Solving stochastic equilibrium models with the extended path method

Joseph E. Gagnon and John B. Taylor

This paper demonstrates that the extended path algorithm of Fair and Taylor [2] can be used to solve the class of asset pricing models introduced by Lucas [5]. The paper begins by describing the application of the extended path method to capital asset pricing. Next, the numerical solution is shown to converge to the true solution as the forecast horizon and the number of stochastic draws are increased. Finally, the method is used to solve a specific functional form of the model in order to test for accuracy and computational intensity.

Keywords: Capital asset pricing; Rational expectations; Solution algorithm

Much recent research in financial economics and monetary economics is based on non-linear stochastic equilibrium models. In these models agents are described as solving intertemporal maximization problems taking prices of assets and goods as given. The models are completed by describing a stochastic endowment or production process. Prices are assumed to be market clearing and expectations are assumed to be rational. An important example of this type of model is the Lucas [5] model of capital asset pricing in an exchange economy.

Except in special cases these capital asset pricing models are non-linear and analytical solutions are rare. This has made estimation of such models, or even calibration and analysis of their dynamic stochastic properties, very difficult.[1]

The purpose of this paper is to investigate the feasibility of using the extended path method – described in Fair and Taylor [2] – to solve capital asset pricing. This method is now being used rather routinely to solve large non-linear rational expectations models, such as the 120-equation multicountry model described in Taylor [8]. However, this multicountry model and other models to which the extended path method has already been applied are considerably different from the typical stochastic equilibrium model described above. To our knowledge the extended path method has not yet been applied to such models.

The paper is organized as follows. The next section briefly describes the Lucas [5] stochastic equilibrium model that we focus on. The third section describes the extended path algorithm and the following section proves that the algorithm converges to the unique rational expectations solution of the Lucas model under fairly general conditions. Finally we solve and

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[1] Many researchers are now engaged in finding ways to solve nonlinear rational expectations models. See Taylor and Uhlig [9] for a summary of several methods as applied to a stochastic growth model. Gagnon [3] describes how the method discussed in this paper can be applied to a stochastic growth model.

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simulate a specific functional form of the model using different parameter values in order to examine the rate of convergence. In simple cases where an analytic solution is feasible we check the accuracy of the numerical solutions. In more complicated cases we test whether the numerical solutions satisfy the theoretical properties presented in Lucas [5].

A representative model

We focus our numerical analysis on a model of capital asset pricing in an exchange economy as presented and examined by Lucas [5]. The economy is described by a representative agent who maximizes a discounted sum of expected utilities of consumption:

$$E_0 \sum_{t=0}^{\infty} \beta^t U(c_t) \quad 0<\beta<1 \tag{1}$$

where β is the subjective discount factor, $U(\cdot)$ is the one-period utility function, c is consumption, and E_0 is the expectations operator conditional on information at time zero.

On the production side, there is one firm which produces a perishable consumption good in the amount y. Production is purely exogenous. Since the good is perishable, c will equal y in all time periods. The firm has one perfectly divisible equity share which entitles the owners to the firm's output. Therefore, the dividend, d, paid on the share will also equal y.

The consumer maximizes (1) subject to his budget constraint:

$$p_t q_t = (p_t + d_t) q_{t-1} - c_t \tag{2}$$

where p_t is the price of the share in period t, and q_t is the proportion of the share that the consumer chooses to hold in period t. Since there is only one consumer, market clearing requires that $q_i = q_j = 1$ for all i, j. The first order condition is

$$p_t = \beta E_t[(p_{t+1} + d_{t+1}) U'(c_{t+1})]/U'(c_t) \tag{3}$$

Setting $c = d$ in Equation (3) yields

$$p_t = \beta E_t[(p_{t+1} + d_{t+1}) U'(d_{t+1})]/U'(d_t) \tag{4}$$

In this paper we suppose that dividends (and output) follow a univariate autoregression:

