On the Properties of Equilibria in Private Value Divisible Good Auctions with Constrained Bidding

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Abstract

I analyze a model of a private value divisible good auction with different payment rules, standard rationing rule pro-rata on-the-margin and both with and without a restriction on the number of bids (steps) bidders can submit. I provide characterization of equilibrium bidding strategies in a model with restricted strategy sets and I show that these equilibria converge to some equilibrium of the model with unrestricted strategy sets as the restrictions are relaxed. In particular, the equilibrium conditions require that the Euler condition characterizing equilibrium strategies in the unrestricted model holds "on average" over the intervals defined by the length of each step of the restricted strategy, where the average is taken with respect to the *endogenous* distribution of the market clearing price. I also prove that the foregone surplus of bidders from using K steps rather than the optimal continuous bids is proportional to $\frac{1}{K^2}$. Sufficient conditions for equilibrium existence are also provided.

Keywords: multiunit auctions, equilibrium existence, uniform price auction, discriminatory auction

JEL Classification: D44

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1 Introduction

Accounting for the differences of real world applications and abstract models of economic theory is an important ingredient in any empirical work. The aim of this paper is to bridge the gap between the theoretical and empirical literature on divisible good auctions. The existing theory literature on divisible good auctions which started with the seminal work of Wilson (1979) usually assumes that the strategies may be continuously differentiable, which is not the case in most (if not all) real life applications. Using the results obtained using these theoretical models to draw inference from the bidding data on the primitives of the model such as the distribution of bidders' valuations can therefore be quite problematic. The goal of this paper is to provide a model which respects the defining features of the real world markets and to obtain equilibrium characterization and existence results which can then constitute basis for empirical work, the body of which has been growing the recent years.

In particular, I consider a classic private value divisible good auction model with the standard rationing rule pro-rata on-the-margin. The new feature of my model is that bidders are restricted to use step functions as their bids, which is a common feature of virtually all auctions in practice. I provide characterization of equilibrium bidding strategies in both a discriminatory (DA) and a uniform price auction (UPA).[1] I further analyze the behavior in auctions with constrained strategy sets by proving two results: First, I prove that the necessary conditions for equilibrium of the restricted model require that the Euler condition characterizing equilibrium strategies in the unrestricted model holds "on average" over the intervals defined by the length of each step of the restricted bidding strategy, where the average is taken with respect to the *endogenously determined* distribution of the market clearing price. This is a result analogous to Wilson (1993)'s result about optimality of

[1] In a discriminatory auction a bidder has to pay her full bid for all units she gets allocated. In a uniform price auction a bidder pays the same market clearing price for all such units.

multipart tariffs, with the main difference being that in the price discrimination setting the distribution (of valuation types) over which the average is taken is exogenous. A corollary of this result is that as we increase the number of steps bidders are allowed to use without bounds, we approach the equilibrium characterization from a model without restrictions as characterized in Wilson (1979) for UPA and Hortaçsu (2002) for DA. Second, I provide a convergence result which bounds the loss of surplus of each bidder due to using only finitely many steps. Regarding equilibrium existence, I prove that in a discriminatory auction with private values and restricted strategy sets there exists an equilibrium in distributional strategies whenever private information is independent across bidders. Extending previous results of McAdams (2006a) to a setting with continuous prices and quantities, I also show that when the restriction on the number of steps that can be used is removed and signals are independent, a monotone pure strategy equilibrium exists in both a uniform price and discriminatory auctions.

This paper is not the first to point out the importance of the discreteness of bidding. Kremer and Nyborg (2004b) analyze a setting where bids have to be specified in terms of discrete quantity units and discrete prices. They show that if the distances between points on each grid are appropriately modified, the auctioneer can eliminate some equilibria which might be quite harmful to auction revenue otherwise. In contrast, I assume in the present paper that both quantity and price are continuous, but that bidding strategies can only consist of finitely many price-quantity pairs. I will argue that this alternative notion of discreteness is also very important for equilibrium characterization. It can in fact be shown that if price and quantity were continuous, rationing rule were pro-rata on-the-margin, and the auctioneer required bidding strategies to be step functions, the resulting set of possible equilibria would also not contain the undesirable underpricing equilibria.

The main reason for the growing interest in the empirical analysis of divisible good auctions is the long-standing debate about whether to use a discriminatory or uniform price auction in order to sell such commodities as government securities or electricity. In case of government securities, many influential economists have argued in favor of a uniform price auction as it seems to be less prone to informational barriers that might preclude some potential bidders from participating. Ausubel and Cramton (2002) established that these two auction mechanisms can be ranked ex ante neither on efficiency nor on revenue grounds. Therefore to be able to answer the question which mechanism might perform better in a given environment, the researcher needs to employ empirical techniques. Several recent papers seek to answer this particular question in a structural framework employing slight variations of the theoretical model proposed in the seminal work of Wilson (1979). Using a slight modification of the necessary conditions characterizing equilibrium bid functions of this model to restrict the bids to lie on a discrete grid, Hortaçsu (2002) estimates marginal valuations of bidders in Turkish treasury bill auctions. However, as mentioned above, Wilson's model restricts the bidders to use only continuously differentiable bid functions so that elegant techniques of the calculus of variations can be applied to solve the model. Yet, in virtually all real world applications bidders are not allowed to use continuously differentiable bid functions. The number of points through which they can characterize their bid functions is finite and very often quite low (for example 10 in case of the Czech treasury bill auctions), moreover bidders almost never approach this upper bound. Kastl (2006) investigates a model of a uniform price auction of a perfectly divisible good in which each bidpoint (step) is costly to submit, and hence, in equilibrium, bidders submit finitely many points as their bids. He finds that necessary conditions and implied bidders' behavior can be quite different in such a setting as the need for a coarser characterization of bid functions forces the bidders to "bundle" bids for several units together and thus introduces new trade-offs. Due to this bundling effect and the resulting trade-off between gain or loss on the last unit in the bundle and higher or lower probability of winning the inframarginal units in the bundle, a rational bidder may submit a bid higher than his marginal valuation for that last unit in a UPA. This then makes empirical work more difficult as the researcher is not able to bound the ex-post revenue of a UPA by a hypothetical UPA with truthful bidding. In this paper I investigate these new trade-offs further. In particular, I show that the optimal bidding behavior in an auction with strategies being required to be step functions involves bidding in a way that the Euler condition characterizing the optimal behavior in a model with strategies being required to be continuously differentiable downward sloping functions holds "on average" over the length of each step (i.e., quantity-bid).

