08	0.12	0.23	0.37	0.79	1.25	1.87	2.69	3.09

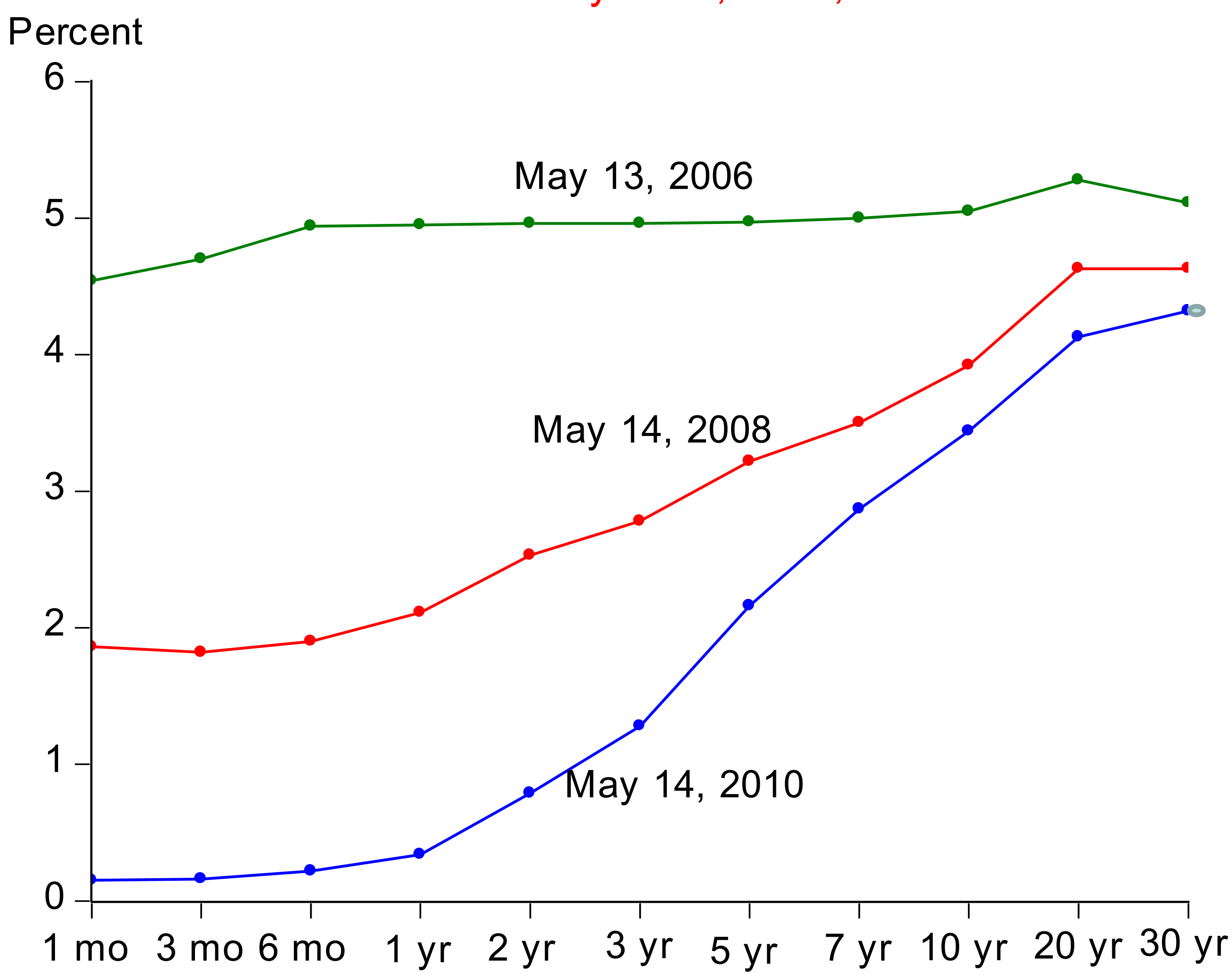
U.S. Treasury Yield Curve: 1-16 May 2012