$$d_t = a_0 + a_1 d_{t-1} + a_2 d_{t-2} + \ldots + a_r d_{t-r} + e_t \tag{5}$$

where e is a random disturbance with zero mean. A rational expectations equilibrium solution for the model is thus a stochastic process for p which satisfies Equations (4) and (5). Mehra and Prescott [6] computed the equilibrium solution for this model when dividends follow a discrete two-state Markov chain rather than a continuously valued process like the one in Equation (5). Sargent [7] showed that when $U(\cdot)$ has the constant relative risk aversion form with a coefficient of unity (logarithmic utility) the solution has the particularly simple form:

$$p_t = \frac{\beta}{1-\beta} d_t \tag{6}$$

regardless of the distribution of dividends. We endeavour later in the paper to compute numerically the equilibrium solution for p for more general values of the coefficient of relative risk aversion and for general processes of the form (5) using the extended path method. In the next section we briefly summarize this method.

The extended path method

Define the variable x as

$$x_t = (p_t + d_t) U'(d_t) \tag{7}$$

Then Equation (4) can be written

$$p_t = \beta U'^{-1}(d_t) E_t x_{t+1} \tag{8}$$

and the entire model consisting of Equations (5), (7) and (8) can be written in the form:

$$f_i(z_t, E_t z_{t+1}, z_{t-1}, z_{t-2}, \ldots, z_{t-r}) = u_{it} \tag{9}$$

$$(u_{1t} = e_t,\ u_{2t} = 0,\ u_{3t} = 0)$$

where $z = (d, p, x)'$. Equation (9) is written in the form used by Fair and Taylor to explain the extended path method. We only briefly summarize the method here. For further details see Fair and Taylor [2].

To explain the solution method suppose first that the model is deterministic ie the variance of e is 0. To start the algorithm we begin with an initial guess of the expectations variable appearing in Equation (9) starting in the period following the initial period and going out $k+1$ periods. In other words, if we are interested in solving the model in period t, we start with guesses of the expectations variables for periods $t+1, t+2, \ldots, t+k+1$. Given these guesses for the expectations variables, the model can be solved for the elements of z in period t through period $t+k$. (The time horizon k is arbitrary at this point but will eventually be chosen so that it does not affect the solution.)

These solutions for z provide new guesses for the expectations variables and permit a second iteration in which a second solution for the z variables is obtained. This second solution is then used to replace the guesses of the future z's and the third iteration begins. (Note that the expectations variables in period $t+k+1$ do not get updated because the model is not solved in period $t+k+1$.) These iterations are repeated until the solved values of z in periods t through $t+k$ converge in the sense that the difference between the elements of z_{t+i} on two successive iterations is within a given tolerance range. These iterations are called Type II iterations by Fair and Taylor [2].[2]

Because it is possible that for a given horizon k the iterations will converge to the wrong value, it is necessary to perform another round of iterations (called Type III iterations). Increase k by unit steps and repeat the whole process described above until increasing k by another unit does not change the solution for z in period t by more than the prescribed tolerance range. The purpose of Type III iterations is to ensure that the guess of the expectations variables in period $t+k+1$ does not affect the solution values in period t. The prescribed tolerance for the Type III iterations must of course be larger than for the Type II iterations.[3]

Now consider the stochastic case where the disturbance e has a positive variance. Stochastic simulation must now be used to calculate the future expectations. As described in Fair and Taylor [2] the disturbance e – and therefore u – in Equation (9) is drawn stochastically. For each Type II iteration a total of $(n \times k)$ draws of the random variable u is taken from its assumed distribution. These draws are used to create n sequences of the disturbances u_{t+1} through u_{t+k}. (The model assumes that the shock in period t is known.) For each random sequence of u_{t+1} through u_{t+k}, Equation (9) is solved iteratively starting in period t and continuing through period $t+k$. The lag values needed to solve each period are taken from the previous period's solution and thus change from sequence to sequence. The lead values, however, are always the initial set of values of z at the beginning of the Type II iteration.