The remainder of this paper is organized as follows. After setting up the general environment of the model in Section 2, I focus on equilibrium characterization in Section 3. I provide implicit characterization of equilibrium bidding strategies via a set of necessary conditions that rule out profitable local deviations in quantity demands for both a discriminatory and uniform price auctions. In Section 4 I investigate the relationship between equilibria and associated payoffs in auctions with restricted strategy sets and those in auctions with no restrictions on strategies. In Section 5 I give sufficient conditions for an equilibrium in distributional strategies to exist in a discriminatory auction with both restricted and unrestricted strategy sets. In case of a uniform price auction I extend results from McAdams (2006a) to prove existence of a pure strategy equilibrium in non-decreasing strategies when strategy sets are not restricted and to show existence of an ε-equilibrium in pure non-decreasing strategies when strategy sets are restricted. Finally, Section 6 concludes.

2 General Setup

I start with the basic share auction framework of Wilson (1979) with private information and private values. There are N potential bidders, who are bidding for a share of a perfectly divisible good and N is commonly known. Each bidder receives a private (possibly multi-dimensional) signal, s_i, which is the only private information about the underlying value of the auctioned good. The joint distribution of signals is denoted by $F(\mathbf{s})$.

Assumption 1 *Bidders signals, $s_1, ..., s_N$, are drawn from a common support $[0,1]^M$ according to an atomless joint d.f. $F(s_1, ..., s_N)$ with strictly positive density f.*

Winning share q of the unit good is valued according to a marginal valuation function $v_i(q, s_i)$. I impose the following assumptions on the marginal valuation function $v(\cdot, \cdot)$:

Assumption 2 *$v_i(q, s_i)$ is measurable and bounded, strictly increasing in each component of s_i $\forall q$, and weakly decreasing and continuous in q $\forall s_i$.*

I denote by $V(q, s_i)$ the gross utility: $V(q, s_i) = \int_0^q v_i(u, s_i)\, du$.

As in practice in most auctions bidders are restricted in the number of points they can use to describe their bid functions, I start analyzing the model with an upper bound on the allowed number of bidpoints $\overline{K}$. Later I consider relaxing this upper bound so that I can analyze the relationship between the model with restricted strategy sets and the traditional model with unrestricted strategy sets considered in Wilson (1979) and Hortaçsu (2002). I also assume that there is a bid l which loses no matter how the rivals behave and there is an upper bound on the maximal bid $\bar{b}$, which for example in the case of treasury bills could be the face value.[2]

Assumption 3 *Each player $i = 1, ..., N$ has an action set:*

$$A_i = \left\{ \begin{array}{c} \left(\vec{b}, \vec{q}, K\right) : \dim\left(\vec{b}\right) = \dim\left(\vec{q}\right) = K \in \left\{0, ..., \overline{K}\right\}, \\ b_{ik} \in B = \{l\} \cup \left[0, \bar{b}\right], \ q_{ik} \in Q = [0,1], \ b_{ik} \geq b_{ik+1}, q_{ik} \leq q_{ik+1} \end{array} \right\}$$

[2] The assumption that bids are bounded is necessary only for the existence results. The bid l can be interpreted as the decision not to participate in a given auction.

Finally, since bidders use step functions, a situation may arise in which multiple prices would clear the market. If that is the case, I assume consistently with most real world auction mechanisms that the auctioneer selects the most favorable price from his perspective, i.e., the highest price.

Assumption 4 *If in any auction* $\exists\ \underline{p}, \overline{p}$ *such that* $\forall p \in \left[\underline{p}, \overline{p}\right] : TD(p) = Q$*, then* $p^c = \overline{p}$*, where* $TD\left(p\right)$ *denotes total demand at price* p*.*

Because bidders' strategies are step functions for any finite $\bar{K}$, the residual supply is a step function and hence but for knife-edge cases any equilibrium involves rationing with probability one. I consider only the rationing rule pro-rata on-the-margin, under which the auctioneer proportionally adjusts the marginal bids so as to equate supply and demand, because this is the only rationing rule I am aware of being used in practice.[3]

Assumption 5 *The rationing rule employed is pro-rata on-the-margin, under which the rationing coefficient satisfies*

$$R\left(p^c\right) = \frac{Q - TD_+\left(p^c\right)}{TD\left(p^c\right) - TD_+\left(p^c\right)}$$

where $TD\left(p^c\right)$ *denotes total demand at price* p^c*, and* $TD_+\left(p^c\right) = \lim_{p \downarrow p^c} TD\left(p\right)$*. Only the bids exactly at the market clearing price are adjusted.*

It is important to notice that while rationing occurs with probability one (as the bidding functions are step functions), we have to distinguish two different situations: (i) only one bidder is marginal, i.e., the residual supply cuts vertically this bidder's bid at the market clearing price and (ii) multiple bidders are marginal, i.e., the residual supply has a horizontal

[3]Kremer and Nyborg (2004a) show that an alternative rationing rule which does not give priority to bids above the market clearing price could also be used. They call such rationing rule "pro-rata" and they show that it may have some desirable properties in that it would eliminate low-revenue equilibria. The downside of implementing such a rationing rule is of course its potentially negative impact on efficiency of the allocation.

segment overlapping with each of these bidders' bids at the market clearing price. Notice that even though rationing occurs in both cases, in case (i) a slight perturbation in the quantity demanded by the rationed bidder at the market clearing price has no effect on his allocation conditional on him being rationed, while in case (ii) his allocation slightly increases. Throughout the paper I refer to the situation described in case (ii) as *a tie*.[4]

Bidders' pure strategies are mappings from private signals to bid functions: $\sigma_i : S_i \to \mathcal{Y}$, where the set $\mathcal{Y}$ includes all possible functions $y : P \to [0, 1]$ when strategy space is unrestricted or the set of step functions satisfying assumption 3 in the case of restricted strategies. A bid function for type s_i can thus be summarized by a function, $y_i(\cdot|s_i)$, which specifies for each admissible price $p \in P$, how big a share $y_i(p|s_i)$ of the securities offered in an auction (type s_i of) bidder i demands. Q denotes the size of the good to be divided between the bidders.[5] Q might itself be a random variable if it is not announced by the auctioneer ex ante or if the auctioneer has the right to augment the supply after he collects the bids. In either case, I assume that the distribution of Q is common knowledge among the bidders. When $\bar{K}$ is finite, we can summarize a bid function for type s_i also by a $\bar{K}$-dimensional vector of price-quantity pairs, where k^{th} pair specifies the height and length of k^{th} step. To distinguish between equilibria in the restricted game and the unrestricted one, I use the term K-step equilibrium to denote a BNE of the restricted game.

Definition 1 *A K-step equilibrium is a Bayesian Nash Equilibrium of a game satisfying Assumption 3 with finite $\bar{K}$.*

In the following sections I look at the characterization of Bayesian Nash Equilibria (and K-step equilibria) of this game with uniform price and discriminatory auction mechanisms with restricted strategy sets and relate these results to previous results which did not impose

[4]Notice that rationing occurs even in case (i), but *a tie* does not occur.

[5]For example for a perfectly divisible unit good, $Q = 1$.

the restrictions on strategies. I investigate the properties of these equilibria and look at the issue of their existence.