Date	1 mo	3 mo	6 mo	1 yr	2 yr	3 yr	5 yr	7 yr	10 yr	20 yr	30 yr
05/01/12	0.07	0.09	0.15	0.19	0.27	0.39	0.84	1.35	1.98	2.76	3.16
05/02/12	0.06	0.08	0.15	0.18	0.27	0.39	0.82	1.33	1.96	2.72	3.11
05/03/12	0.05	0.09	0.15	0.19	0.28	0.40	0.82	1.34	1.96	2.72	3.12
05/04/12	0.05	0.07	0.14	0.18	0.27	0.37	0.78	1.28	1.91	2.67	3.07
05/07/12	0.06	0.10	0.15	0.18	0.27	0.37	0.79	1.29	1.92	2.67	3.07
05/08/12	0.08	0.09	0.15	0.18	0.27	0.36	0.77	1.26	1.88	2.63	3.03
05/09/12	0.07	0.09	0.15	0.18	0.27	0.36	0.77	1.26	1.87	2.63	3.03
05/10/12	0.08	0.10	0.15	0.18	0.27	0.37	0.79	1.28	1.89	2.64	3.07
05/11/12	0.07	0.10	0.15	0.18	0.27	0.36	0.75	1.24	1.84	2.59	3.02
05/14/12	0.07	0.10	0.15	0.19	0.29	0.37	0.73	1.20	1.78	2.53	2.95
05/15/12	0.08	0.09	0.15	0.19	0.29	0.38	0.74	1.19	1.76	2.50	2.91
05/16/12	0.08	0.10	0.15	0.20	0.30	0.40	0.75	1.19	1.76	2.48	2.90

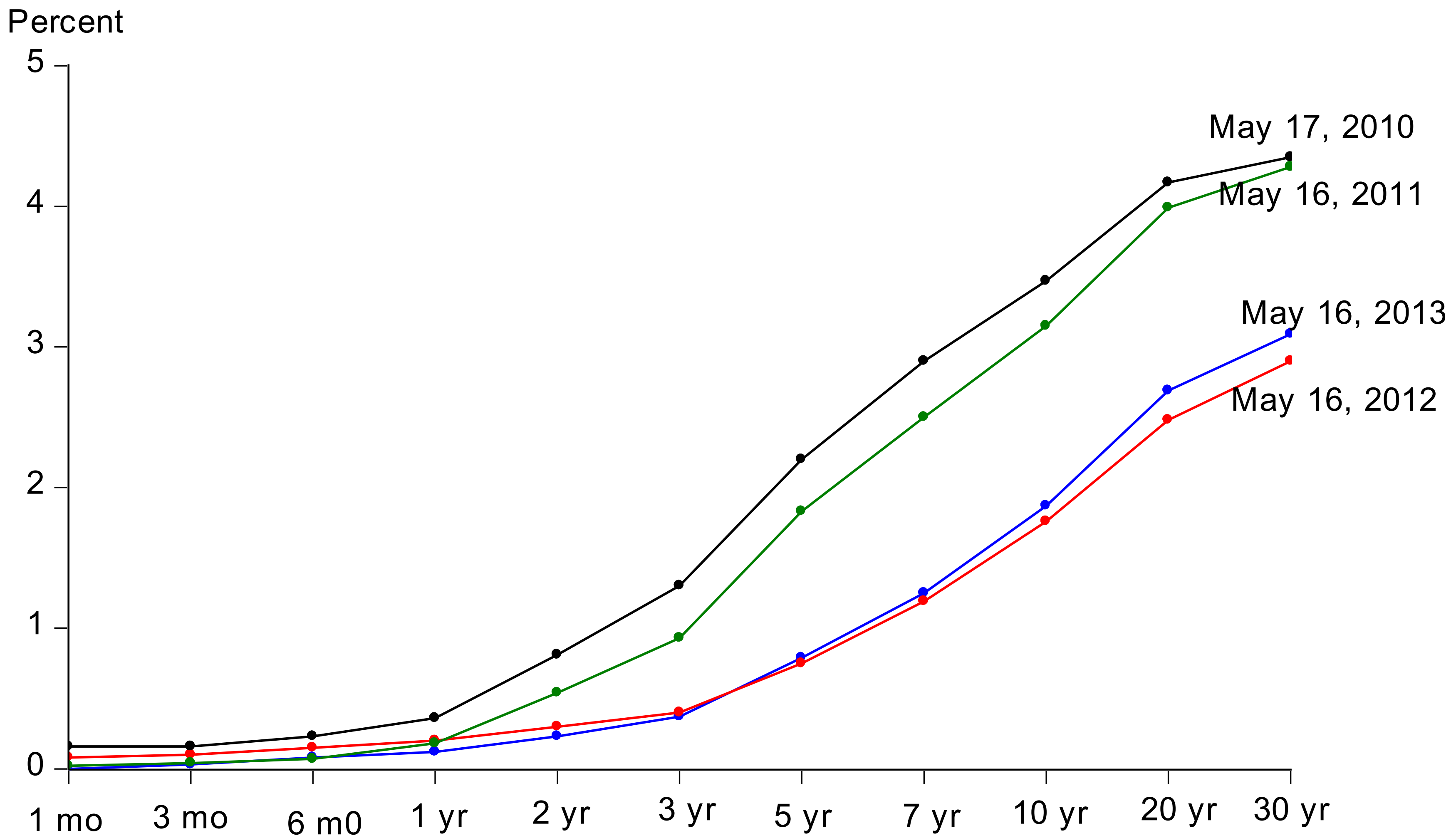
U.S. Treasury Yield Curve: 2-13 May 2011