The solved values of z_{t+i}, $i=1, \ldots, k$, on each of these stochastic sequences of u are then averaged across all draws of u and used for the initial set of values of z on the next Type II iteration. The Type II iterations are repeated until convergence for a fixed k. Finally a series of Type III iterations are performed until the solution converges.

[2] Type I iterations are the Gauss-Seidel iterations necessary to solve a simultaneous system for a given set of expectations variables. In the Lucas [5] model the equations are recursive so no Type I iterations are needed.

[3] In order to speed convergence, we have found it useful to employ the final values from the previous Type III iteration as the initial guesses of the expectations variables on the next Type III iteration. Note, however, that fresh guesses are needed for the expectations beyond the horizon of the previous iteration.

Proof of convergence

In this section we demonstrate that the extended path algorithm converges to the rational expectations solution for the model presented above. Let n equal the number of draws of the stochastic process, e_{t+i}, drawn for each period, $i = 1, \ldots, k$. Let k be the forecast horizon for each Type III iteration as described above. The following paragraphs show that the extended path solution converges to the rational expectations solution as $n, k \to \infty$.

In order to demonstrate the convergence of the solved value of z_t to its true value, it is first necessary to establish some notation. Let $z_{t+i}(j, k)$ denote the solved value of z in period $t+i$ after the jth Type II iteration within a Type III iteration with forecast horizon k. Thus, the first argument in parentheses refers to the number of Type II iterations and the second argument refers to the forecast horizon of the current Type III iteration. $z_{t+i}(0, k)$, $i = 1, \ldots, k+1$, denotes the initial set of guesses of the z vectors for this Type III iteration.

The first Type II iteration begins by drawing a realization of the process, e^1_{t+i}, for each period, $i = 1, \ldots, k$. The superscript 1 refers to the first stochastic draw. $z^h_{t+i}(j, k)$ therefore denotes the value of z in period $t+i$ on the hth stochastic draw *before* the completion of the jth Type II iteration. The superscript is dropped when all the results of the stochastic draws have been averaged ie

$$n^{-1} \sum_{h=1}^{n} z^h_{t+i}(j, k) = z_{t+i}(j, k) \tag{10}$$

The first sequence of shocks is combined with the observed values of d in periods $t-r$ through t to compute a path of the dividend process, $d^1_{t+i}(1, k)$, according to Equation (5).

$$d^1_{t+1}(1, k) = a_0 + a_1 d_t + \ldots + a_r d_{t-r+1} + e^1_{t+1} \tag{11}$$

$$d^1_{t+2}(1, k) = a_0 + a_1 d^1_{t+1}(1, k) + a_2 d_t + \ldots + a_r d_{t-r+2} + e^1_{t+2}$$

$$\vdots$$

$$d^1_{t+k}(1, k) = a_0 + a_1 d^1_{t+k-1}(1, k) + a_2 d^1_{t+k-2}(1, k) + \ldots + a_r d^1_{t+k-r}(1, k) + e^1_{t+k}$$

The path $p^1_{t+i}(1, k)$ is computed using Equation (8) and $d^1_{t+i}(1, k)$, where $E_{t+i}x_{t+i+1}$ is taken from the vector $z_{t+i+1}(0, k)$. Finally, the path of $x^1_{t+i}(1, k)$ is given by Equation (7), $d^1_{t+i}(1, k)$, and $p^1_{t+i}(1, k)$.

New realizations of the process $e^h_{t+1}, \ldots, e^h_{t+k}$ are drawn, and the path of $z^h_{t+i}(1, k)$ is computed a total of n times. Note that the initial set of expectations guesses, $z_{t+i}(0, k)$, are always used when future expectations are required to solve an equation. Unlike the values of the lagged variables, expectations values do *not* change across stochastic draws of e.