3 Equilibrium Characterization

3.1 Discriminatory Auctions

The expected utility of a bidder i who is employing a strategy $y_i(\cdot|s_i)$ in a discriminatory auction given that other bidders are using $\{y_j(\cdot|\cdot)\}_{j\neq i}$ can be written as:

$$EU_i(s_i) = E_{Q,s_{-i}|s_i}\left[\begin{array}{c} \int_0^{q_i^c(Q,\mathbf{s},\mathbf{y}(\cdot|s))} v_i(u,s_i)\,du \\ -\sum_{k=1}^{K} \mathbf{1}\left(q_i^c(Q,\mathbf{s},\mathbf{y}(\cdot|s)) > q_k\right)(q_k - q_{k-1})\,b_k \\ -\sum_{k=1}^{K} \mathbf{1}\left(q_k \geq q_i^c(Q,\mathbf{s},\mathbf{y}(\cdot|s)) > q_{k-1}\right)\left(q_i^c(Q,\mathbf{s},\mathbf{y}(\cdot|s)) - q_{k-1}\right)b_k \end{array}\right]$$

where $q_i^c(Q,\mathbf{s},\mathbf{y}(\cdot|s))$ is the (market clearing) quantity bidder i obtains if the state (bidders' private information and the supply quantity) is $(Q,\mathbf{s})$ and bidders submit bids specified in the vector $\mathbf{y}(\cdot|s) = [y_1(\cdot|s_1),...,y_N(\cdot|s_N)]$. $\mathbf{1}(\cdot)$ is an indicator function equal to 1 if the argument is true and 0 otherwise. A Bayesian Nash Equilibrium in this setting is thus a collection of functions such that almost every type s_i of bidder i is choosing his bid function so as to maximize his expected utility: $y_i(\cdot|s_i) \in \arg\max EU_i(s_i)$ for a.e. s_i and all bidders i.

It is straightforward to see that a bidder never submits a bid for any q that has a chance of being accepted and that is above his marginal valuation for that q in a discriminatory auction without restriction on strategies. All such bids are strictly dominated by bidding the marginal value for that q instead. When strategies are restricted, a similar result can be established and is stated as lemma 2 in the appendix.

Let $p^c(Q,\mathbf{s},\mathbf{y}(\cdot|s))$ be the market clearing price associated with state $(Q,\mathbf{s})$. Hortaçsu (2002) shows that when strategy sets are restricted to include only continuously differentiable functions, every equilibrium strategy $y(p|s_i)$ has to obey the Euler equation:

$$v(y(p|s_i), s_i) = p + \frac{H(p, y(p|s_i))}{H_p(p, y(p|s_i))} \tag{1}$$

where $H(p, x)$ is the probability distribution of the market clearing price when x units are demanded by bidder i (at price p) and all other bidders $j \neq i$ submit the equilibrium bid functions, i.e., $H(p, x) \equiv \Pr(p^c \leq p|x) = \Pr\left(x \leq Q - \sum_{j \neq i} y(p|s_j)\right)$ (H_p is the derivative of $H(\cdot, \cdot)$ with respect to the first argument, i.e., the density of the market clearing price.)

Now I focus on equilibria within the restricted class of strategies as specified in Assumption 3. I begin by stating sufficient conditions for no (payoff relevant) ties[6] to occur with positive probability in a K-step equilibrium.

Lemma 1 *Under assumptions 1-5 ties occur with zero probability for a.e. s_i in any K-step equilibrium of a discriminatory auction, except possibly at the last step, where s_i must be indifferent between winning or losing all units between the lowest share he gets allocated after rationing in the event of a tie and the last infinitesimal unit he may be allocated in equilibrium, i.e., $b_K = v(q, s_i)\ \forall q \in \left[\underline{q}^{RAT}, \overline{q}\right]$, where $\underline{q}^{RAT} \equiv \inf_{(Q,s_{-i}) \in TIE(s_i)} q_i^c(Q, s_{-i}, s_i, \mathbf{y}(\cdot|s))$, $TIE(s_i) \equiv \{(Q, s_{-i}) : \exists j, m : b_{i,K}(s_i) = b_{j,m}(s_j) = p^c(Q, s_{-i}, s_i, \mathbf{y}(\cdot|s))\}$, and $\overline{q} \equiv \sup_{Q,s_{-i}} q_i^c(Q, s_{-i}, s_i, \mathbf{y}(\cdot|s))$.*

In order to grasp the intuition behind the last lemma, let us think about a type who is tying with positive probability. Being involved in a tie implies that the share allocated after rationing is strictly less than the share demanded. Whenever there is a positive probability of winning the last infinitesimal unit demanded at a given step, bidding above one's marginal

[6] Recall that a tie occurs when demands of at least two bidders are rationed at the market clearing price.

valuation for that last infinitesimal unit is weakly dominated by bidding the marginal valuation itself.[7] Therefore, if the surplus enjoyed from obtaining the last infinitesimal unit allocated after rationing is strictly positive and by continuity of $v(\cdot, s_i)$ so is the surplus for all units very close to this one, then a slight deviation in i's bid up, which would break the tie, would result in a strict increase in i's payoff due to a strict increase in surplus from allocation and arbitrarily small increase in payment.

Therefore, were a bidder to tie with positive probability, the surplus on the last infinitesimal unit demanded would have to be weakly negative, as otherwise he would prefer to avoid the tie. Therefore, a tie could occur with positive probability only at the last step, because otherwise the bidder could just shift a small enough share demanded from the step at which tie has a positive probability of occurring to the next one at which he is bidding less. As in all states, in which he was allocated the share that he now shifted to the neighboring step, he had to pay the full bid, i.e., he earned no surplus on those (otherwise there would have been a simpler profitable deviation of increasing his bid by ε), and there is positive probability that the market clearing price will be weakly lower than his next bid (otherwise relabel the bids so that we call the last bid weakly below which market can actually clear the last one), this deviation would be strictly profitable (the expected quantity after rationing at k^{th} step is continuous in i's demand q_k and hence for a small enough deviation, the negative effect on the allocated quantity share after rationing can be made arbitrarily small).