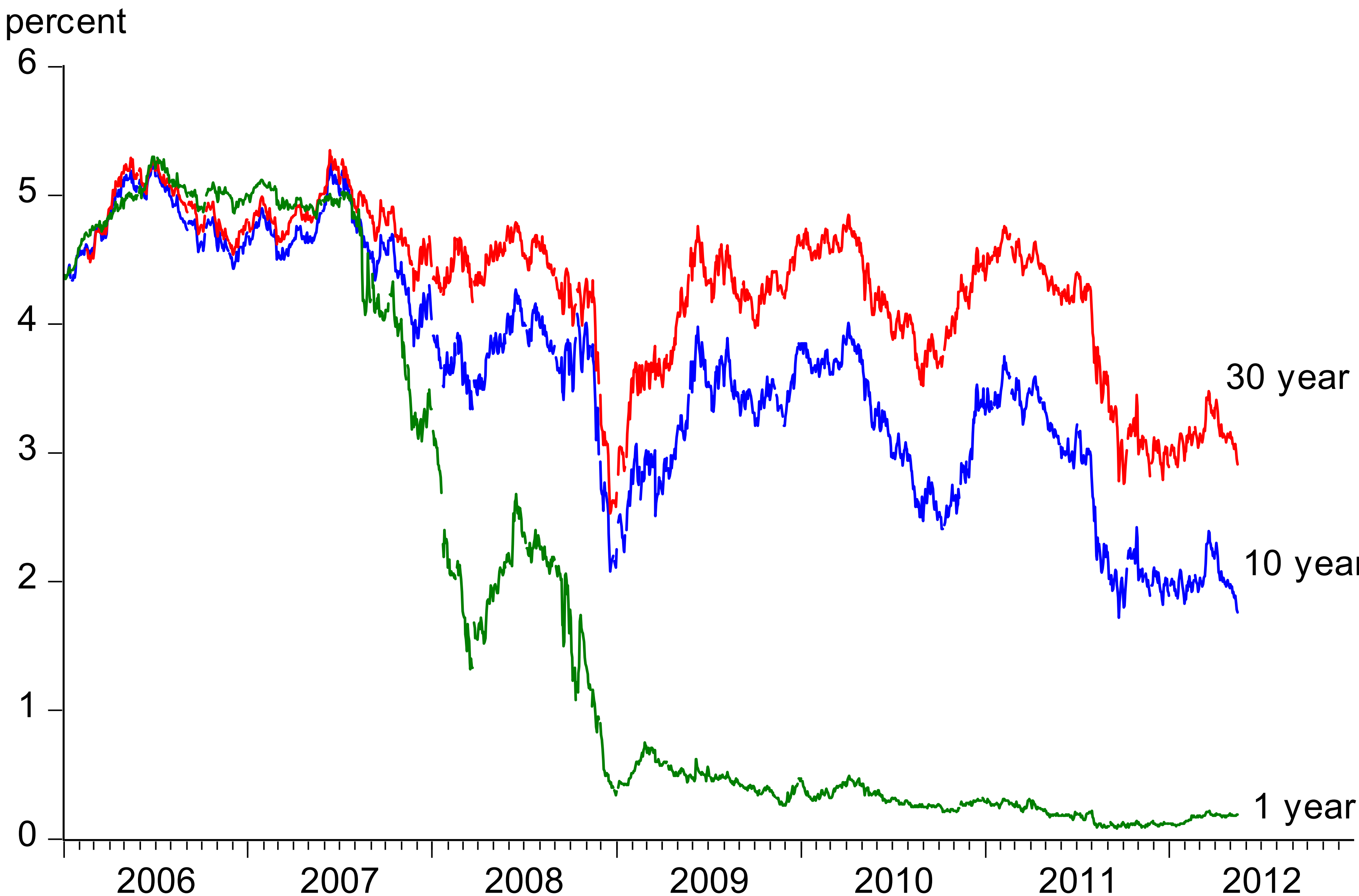
Date	1 mo	3 mo	6 mo	1 yr	2 yr	3 yr	5 yr	7 yr	10 yr	20 yr	30 yr
05/02/11	0.02	0.05	0.10	0.22	0.61	1.01	1.96	2.66	3.31	4.14	4.38
05/03/11	0.02	0.03	0.09	0.20	0.61	1.01	1.96	2.64	3.28	4.11	4.36
05/04/11	0.02	0.03	0.07	0.19	0.60	1.00	1.95	2.61	3.25	4.08	4.33
05/05/11	0.01	0.02	0.07	0.20	0.58	0.97	1.88	2.54	3.18	4.00	4.26
05/06/11	0.02	0.02	0.07	0.18	0.57	0.96	1.87	2.54	3.19	4.03	4.29
05/09/11	0.01	0.03	0.07	0.17	0.57	0.94	1.84	2.51	3.17	4.03	4.30
05/10/11	0.02	0.03	0.07	0.19	0.59	1.03	1.91	2.57	3.23	4.07	4.34
05/11/11	0.01	0.03	0.07	0.18	0.56	0.96	1.87	2.53	3.19	4.04	4.31
05/12/11	0.01	0.02	0.07	0.18	0.57	0.98	1.89	2.56	3.22	4.07	4.37
05/13/11	0.01	0.03	0.08	0.19	0.57	0.96	1.86	2.53	3.18	4.03	4.32