To complete the first Type II iteration, take the average of these paths:

$$
\begin{aligned}
z_{t+i}(1, k) &= n^{-1} \sum_{h=1}^{n} z^h_{t+i}(1, k) \\
&= n^{-1} \sum_{h=1}^{n} \begin{pmatrix} d^h_{t+i}(1, k) \\ p^h_{t+i}(1, k) \\ x^h_{t+i}(1, k) \end{pmatrix} \qquad (12) \\
&= n^{-1} \sum_{h=1}^{n} \begin{pmatrix} d^h_{t+i}(1, k) \\ \beta U'^{-1}[d^h_{t+i}(1, k)]x_{t+i+1}(0, k) \\ [p^h_{t+i}(1, k) + d^h_{t+i}(1, k)]U'[d^h_{t+i}(1, k)] \end{pmatrix} \\
&= n^{-1} \sum_{h=1}^{n} \begin{pmatrix} d^h_{t+i}(1, k) \\ \beta x_{t+i+1}(0, k)U'^{-1}[d^h_{t+i}(1,k)] \\ \beta x_{t+i+1}(0, k) + d^h_{t+i}(1, k)U'[d^h_{t+i}(1,k)] \end{pmatrix}
\end{aligned}
$$

$$i = 0, \ldots, k$$

Successive Type II iterations proceed exactly like the first. Within each Type II iteration the expectations values are constant across stochastic draws, and these values are taken from the results of the preceding Type II iteration. A little substitution reveals that the general expression for a Type II iteration is

$$
z_{t+i}(j, k) = n^{-1} \sum_{h=1}^{n} \begin{pmatrix} d^h_{t+i}(j, k) \\ \beta x_{t+i+1}(j-1, k)U'^{-1}[d^h_{t+i}(j, k)] \\ \beta x_{t+i+1}(j-1, k) + d^h_{t+i}(j, k) \\ \times U'[d^h_{t+i}(j, k)] \end{pmatrix} \qquad (13)
$$

for $i = 0, \ldots k$.

Note that since z is not solved in period $t + k + 1$, the value of x in period $t + k + 1$ never changes during this Type III iteration. An immediate implication is that the vector z_{t+k} achieves Type II convergence after the first Type II iteration as long as the same sequences of shocks are drawn on each Type II iteration.[4] Note also that the value of d in each period does not change from one Type II iteration to the next, given the same draws of the shocks. The (Type II) converged value of z in period $t + k$ is given by

$$
z_{t+k}(1, k) = n^{-1} \sum_{h=1}^{n} \begin{pmatrix} d^h_{t+k}(1, k) \\ \beta x_{t+k+1}(0, k)U'^{-1}[d^h_{t+k}(1, k)] \\ \beta x_{t+k+1}(0, k) + d^h_{t+k}(1, k) \\ \times U'[d^h_{t+k}(1, k)] \end{pmatrix} \qquad (14)
$$

Once the vector z_{t+k} has converged, the expected value of x_{t+k} used to solve for z_{t+k-1} in subsequent Type II iterations will not change. Thus, z_{t+k-1} achieves Type II convergence after only two Type II iterations.

$$
\begin{aligned}
z_{t+k-1}(2, k) &= n^{-1} \sum_{h=1}^{n} \begin{pmatrix} d^h_{t+k-1}(2, k) \\ \beta x_{t+k}(1, k)U'^{-1}[d^h_{t+k-1}(2, k)] \\ \beta x_{t+k}(1, k) + d^h_{t+k-1}(2, k) \\ \times U'[d^h_{t+k-1}(2, k)] \end{pmatrix} \qquad (15) \\
&= n^{-1} \sum_{h=1}^{n} \begin{pmatrix} d^h_{t+k-1}(1, k) \\ \{\beta^2 x_{t+k+1}(0, k) + n^{-1} \sum_{h=1}^{n} \beta d^h_{t+k}(1, k) \\ \times U'[d^h_{t+k}(1, k)]\} \\ \times U'^{-1}[d^h_{t+k-1}(1, k)] \\ \beta^2 x_{t+k+1}(0, k) + \sum_{j=0}^{1} \beta^j d^h_{t+k-1+j}(1, k) \\ \times U'[d^h_{t+k-1+j}(1, k)] \end{pmatrix}
\end{aligned}
$$