Notice that we can rewrite the expected utility of a bidder of type s_i in a discriminatory auction as:

$$\begin{aligned} EU(s_i) &= \sum_{k=1}^{K} \left[\Pr(b_k > p^c > b_{k+1} | s_i) V(q_k, s_i) - \Pr(b_k > p^c | s_i) b_k (q_k - q_{k-1}) \right] \\ &+ \sum_{k=1}^{K} \Pr(b_k = p^c | s_i) E_{Q, s_{-i} | s_i} \left[V(q_i^c(Q, \mathbf{s}, \mathbf{y}(\cdot|s)), s_i) - b_k q_i^c(Q, \mathbf{s}, \mathbf{y}(\cdot|s)) | b_k = p^c \right] \end{aligned} \tag{2}$$

[7]For a formal statement see lemma 2 in the appendix.

where as before p^c is the (random) market clearing price, $q_i^c(Q, \mathbf{s}, \mathbf{y}(\cdot|s))$ is the quantity share allocated to i if the state of the world is $(Q, \mathbf{s})$ and bidders use strategies $\mathbf{y}(\cdot|s)$, and $q_0 = b_{K+1} = 0$. Note that maximization of this expected utility with respect to quantity demanded at k^{th} step, q_k, results in expressions involving realizations of the market clearing price only in the interval $[b_{k+1}, b_k]$. In particular, as I emphasize later, the problem is slightly reminiscent of the solution to the maximization problem with unconstrained strategies - only there the optimality condition would involve only one realization of the market clearing price p^c at b_k rather than a whole interval.

Strategies in any equilibrium must be locally optimal, in other words, any local deviation must not be profitable. Using a local perturbation argument, for a discriminatory auction I obtain the following necessary conditions that the quantity requested in any step of a pure strategy that is a part of a K-step equilibrium has to satisfy.

Proposition 1 *Under assumptions 1-5 in any K-step Equilibrium of a discriminatory auction, for almost all s_i, every step $k < K$ in the equilibrium bid function $y_i(\cdot|s_i)$ has to satisfy*

$$\Pr(b_k > p^c > b_{k+1}|s_i)[v(q_k, s_i) - b_k] = \Pr(b_{k+1} \geq p^c|s_i)(b_k - b_{k+1}) \quad (3)$$

and if the marginal valuation function is continuous also at $q_K < 1$, the last step K has to satisfy $v(\overline{q}, s_i) = b_K$ where $\overline{q} = \sup_{Q,s_{-i}} q_i^c(Q, s_{-i}, s_i, \mathbf{y}(\cdot|s))$.

The optimality condition with respect to the bid b_k can be derived in a straightforward manner by differentiating (2), but it cannot be simplified and interpreted as naturally as equation (3).[8] The intuition for the necessary condition (3) is nicely obtained once we think about in which states of the world varying i's demanded quantity can actually affect his

[8]For empirical work in the spirit of the recent literature on non-parametric estimation of auction models pioneered by Guerre, Perrigne and Voung (2000), equation (3) is exactly what a researcher needs in order to obtain identification of marginal valuation at q_k using data on bids in a discriminatory auction.

payoff. Changing quantity demanded at k^{th} step affects i's payoff only in the states that he is not rationed in, as in the event he is rationed either he is the only bidder that is rationed and changing his demanded quantity will thus not affect his payoff, or if there is a tie, he is indifferent between winning or not winning the last units by lemma 1, and changing the quantity he wins marginally thus again has no effect on his payoff. Therefore the cost of quantity shading is losing the surplus on the last unit $v(q_k, s_i) - b_k$ which happens only in the case that the quantity demanded at this step is actually marginal, i.e., market clears strictly between b_k and b_{k+1}. On the other hand by shading his demanded quantity by a unit, he saves the difference between his k^{th} and $(k+1)^{st}$ bid on this unit whenever the price will be weakly lower than the $(k+1)^{st}$ bid.

Notice an important distinction between the discretized version of the Euler equation from the model with unrestricted bidding given by equation (8) in Hortaçsu (2002) and our optimality condition (3). In Hortaçsu's model, the indexes k and $k+1$ refer to adjacent prices on the discretized grid, whereas in the optimality condition (3) they refer to bids at subsequent steps of the submitted bid curve.

3.2 Uniform Price Auctions

Let us now turn to the case of a uniform price auction. The expected utility of a bidder i who is employing a strategy $y_i(\cdot|s_i)$ in a uniform price auction given that other bidders are using $\{y_j(\cdot|\cdot)\}_{j\neq i}$ can be written as:

$$\begin{aligned} EU_i(s_i) &= E_{Q,s_{-i}|s_i} u(s_i, s_{-i}) \\ &= E_{Q,s_{-i}|s_i} \left[\int_0^{q_i^c(Q,\mathbf{s},\mathbf{y}(\cdot|s))} v_i(u, s_i, s_{-i})\, du - p^c(Q, \mathbf{s}, \mathbf{y}(\cdot|s))\, q_i^c(Q, \mathbf{s}, \mathbf{y}(\cdot|s)) \right] \end{aligned}$$

where as before $q_i^c(Q, \mathbf{s}, \mathbf{y}(\cdot|s))$ is the (market clearing) quantity bidder i obtains if the state (bidders' private information and the supply quantity) is $(Q, \mathbf{s})$ and bidders bid according to strategies specified in the vector $\mathbf{y}(\cdot|s) = [y_1(\cdot|s_1), ..., y_N(\cdot|s_N)]$, and similarly $p^c(Q, \mathbf{s}, \mathbf{y}(\cdot|s))$ is the market clearing price associated with state $(Q, \mathbf{s})$. A Bayesian Nash Equilibrium in this setting is thus a collection of functions such that almost every type s_i of bidder i is choosing his bid function so as to maximize his expected utility: $y_i(\cdot|s_i) \in \arg\max EU_i(s_i)$ for a.e. s_i and all bidders i.

In most of the previous literature, starting with Wilson (1979), the set $\mathcal{Y}$ of admissible strategies was restricted to continuously differentiable functions so that calculus of variations techniques could be applied. These techniques enable us to show that in an IPV model and within this restricted class of strategies a symmetric BNE $y(\cdot|\cdot)$ has to satisfy the following necessary condition for all (p, s_i):

$$v(y(p|s_i), s_i) = p - y(p|s_i)\frac{H_y(p, y(p|s_i))}{H_p(p, y(p|s_i))} \tag{4}$$

where as above $H(p, x)$ is the probability distribution of the market clearing price when x units are demanded by bidder i and all other bidders $j \neq i$ submit the equilibrium bid functions, i.e., $H(p, x) \equiv \Pr(p^c \leq p|x) = \Pr\left(x \leq Q - \sum_{j \neq i} y(p|s_j)\right)$ (H_p and H_y are the derivatives of $H(\cdot, \cdot)$ with respect to the first and second argument respectively). As Wilson points out, the auction game might have multiple equilibria, some of which lead to low revenue for the auctioneer. Back and Zender (1993) also noted that a UPA might posses equilibria that are extremely "bad" for the auctioneer, because under rationing pro-rata on-the-margin any price above the reservation price could potentially be supported as a market clearing price at all states of the world, i.e., independent of the realization of the private information, by suitable choice of strategies (which are not continuously differentiable). Such equilibria, while achieved in a non-cooperative way, are usually called "seemingly collusive" and several authors (e.g., Kremer and Nyborg (2004b), LiCalzi and Pavan (2004) and McAdams (2006b)) show how the auctioneer would eliminate at least some of these undesirable equilibria.