Cyclical changes in the yield curve:
Steepening during the recession:
U.S. Treasuries – May 2006, 2008, 2010



But flattened during 2011 and has stayed flat





US Treasury Constant Maturity Yields

Empirical effect of output-inflation on yield curve

$$i_t^{(n)} = a_n + b_{1,n} y_t + b_{2,n} \pi_t$$

$i_t^{(n)}$ is the yield to maturity on bonds with maturity *n.*

Note that for the shortest maturity (one-day), this is a monetary policy rule.

$$i_t = 1 + .5 y_t + 1.5 \pi_t$$

Empirical Estimates of the "Rules" for Longer Rates

Mat'ity (Years)	1960Q1 – 1979Q4				1984Q1 – 2006Q4			
	a_n	$b_{1,n}$	$b_{2,n}$		a_n	$b_{1,n}$	$b_{2,n}$	
Fed Funds	2.180	0.623	0.760	0.777	2.022	1.322	1.234	0.504
	(0.278)	(0.118)	(0.080)		(0.921)	(0.265)	(0.362)	
1	2.874	0.454	0.604	0.847	2.156	1.244	1.224	0.513
	(0.194)	(0.078)	(0.051)		(0.856)	(0.252)	(0.343)	
2	2.960	0.335	0.602	0.860	2.394	1.124	1.273	0.478
	(0.186)	(0.076)	(0.053)		(0.849)	(0.257)	(0.356)	
3	3.166	0.259	0.580	0.865	2.652	1.000	1.279	0.451
	(0.181)	(0.071)	(0.052)		(0.818)	(0.254)	(0.355)	
4	3.263	0.222	0.572	0.868	2.815	0.888	1.304	0.426
	(0.183)	(0.071)	(0.053)		(0.798)	(0.254)	(0.358)	
5	3.303	0.191	0.573	0.872	2.913	0.806	1.318	0.416
	(0.186)	(0.07)	(0.054)		(0.777)	(0.244)	(0.356)	

Note how $b_{1,n}$ declines with n

No-Arbitrage Affine Model of the Term Structure

Let $P_t^{(n)}$ = price at time t of a zero coupon bond with maturity n quarters.