It is easy to see that the vector z_{t+k-2} converges after the third Type II iteration and z_{t+k-i} converges after the $(i + 1)$th iteration. Furthermore, $d^h_{t+i}(j, k)$ does not depend on the Type II or Type III iteration, so we will abbreviate it as d^h_{t+i}. By successive substitutions we obtain the expression for the (Type II) converged value of z_t:

$$
z_t(k + 1, k) = n^{-1} \sum_{h=1}^{n} \begin{pmatrix} d_t \\ [\beta^{k+1} x_{t+k+1}(0,k) + \sum_{j=1}^{k} n^{-1} \sum_{h=1}^{n} \beta^j d^h_{t+j} \\ \times U'(d^h_{t+j})]U'^{-1}(d_t) \\ \beta^{k+1} x_{t+k+1}(0, k) + \sum_{j=0}^{j} \beta^j d^h_{t+j} U'(d^h_{t+j}) \end{pmatrix} \qquad (16)
$$

[4] Equation (17) below demonstrates that the assumption of identical sequences of shocks is not necessary when the number of draws is large.

Next, consider the effect of increasing the number of stochastic draws, n. For a fixed guess of $z_{t+k+1}(0, k)$, and for a real-valued $U(\cdot)$ that is continuous over the range of d, we have:[5]

$$\lim_{n\to\infty} z_t(k+1, k) = E_t\left(\frac{d_t}{\dfrac{\left[\beta^{k+1}x_{t+k+1}(0,k)+\sum_{j=1}^{k}\beta^j d_{t+j}U'(d_{t+j})\right]\times U'^{-1}(d_t)}{\beta^{k+1}x_{t+k+1}(0,k)+\sum_{j=0}^{k}\beta^j d_{t+j}U'(d_{t+j})}}\right) \tag{17}$$

where the convergence is almost everywhere, or strong convergence. Equation (17) states that the value of $z_t(k+1, k)$ approaches the true expected value of the expression inside the large parentheses almost surely as n approaches infinity.

To understand this convergence result better, consider each element of $z_t(k+1, k)$ separately. The dividend, $d_t(k+1, k)$, is known and therefore does not depend on the n stochastic draws. The asset price, $p_t(k+1, k)$, consists of two terms. The first term

$$n^{-1}\sum_{h=1}^{n}\beta^{k+1}x_{t+k+1}(0,k)U'^{-1}(d_t)$$

is fixed and does not depend on the stochastic draws. The second term,

$$n^{-1}\sum_{h=1}^{n}\sum_{j=1}^{k}n^{-1}\sum_{h=1}^{n}\beta^j d^h_{t+j}U'(d^h_{t+j})U'^{-1}(d_t)$$

can be rearranged as

$$U'^{-1}(d_t)\sum_{j=1}^{k}\beta^{jh-1}\sum_{h=1}^{n}d^h_{t+j}U'(d^h_{t+j})$$

For each j, the expression

$$n^{-1}\sum_{h=1}^{n}d^h_{t+j}U'(d^h_{t+j})$$

is the average value of $d_{t+j}U'(d_{t+j})$ over the stochastic draws of u. The Kolmogorov law of large numbers states that the average of any continuous, real-valued function of normally distributed independent random variables approaches the true expected value of that function almost surely as the number of observations increases. Because the draws of the disturbance are taken over an independent normal distribution, the law of large numbers operates to yield Equation (17). Finally, convergence of $x_t(k+1, k)$ is trivial given the convergence of $p_t(k+1, k)$.