Let us now state the set of necessary conditions that pure strategies played by bidders in any equilibria have to satisfy with probability one. The argument is again based on a requirement that any local deviation must not be profitable. When bidders are restricted in the number of bids they are allowed to submit, Kastl (2006) proves the following result:

Proposition 2 *Under assumptions 1-5 in any K-step equilibrium of a Uniform Price Auction, for almost all s_i, every step k in the equilibrium bid function $y_i(\cdot|s_i)$ has to satisfy:*
(i) If $v(q_k, s_i) > b_k$

$$\Pr(b_k > p^c > b_{k+1}|s_i)\left[v(q_k, s_i) - E(p^c|b_k > p^c > b_{k+1}, s_i)\right] = q_k \frac{\partial E(p^c I[b_k \geq p^c \geq b_{k+1}]|s_i)}{\partial q_k} \tag{5}$$

(ii) If $v(q_k, s_i) \leq b_k$

$$\begin{aligned} &\Pr(b_k > p^c > b_{k+1}|s_i)\left[v(q_k, s_i) - E(p^c|b_k > p^c > b_{k+1}, s_i)\right] + \\ &+\Pr((b_k = p^c \vee b_{k+1} = p^c|s_i) \ \wedge \ Tie)\, E\left(\left(v\left(q^{RAT}, s_i\right) - p^c\right)\frac{\partial q^{RAT}}{\partial q_k}|\,(b_k = p^c \vee b_{k+1} = p^c) \ \wedge \ Tie, s_i\right) \\ &= q_k \frac{\partial E(p^c I[b_k \geq p^c \geq b_{k+1}]|s_i)}{\partial q_k} \end{aligned} \tag{6}$$

Once again the necessary condition for each bid b_k can be derived in a straightforward manner, but it does not lend itself to a nice interpretation as (5).[9] The intuition for (5) is again quite simple. If a bidder bids below his marginal valuation of the last unit, he strictly prefers to win all units he requests at his bid, and thus cannot tie in equilibrium with positive probability. In this case, shading his demanded quantity by one unit results in a loss of his surplus on that unit, in the case that the market clearing price is between his k^{th} and $k+1^{st}$

[9] As equation (3) in the case of a discriminatory auction, in order to conduct non-parametric estimation, equation (5) provides the link between the (unobserved) primitives of the model, marginal valuation at q_k, and the observables - bids.

bid, which is captured on the LHS of (5). On the other hand, if his bid is marginal and sets the market clearing price, withholding the demand for the last unit can result in a decrease in the market clearing price, and this savings will be realized on all inframarginal units, and precisely this effect is captured on the RHS of (5). If a bidder bids weakly above his marginal valuation for some units, we can no longer guarantee that he does not tie with positive probability in equilibrium, and hence the cost of shading his demand now includes the effect on the surplus at the expected quantity after rationing in the event of a tie at one of his bids.

4 Properties of K-step equilibria

I begin the analysis of the properties of K-step equilibria by examining the relationship between K-step equilibrium optimality conditions derived in previous sections and the optimality conditions from the unrestricted model analyzed by Wilson (1979) and Hortaçsu (2002). Let $E^{DA}(b,q|s) \equiv v(q,s) - b - \frac{H(b,q)}{H_b(b,q)}$ be the Euler condition for optimal bidding in a discriminatory auction in the unrestricted model. Let E^{DA}_{K,q_k} be the optimality conditions for quantity bids at k^{th} step in a K-step equilibrium. Similarly, let $E^{UPA}(b,q|s) \equiv v(q,s) - b + q\frac{H_q(b,q)}{H_b(b,q)}$ be the corresponding Euler equation for a uniform price auction in the unrestricted model and E^{UPA}_{K,q_k} be the optimality condition for quantity bid in a K-step equilibrium. The optimality conditions for quantity demands in a K-step equilibrium are given by equations (3) and (5).

Proposition 3 *Under assumptions 1-5, the optimality conditions for bidding in a K-step equilibrium require the bids in both a uniform price and discriminatory auctions have to satisfy the Euler condition from the unrestricted model at each step* ***on average*** *for a.e. s_i , where the average is taken with respect to the distribution of market clearing price, $H(p,q)$.*

The last proposition can also be written (with some abuse of notation) as

$$E^{a}_{K,q_k}\left(\vec{b},\vec{q}|s_i\right)=\int_{b_{k+1}}^{b_k}E^{a}\left(p,q_k|s_i\right)H_p\left(p,q_k\right)dp=0$$

for $a\in\{UPA,DA\}$. This notation is not quite correct, since at the upper bound of the integral, b_k, the distribution of the market clearing price has a mass point and hence the density is not well-defined.

A natural related question to investigate is what happens to equilibrium characterization as we increase the limit on the number of steps that bidders can use to characterize their bids. Fortunately, a simple corollary to the previous proposition confirms the intuition that as we increase the number of steps without bounds, our conditions for both auction mechanisms converge to the conditions derived in Wilson (1979) for a UPA and in Hortaçsu (2002) for a DA.

Corollary 1 *As $K\rightarrow\infty$, any K-step equilibrium approaches an equilibrium from the differentiable model:*

$$\lim_{K\rightarrow\infty}E^{a}_{K,j}\left(\vec{b},\vec{q}|s\right)=E^{a}\left(p,j|s\right)\ \ for\ a\in\{UPA,DA\}\ \forall j\in\{q_k\}_{k=1}^{K}$$

An interesting question to ask is therefore whether using only few steps is enough to capture almost the full expected surplus. This question is similar to analyzing a price discrimination problem, where the seller chooses few multipart tariffs. Wilson (1993, §8.3) established that multipart tariffs are approximately optimal, in the sense that the loss associated with using a n multipart tariffs rather than an optimal nonlinear tariff is of the order $\frac{1}{n^2}$. In related work, Chao and Wilson (1987) showed a similar result in the context of a model with uncertain probability of service and few priority classes. The main difference from our setup is that in the price discrimination problem, the distribution of types is exogenously given, whereas in multiunit auctions, the distribution of the market clearing price is endogenously determined by equilibrium strategies of the participants. The following proposition establishes a similar result for bidding in multiunit auctions with restricted strategy sets.

Proposition 4 *Assume that $v(q, s_i)$ and $\Pr(q^c > X)$ have bounded second derivatives w.r.t. q_k for all k and a.e. s_i and hold rivals' behavior fixed. Then the loss in expected payoff from using a bid with K steps rather than an optimal continuous bid is of the order $\frac{1}{K^2}$.*

Note that I consider the asymptotics as the rivals' behavior is held fixed due to the problem of potential multiplicity equilibria. The previous result might provide an explanation why we observe so few steps in bid functions submitted in practice. In particular, suppose that a bid function with just one step leads to 50% of the optimal expected surplus, then a bid function with just three steps already leads to about 95% of the optimal surplus! Therefore using only a few steps might suffice for practical purposes. In light of the last proposition, it seems a little clearer why in practice bidders very often do not use all points that they are allowed to use. Moreover, the number of points actually submitted is quite often much lower than this upper bound. To rationalize this feature of the data Kastl (2006) assumes that bidding is costly. Using the estimates of marginal valuations he is then able to compute bounds on this implied cost of bidding, which in turn provide bounds on surplus that bidders could have extracted by using a larger number of points. His results suggest that the returns to the additional submitted point seem to be quite low. Similarly, Chapman, McAdams and Paarsch (2006) find that the loss of surplus due to not playing the best response (when costs of bidding are absent) seems to be surprisingly low. Now I turn to the issue of equilibrium existence.