Assume that face value is 1.

Define $y_t^{(n)}$ as the yield on the zero coupon bond: $P_t^{(n)} = \exp(-ny_t^{(n)})$ and $y_t^{(1)} = r_t$

The "no-arbitrage" condition is defined as

$$P_t^{(n)} = E_t[m_{t+1}P_{t+1}^{(n-1)}]$$

We will assume that the "pricing kernel" has this convenient functional form:

$$m_{t+1} = \exp(-r_t - .5\boldsymbol{\lambda}_t'\boldsymbol{\lambda}_t - \boldsymbol{\lambda}_t'\boldsymbol{\varepsilon}_{t+1})$$

where the risk term $\boldsymbol{\lambda}_t$ could depend on other endogenous variables

A Special Case:
The no-arbitrage model without risk

Consider implications of $P_t^{(n)} = E_t[m_{t+1} P_{t+1}^{(n-1)}]$ in the case where $m_{t+1} = \exp(-r_t)$

$P_t^{(1)} = \exp(-r_t)$

$P_t^{(2)} = m_{t+1} P_{t+1}^{(1)} = \exp(-r_t)\exp(-r_{t+1}) = \exp(-r_t - r_{t+1}) = \exp(-2 y_t^{(2)})$

where $y_t^{(2)} = (r_t + r_{t+1})/2$

$P_t^{(3)} = m_{t+1} P_{t+1}^{(2)} = \exp(-r_t - r_{t+1} - r_{t+2}) = \exp(-3 y_t^{(3)})$

where $y_t^{(3)} = (r_t + r_{t+1} + r_{t+2})/3$

And so on.

Equations used in "pure expectations model"

In general under no arbitrage and pricing kernel assumptions, the term structure is affine:

First summarize macro data with a VAR and a policy rule

$$\mathbf{X}_t = \boldsymbol{\mu} + \boldsymbol{\Phi}\mathbf{X}_{t-1} + \boldsymbol{\Sigma}\boldsymbol{\varepsilon}_t$$

$$r_t = \delta_0 + \boldsymbol{\delta}_1'\mathbf{X}_t$$

$$P_t^{(n)} = E_t[m_{t+1}P_{t+1}^{(n-1)}] \qquad \text{with yield defined from } P_t^{(n)} = \exp(-ny_t^{(n)})$$

$$m_{t+1} = \exp(-r_t - .5\boldsymbol{\lambda}_t'\boldsymbol{\lambda}_t - \boldsymbol{\lambda}_t'\boldsymbol{\varepsilon}_{t+1})$$

$$\boldsymbol{\lambda}_t = \boldsymbol{\lambda}_0 + \boldsymbol{\lambda}_1\mathbf{X}_t$$

Objective is to derive term structure equations of the form

$$y_{\mathrm{t}}^{(\mathrm{n})} = a_n + \mathbf{b}_n'\mathbf{X}_t$$

This is the reason for the term “affine” model

that incorporate the no-arbitrage conditions and connect the macro variables to the whole term structure rather than just to the one period rate through the policy rule

First, here is the answer:

$a_n = -A_n / n$ and $\mathbf{b}_n = -\mathbf{B}_n / n$ where

$A_1 = -\delta_0$ and $\mathbf{B}_1 = -\boldsymbol{\delta}_1$

$$\left.\begin{aligned} A_{n+1} &= A_n + \mathbf{B}_n'(\boldsymbol{\mu} - \boldsymbol{\Sigma}\boldsymbol{\lambda}_0) + .5\mathbf{B}_n'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\mathbf{B}_n - \delta_0 \\ \mathbf{B}_{n+1}' &= \mathbf{B}_n'(\boldsymbol{\Phi} - \boldsymbol{\Sigma}\boldsymbol{\lambda}_1) - \boldsymbol{\delta}_1' \end{aligned}\right\} \quad \text{for } n = 1,2,\ldots$$

Proof of Affine Term Structure Equations

Derivation of $y_t^{(n)} = a_n + \mathbf{b}_n' \mathbf{X}_t \qquad a_n = -A_n / n$ and $\mathbf{b}_n = -\mathbf{B}_n / n$