Equation (17) demonstrates that the (Type II) converged value of $z_t(k+1, k)$ still depends on the initial guess of $x_{t+k+1}(0, k)$. The Type III iterations are designed to ensure that the final converged solution does not depend on this initial guess to any significant extent. Each successive Type III iteration proceeds by raising the value of k and restarting the entire Type II convergence process. The ultimate result is to augment the value of k in Equation (17) by one. As long as the sequence of endpoint guesses, $x_{t+k+1}(0, k)$, is not increasing in k at an exponential rate equal to or greater than $1/\beta$, the following limit holds for the variable of interest:

$$\lim_{k,n\to\infty} p_t(k+1,k) = U'^{-1}(d_t)\sum_{j=1}^{\infty}\beta^j E_t[d_{t+j}U'(d_{t+j})] \tag{18}$$

By successively shifting Equation (4) forward, substituting for p_{t+1} in the righthand side, and applying the law of iterated expectations, it is easy to obtain the following expression for the asset price in period t:

$$p_t = U'^{-1}(d_t)\sum_{j=1}^{\infty}\beta^j E_t[d_{t+j}U'(d_{t+j})] \tag{19}$$

By inspection of Equations (18) and (19) it is clear that the extended path solution converges to the true rational expectations solution as the number of stochastic draws and the number of Type III iterations increase.

Solving the model

In this section we report a small selected number of the experiments we performed to test accuracy and computation time for the extended path as a solution method for the capital asset pricing model. For the remainder of the section we assume that $U(\cdot)$ has the constant relative risk aversion form:

$$U(c) = \frac{c^{1-\alpha}}{1-\alpha} \quad 0 < \alpha \tag{20}$$

The first test of the method is for the case of log utility ($\alpha = 1$). In this case there is a simple analytic solution (Equation (6)) regardless of the distribution of dividends. This gives a simple test of the algorithm. We calculated the asset price for hypothetical dividend series with

[5] Equation (17) follows from the Kolmogorov law of large numbers under the assumption that the random draws of the disturbance, e, are independent and identically distributed. See Amemiya [1] Theorem 3.3.2.

Table 1. Solution values for asset prices with a discount factor of 0.95.

$d_t = 8.0 + e_t \quad \sigma_e^2 = 1.0$

Number of stochastic draws: $n = 50$

Number of periods in forecast horizon: $k = 200$

Case 1

Coefficient of relative risk aversion (α) equals 1.5

Trial	Price (p)	Dividend (d)	Ratio (p/d)
1	124.8	7.0	17.83
2	152.5	8.0	19.06
3	182.0	9.0	20.22

Elasticity of price to dividend: 1.501

Case 2

Coefficient of relative risk aversion (α) equals 0.5

Trial	Price (p)	Dividend (d)	Ratio (p/d)
1	142.4	7.0	20.35
2	152.2	8.0	19.03
3	161.4	9.0	17.94

Elasticity of price to dividend: 0.499

Table 2. Solution values for asset prices with a discount factor of 0.95.

$d_t = 4.0 + 0.5d_{t-1} + e_t \quad \sigma_e^2 = 1.0$

Number of stochastic draws: $n = 50$

Number of periods in forecast horizon: $k = 200$

Case 1

Coefficient of relative risk aversion (α) equals 1.5

Trial	Price (p)	Dividend (d)	Ratio (p/d)
1	125.0	7.0	17.85
2	152.3	8.0	19.04
3	181.3	9.0	20.14

Elasticity of price to dividend: 1.479

Case 2

Coefficient of relative risk aversion (α) equals 0.5

Trial	Price (p)	Dividend (d)	Ratio (p/d)
1	142.4	7.0	20.34
2	152.6	8.0	19.07
3	162.3	9.0	18.03

Elasticity of price to dividend: 0.521

various dynamic specifications. (Throughout this section the discount factor was fixed at 0.95.) With $k = 100$ the ratio of the price p to the dividend d was about 1% away from the exact value of 19. When $k = 200$ the error drops below 0.01%. When $k = 50$ the error rises to about 50%. These results are insensitive to the specification of the dividend process; consequently, the accuracy is not improved by taking more stochastic draws.