5 Equilibrium Existence

5.1 Discriminatory Auctions

Before proceeding I need one additional assumption:

Assumption 6 *Signals are independent, i.e.,* $F(\mathbf{s}) = \prod_i f_i(s_i)$.

I begin this section with establishing a sufficient condition for existence of a K-step equilibrium in distributional strategies[10] of a discriminatory auction with private values.

Proposition 5 *Under assumptions 1-6 there exists a K-step equilibrium in distributional strategies of a discriminatory auction.*

In order to establish existence using a sequence of perturbed games that limits to our game, we need to argue that the (ex post) payoffs are locally continuous at the limiting strategies. Using lemma 1, no ties occur except for the last step, at which the potentially tying bidder earns no surplus on the marginal allocated units. Hence it is easy to find a contradiction if a tie were to occur at some other step than the last one under the limiting strategies. In particular, the proposed strategies used in some nearby perturbed game where ties were impossible would fail to constitute an equilibrium.

Ties could occur, however, at the last step. Yet as only countably many types of a given bidder can actually tie at the same price and be indifferent between winning or not the last allocated units (because v is strictly increasing in s) together with the observation that only countably many prices can clear the market with positive probability ex ante (due to independence of types), ties are again a zero probability event for a.e. type, and continuity thus obtains.

It is straightforward to extend this existence result to a game with no restriction on the number of steps. The strategy set in the limiting game is a closure of the strategy sets along

[10] See Milgrom and Weber (1985) for a definition.

the sequence of games in which we increase the upper bound on the number of steps, and hence existence obtains by using the upper-hemicontinuity of the best reply correspondence. In fact, without restrictions on the number of steps, one can prove (as illustrated in the next subsection for a UPA) a stronger result: existence of a pure strategy equilibrium in non-decreasing strategies (see also Reny (1999)).

5.2 Uniform Price Auction

Several recent papers (e.g. Reny (1999), Athey (2001), McAdams (2003, 2006a), Reny and Zamir (2004)) have provided very elegant existence results for games of incomplete information that guarantee in many cases existence of a pure strategy equilibrium that is non-decreasing in private information. I build upon these results to show existence of an equilibrium in the unrestricted version of our model. McAdams (2006a) provides an existence theorem for a standard multiunit uniform price auction, in which the quantity for sale is not perfectly divisible, a particular (and not quite realistic) rationing rule is used, and bids are restricted to come from a finite lattice. The difficulty in applying McAdams's result on isotone equilibrium thus lies in the continuous action spaces and in the rationing rule. While obtaining existence when the bidders are restricted to particular discrete action spaces in which ties are impossible is not difficult in the current model, the extension from discrete to continuous action spaces is not trivial, since under some rationing rules a discontinuity in ex post payoffs arises in the event of a tie.[11] It is precisely the continuity of payoffs in own actions that is needed in order to apply a result similar to Theorem 2 of Athey (2001) for extension of the existence result to continuous action spaces. Therefore, similarly to Reny and Zamir (2004), I first show existence of an equilibrium in non-decreasing strategies in a

[11]Kazumori (2003) was the first to attempt to generalize McAdams' proof to continuous action spaces. Kazumori uses a different approach to proving existence of an isotone equilibrium. He looks for equilibria in perturbed games, in which payoff-continuity is restored due to a different way rationing is performed, and then takes the perturbations to zero.

particular version of the current game with carefully selected discrete price-quantity grids. Then, in the second step, I consider a sequence of such games with the grids becoming finer and finer and limiting to a dense subset of $B \times Q$, and I show that the limit of these equilibria is indeed an equilibrium of the limiting game. With rationing pro-rata on-the-margin, the payoff is potentially discontinuous in the bid in the event of a tie. If the bidder would increase his price-bid slightly, he would break the tie, and thus the quantity he receives would increase discontinuously, leading to a discontinuous jump in his payoff. Fortunately, by considering a sequence of perturbed games I can show existence even with this rationing rule with private values.

The following proposition establishes a stronger existence result than Proposition 5 for a discriminatory auction, because it guarantees existence of a pure strategy equilibrium in which the strategies are non-decreasing in private information. The cost of this result, however, is that it applies only in games in which the bidders are not restricted in their bidding strategies. In particular, they can submit an arbitrary number of steps. I also require the private signals to be independently distributed in order to guarantee non-decreasing differences in interim payoffs (see lemma 5 in the appendix).

Proposition 6 *Under assumptions 1-6 and no restriction on the number of steps, i.e., $\overline{K} = \infty$, there exists an isotone pure strategy equilibrium of a uniform price auction.*

Having established existence for the case that the upper bound on the number of bid-points, K, is not set, this existence proof can be extended to guarantee existence of a Bayesian Nash Equilibrium in non-decreasing pure strategies in the model with continuously differentiable functions using the same argument as for the existence in a discriminatory auction.

When the upper bound on the number of bidpoints is set exogenously (or endogenously because of some cost per bidpoint as in Kastl (2006)), the approach used for proving Proposition 6 cannot be used, since the payoff function ceases to be modular. To see this, consider the special case of $K = 1$, i.e., bidders are allowed to submit at most one step. Such strategy set is still a lattice, but because the point defining the step is two-dimensional, if we take any two bid curves which differ in each coordinate of the step such that $q_1 < q_2$ and $b_1 > b_2$, the strategy corresponding to the component-wise greatest element (sup) (q_2, b_1) is different than either of the submitted ones. The consequence of this observation is that the set of allocations and market clearing prices under the two strategies can now be different from the corresponding set for the strategies constructed by using their meet and join. The equivalence of these sets in the case that bid functions are unrestricted (which has been observed by McAdams (2003)) is what allows us to obtain modularity. We cannot thus guarantee quasisupermodularity of interim payoffs in general. I can, however, prove a weaker result - existence of an ε-equilibrium, which makes use of our earlier result about the loss from coarse bidding summarized in Proposition 4.