Start with $n = 1$

$P_t^{(1)} = \exp(-r_t) \Rightarrow y_t^{(1)} = \delta_0 + \boldsymbol{\delta}_1' \mathbf{X}_t \Rightarrow a_1 = -A_1 = \delta_0$ and $\mathbf{b}_1 = -\mathbf{B}_1 = \boldsymbol{\delta}_1$

Now for any n

$$P_t^{(n+1)} = E_t[m_{t+1} P_{t+1}^{(n)}]$$
$$= E_t \exp(-r_t - .5\boldsymbol{\lambda}_t'\boldsymbol{\lambda}_t - \boldsymbol{\lambda}_t'\varepsilon_{t+1} - n y_{t+1}^{(n)})$$
$$= E_t \exp(-r_t - .5\boldsymbol{\lambda}_t'\boldsymbol{\lambda}_t - \boldsymbol{\lambda}_t'\varepsilon_{t+1} + A_n + \mathbf{B}_n'\mathbf{X}_{t+1})$$
$$= E_t \exp(-r_t - .5\boldsymbol{\lambda}_t'\boldsymbol{\lambda}_t - \boldsymbol{\lambda}_t'\varepsilon_{t+1} + A_n + \mathbf{B}_n'(\boldsymbol{\mu} + \boldsymbol{\Phi}\mathbf{X}_t + \boldsymbol{\Sigma}\boldsymbol{\varepsilon}_{t+1}))$$
$$= E_t \exp(-\delta_0 - \boldsymbol{\delta}_1'\mathbf{X}_t - .5\boldsymbol{\lambda}_t'\boldsymbol{\lambda}_t - \boldsymbol{\lambda}_t'\varepsilon_{t+1} + A_n + \mathbf{B}_n'(\boldsymbol{\mu} + \boldsymbol{\Phi}\mathbf{X}_t + \boldsymbol{\Sigma}\boldsymbol{\varepsilon}_{t+1}))$$
$$= E_t \exp(-\delta_0 + A_n + \mathbf{B}_n'\boldsymbol{\mu} + (\mathbf{B}_n'\boldsymbol{\Phi} - \boldsymbol{\delta}_1')\mathbf{X}_t - .5\boldsymbol{\lambda}_t'\boldsymbol{\lambda}_t - (\boldsymbol{\lambda}_t' - \mathbf{B}_n'\boldsymbol{\Sigma})\boldsymbol{\varepsilon}_{t+1}))$$
$$= \exp(-\delta_0 + A_n + \mathbf{B}_n'\boldsymbol{\mu} + (\mathbf{B}_n'\boldsymbol{\Phi} - \boldsymbol{\delta}_1')\mathbf{X}_t - .5\boldsymbol{\lambda}_t'\boldsymbol{\lambda}_t) E_t \exp(-(\boldsymbol{\lambda}_t' - \mathbf{B}_n'\boldsymbol{\Sigma})\boldsymbol{\varepsilon}_{t+1}))$$

Now take expectations and this becomes :

$$\exp(A_{n+1} + \mathbf{B}_{n+1}'\mathbf{X}_t) = \exp(-(n+1)a_{n+1} - (n+1)\mathbf{b}_{n+1}\mathbf{X}_t) = \exp(-(n+1)y_t^{(n+1)})$$

where

$$\left.\begin{aligned} A_{n+1} &= A_n + \mathbf{B}_n'(\boldsymbol{\mu} - \boldsymbol{\Sigma}\boldsymbol{\lambda}_0) + .5\mathbf{B}_n'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\mathbf{B}_n - \delta_0 \\ \mathbf{B}_{n+1}' &= \mathbf{B}_n'(\boldsymbol{\Phi} - \boldsymbol{\Sigma}\boldsymbol{\lambda}_1) - \boldsymbol{\delta}_1' \end{aligned}\right\} \quad \text{for } n = 1,2,\ldots$$

E(exp(z))= exp(μ+σ²/2))

Other Models of the Term Structure

- Note that according to the no-arbitrage affine model, QE has no effect
- You need to put non-rational expectations or other rigidities into the theory
- Preferred habitat or market segmentation
  - Vayanos and Vila (2009)

$$D_{t,\tau} = \alpha_\tau (i_{t,\tau} - \beta_\tau)$$

- But little evidence until recently