We also experimented with values of the coefficient of relative risk aversion greater or less than one. For these other values there is no analytic solution with which to compare the numerical solution. However, Section 7.2 in Lucas [5] shows that when the dividend process is not autocorrelated, the elasticity of price to dividend is equal to the coefficient of relative risk aversion, α. Table 1 presents the simulated asset prices and elasticities arising from three different realizations of the dividend process, given a normal iid process for future dividends. As predicted, the elasticity of price with respect to dividends is no longer unity. With $k = 200$ and $n = 50$, the simulated elasticities in Table 1 are equal to the coefficient of relative risk aversion with a tolerance of about 0.1%.[6]

The solution accuracy appears more sensitive to the forecast horizon than to the number of stochastic draws. Cutting the forecast horizon k by half increases the error to over 1% ($k = 100$, $n = 50$). On the other hand, taking only one-fifth as many draws raises the error by just under 1% ($k = 200$, $n = 10$).

We also present solution results with dividends following a first order autoregressive process (see Tables 2 and 3). Once again $k = 200$ and $n = 50$. The serial correlation in dividends tended to increase the elasticity of prices with respect to dividends (compared with the independent dividends case) when the coefficient of relative risk aversion is less than one, and

Table 3. Solution values for asset prices with a discount factor of 0.95.

$d_t = 0.8 + 0.9d_{t-1} + e_t \quad \sigma_e^2 = 1.0$

Number of stochastic draws: $n = 50$

Number of periods in forecast horizon: $k = 200$

Case 1

Coefficient of relative risk aversion (α) equals 1.5

Trial	Price (p)	Dividend (d)	Ratio (p/d)
1	127.1	7.0	18.16
2	152.2	8.0	19.03
3	178.4	9.0	19.82

Elasticity of price to dividend: 1.347

Case 2

Coefficient of relative risk aversion (α) equals 0.5

Trial	Price (p)	Dividend (d)	Ratio (p/d)
1	141.8	7.0	20.26
2	154.4	8.0	19.30
3	166.6	9.0	18.51

Elasticity of price to dividend: 0.641

[6] The numerical solutions presented in Table 1 required a total of 5.3 minutes of CPU time on an Amdahl 5840 (8 MIPS) and 9.8 minutes on an IBM 4381. The algorithm was programmed in FORTRAN and is available upon request.

to decrease it when that coefficient is greater than one.[7] These effects can also be expected from the calculations presented in Lucas [5].

Overall, the experiments seem to indicate that the extended path method gives accurate solutions for the asset price in this type of stochastic simulation model. The horizon length of 200 periods to get accuracy of around 0.1 % seems very large to us.[8] However, the number of stochastic draws is rather modest.

Concluding remarks

The above solution results illustrate the feasibility of using the extended path method in this type of application. There are still several issues that need to be resolved including whether the method converges to the correct value in a wider class of models than the one we checked here.

Gagnon [3] describes the use of the extended path algorithm to solve a stochastic growth model. As of now, there is no proof that the extended path algorithm converges to the true rational expectations solution in this class of models. However, by comparing the solutions with those of alternative algorithms, it appears that the extended path algorithm yields quite accurate solutions with a modest amount of computation time.

Computation time may be an issue in larger systems, although we could do all the computations described here with only a moderate amount of computer time. A careful comparison of computation time of this method with other methods would also be of practical value.

[7] The models with serially correlated dividends required somewhat larger computation times than the model with uncorrelated dividends. For example, the solutions in Table 3 used twice as much CPU time as those in Table 1.

[8] As can be seen in Equation (17), the speed of convergence as the forecast horizon, k, increases is of exponential order β. When β is increased from 0.95 to 0.99, the margin of error in Table 1 rises from 0.1 % to 1 %.

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