Corollary 2 *Assume independent private values and rationing pro-rata on-the-margin, then for any* $\varepsilon > 0$: $\exists \underline{K}$ *such that* $\forall K \geq \underline{K}$ *there exists a* K*-step* ε*-equilibrium in non-decreasing pure strategies of the uniform price auction.*

Proving existence of an equilibrium even in distributional strategies in the setting with restricted strategies remains an open question. The reason why this problem is difficult is that with rationing pro-rata on-the-margin and restricted strategy sets some types can actually prefer to tie at some market clearing price to both winning and losing at that price, because they may submit bids above their marginal valuation schedules as discussed earlier. This possibility then makes it very hard to argue that in any limit of a sequence of perturbed games we cannot have a tie between a type who would prefer not to tie and a type who would prefer to tie.

Jackson and Swinkels (2005) analyze a wide range of multiunit auctions with discrete number of units and private values and establish existence of an equilibrium in distributional strategies for a variety of rationing rules. The main underlying argument of Jackson and Swinkels is that (positive probability) ties at the market clearing price are not compatible with equilibrium. This, however, requires that no bidder ever submits a bid above his marginal valuation, because otherwise he could potentially prefer tying over winning at his bid. It is exactly the impossibility of ties with positive probability that establishes continuity of payoffs in own bids.

Also notice that while Reny and Zamir (2004) prove existence of an isotone pure strategy equilibrium and the no-ties result for a first price auction with affiliated values, we cannot obtain such a strong result for multiunit auctions.[12] The reason is that with multiple units players' actions are not necessarily complements. In a first price auction with affiliated values, Reny and Zamir note that there is a sense that if others bid higher, i has an incentive to bid higher as well as it is "more likely" that the information of other bidders is more favorable. It turns out, however, that in a uniform price auction (as in a second price auction), i's payoff is not necessarily non-decreasing in the signals of other bidders, since the price paid may depend positively on those.

Therefore to make progress in empirical work with a model of a uniform price auction with restricted strategy sets as in assumption 3, we might need to perform the empirical exercise conditional on equilibrium existence. Alternatively, we could think about a model in which prices are discrete, but on a very fine grid. In that case, an equilibrium would be guaranteed to exist (a we have a finite game) and the necessary conditions derived from the model with continuous price space in this paper would hold approximately.

[12]In fact, they were able to show that the "good news" affect of affiliation outweighs the possible winner's curse.

6 Conclusion

I introduce a model of an auction of a perfectly divisible good, in which the bidders are restricted in the number of points through which they can characterize their bid functions. I provide equilibrium characterization for both uniform price and discriminatory auction mechanisms when values are private. I give sufficient conditions for equilibrium existence in a discriminatory auction for both settings: with and without a restriction on the number of steps. For a uniform price auction, I prove existence of a pure strategy equilibrium in non-decreasing strategies, but only for the case of unrestricted strategy sets and independent private values. With restricted strategy sets I prove existence of ε-equilibrium. I further demonstrate that the equilibrium characterization from the restricted model limits to the equilibrium characterization of the unrestricted model as the number of points bidders are allowed to use grows large. I also show that the loss from using only K steps is of the order of $\frac{1}{K^2}$, which may provide an explanation why in practice we observe even fewer steps in submitted bid functions than the maximal number bidders are allowed to use. Even though bidders might not be best responding in many of the auctions as empirically documented in Chapman, McAdams and Paarsch (2006) and Kastl (2006), both of these papers find that the losses associated with not fine-tuning the bid a little more by submitting more steps seem to be very low.

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A Appendix

A.1 Proof of Lemma 1

I begin with a preliminary lemma, which guarantees that no bidder submits a bid strictly above his marginal value in a discriminatory auction, whenever there is a positive probability of this bid being accepted.

Lemma 2 *In a discriminatory auction, if for a bidder of type s_i at some step k,* $\Pr\left(q_i^c\left(Q, s_{-i}, s_i, \mathbf{y}\left(\cdot|s\right)\right) \geq q_k\right) > 0$*, then* $b_k \leq v\left(q_k, s_i\right)$*.*

Proof. Suppose for contradiction that $b_k > v(q_k, s_i)$ for some step k, such that $\pi \equiv \Pr(q_i^c(Q, s_{-i}, s_i, \mathbf{y}(\cdot|s)) \geq q_k) > 0$. Consider the following deviation: Let $\overline{q} = \sup\{q : v(q, s_i) > b_k\}$. Change k^{th} step to $(b_k + \varepsilon, \overline{q} - \varepsilon^2)$. The upper bound on the loss from this deviation is: $\varepsilon\overline{q} + \varepsilon^2 v(\overline{q} - \varepsilon^2, s_i)$ where the first part is due to an increased payment and the second part due to a potential loss of surplus on quantities in $[\overline{q} - \varepsilon^2, \overline{q}]$. The lower bound on the gain from this deviation is $\pi(b_k - v(q_k, s_i)) > 0$, which is independent of ε. Therefore, for ε small enough the deviation is strictly profitable since $v(\cdot, s_i)$ is bounded by assumption. ■

The only case a bidder might submit a bid above his marginal value is at the last step (where we label the step as last when the probability of an allocation being larger than the one requested at this step is zero). Moreover, the only reason for submitting such a bid is if the bidder might tie with another bidder(s) and thus be rationed, in which case submitting a larger marginal demand might increase his allocation.

Now we are ready to prove lemma 1.

Suppose that there exists an equilibrium, in which for a type s_i of bidder i a tie between at least two bidders can occur with positive probability $\pi > 0$. Since there can be only finitely many prices that can clear the market with positive probability, in order for a tie to be a positive probability event, it has to be the case that there exists a positive measure subset of types $\hat{S}_{-i} \in [0,1]^{N-1}$ such that for some bidder j, and all profiles of types $s_{-i} \in \hat{S}'_{-i} \subseteq \hat{S}_{-i}$ (another positive measure subset) and some steps k and l we have $b_{ik}(s_i) = b_{jl}(s_j) = p^c(Q, s_{-i}, s_i)$. Without loss suppose that this event occurs at the bid (b_{ik}, q_{ik}), and that the maximum quantity allocated to i after rationing is $\overline{q}_i^{RAT} < q_{ik}$. Let $\overline{S}_\pi^R$ denote the maximal level of the residual supply at b_{ik} in the states leading to rationing at b_{ik}.

Consider a deviation to a step $b'_{ik} = b_{ik} + \varepsilon$ and $q'_{ik} = q_{ik}$ where ε is sufficiently small. This deviation increases the probability of winning $q_{ik} - q_{ik-1}$ units. Most importantly in the states that led to rationing under the original bid, type s_i of bidder i will now obtain $q^* > \overline{q}_i^{RAT}$, where $q^* \geq \min\left\{q_{ik}, \overline{S}_\pi^R\right\}$. Notice that since we hypothesized a positive probability of a tie at b_{ik}, we need to have $q_{ik-1} < q_i^{RAT} < q_{ik}$ due to rationing pro-rata on-the-margin. Therefore, there is indeed room for a deviation. The probabilities of winning other units remain unchanged. Therefore the lower bound on the increase in s_i's expected gross surplus from such a deviation is $\pi\left(V(q^*, s_i) - V\left(\overline{q}_i^{RAT}, s_i\right)\right) \geq 0$

To continue, let us first focus on steps other than the last one, $k < K_i$ and suppose that marginal valuation function is strictly decreasing. Then we have that

$$\pi\left(V(q^*, s_i) - V\left(\overline{q}_i^{RAT}, s_i\right)\right) > 0$$

as $v(q^*, s_i) > v(q_k, s_i) \geq b_{ik}$ where the last inequality follows from lemma 2. The increased bid $b_k + \varepsilon$ also results in an increase in the payment for the share requested at this step. This increase, however, is bounded by $(q_k - q_{k-1})\varepsilon$. Comparing the upper bound on the change in expected payment with the lower bound on the change in expected gross utility, in order for this deviation to be strictly profitable we need to obtain

$$(q_k - q_{k-1})\varepsilon < \pi\left(V(q^*, s_i) - V\left(\overline{q}_i^{RAT}, s_i\right)\right) \quad \text{(A-1)}$$

Since

$$0 < \frac{\pi\left(V(q^*, s_i) - V\left(\overline{q}_i^{RAT}, s_i\right)\right)}{(q_k - q_{k-1})}$$

for ε small enough, the inequality (A-1) will hold, and thus the proposed deviation would indeed be strictly profitable for the type s_i. There can be only countably many types s_i with a profitable deviation otherwise bidder i could implement this deviation jointly and thus for a.e. type s_i ties have zero probability in equilibrium for all bidders i.

Suppose that marginal valuation function is not strictly decreasing and the above described deviation is not strictly profitable, i.e., $V(q^*, s_i) - V\left(\overline{q}_i^{RAT}, s_i\right) = 0$. Let $\overline{q} = \sup\{q : v(q, s_i) > b_k\}$. Then consider the following deviation instead: $(b_k + \varepsilon, \overline{q} - \varepsilon^2)$. The lower bound on the gain is: $\pi(q_k - \overline{q})(b_k - b_{k+1})$ due to a lower payment whenever the allocation under the original strategy was bigger than q_k (which is independent of ε) and the upper bound on the loss is: $\varepsilon\overline{q} + \varepsilon^2 v(\overline{q} - \varepsilon^2, s_i)$ due to a higher payment and loss of surplus. Again, for ε small enough, this deviation is strictly profitable.

Now focus on the last step, K_i. Suppose that s_i's expected gross surplus from the deviation described above is non-positive, i.e., that $b_k \geq v(q, s_i)$ for some $q \in \left[\overline{q}^{RAT}, q_k\right]$. Suppose that $\exists q' \in \left(\overline{q}^{RAT}, q_k\right)$ such that $v(q', s_i) > b_k$ and for some arbitrarily small $\delta > 0$, $v(q' + \delta, s_i) \leq b_k$.
Let $q'' = \sup_q \left\{q : v(q, s_i) > b_k \wedge q \in \left[\overline{q}_i^{RAT}, q_k\right]\right\}$. If $v(q'', s_i) > b_k$, then consider the following deviation instead: Submit the same bid with k^{th} step being replaced with $(b_k + \varepsilon, q'')$. By the same argument as above, such a deviation strictly increases s_i's expected gross utility since the loss from not winning $q \in (q'', q_k]$ is zero, which follows from definition of q'' and since the marginal valuation function is weakly decreasing. If $v(q'', s_i) = b_k$, then consider the following deviation: submit the same bid with k^{th} step being replaced with $(b_k + \varepsilon, q'' - \varepsilon^2)$. Now the loss from not winning units in $(q'' - \varepsilon^2, q'']$ is bounded by $\varepsilon^2 v(q'' - \varepsilon^2, s_i)$, the loss from increased payment is bounded as before by $(q_k - q_{k-1})\varepsilon$, and the gain from increased allocation in the event of a tie is as before (from an ex ante perspective) at least

$\pi\left(V\left(q^{*},s_{i}\right)-V\left(\overline{q}_{i}^{RAT},s_{i}\right)\right)$ where $q^{*}\geq\min\left\{q''-\varepsilon^{2},\overline{S}_{\pi}^{R}\right\}$. As in the argument underlying the proof above, for sufficiently small ε, the proposed deviation results in a strict increase in s_i's payoff.

Now suppose that such q' does not exist, i.e., $v(q,s_i)\leq b_k\ \forall q\in\left[\overline{q}^{RAT},q_k\right]$. Let $q''=\inf_q\{q:v(q,s_i)\leq b_k\wedge q\in(q_{k-1},q_k)\}$. If $\overline{q}^{RAT}>q''>\underline{q}_i^{RAT}$ where $\underline{q}_i^{RAT}$ is the minimal allocation obtained by s_i in the event of a tie, then consider a deviation in k^{th} step to $(b_k+\varepsilon,q''-\varepsilon^2)$. This results in a profitable deviation by the same argument as above, with some $0<\pi'<\pi$ where π' is the probability of a tie which results in an allocation after rationing which is less than q''.

If instead $q''\leq\underline{q}_i^{RAT}$, s_i wins all units on which he gains strictly positive surplus even when tying at k^{th} step. Therefore, a tie may occur with positive probability only at the last step and the bidder must not prefer winning any units in $\left[\underline{q}_i^{RAT},q^{*}\right]$, where $\overline{q}\equiv\sup_{s_{-i},Q}y_i^c(Q,s_{-i},s_i)$, i.e., the maximal quantity s_i may be allocated in an equilibrium. Since units in $(\overline{q},q_K]$ are never won, the marginal value for those may be lower than the bid. QED

A.2 Proof of Proposition 1

We begin by a lemma which guarantees that under assumption 4 for a.e. type s_i the residual supply cannot in equilibrium vertically overlap with this type's bid with positive probability, except possibly at the last step, in which case it is payoff irrelevant.

Lemma 3 *In a discriminatory auction with private values in which the market clearing price is the highest price that would clear the market, for a.e. s_i, the measure of states in which multiple prices would clear the market is zero at all steps $k<K$.*

Proof. Suppose for contradiction that for s_i such event could occur with probability π at step (b_k,q_k). Since by lemma 1 a tie at b_k is a zero probability event at $k<K$, then as the residual supply is a step function, the type s_i could slightly decrease his bid to $(b_k-\varepsilon,q_k)$ and achieve a strict increase in his payoff, where the argument is essentially the same as in the proof of lemma 1. The reason being that without a tie at b_k, there is ε small enough that the effect on allocation to s_i in other states of the world is arbitrarily small, yet the saving resulting from the deviation is strictly positive, and hence the deviation is strictly profitable. ■

Now we are ready to prove the characterization. I begin with steps $k<K$ so that lemma 1 and lemma 3 can be applied.

We will perturb the k^{th} step to $q' = q_k - \varepsilon$ and take the limit as $q' \rightarrow q_k$.

Let $\theta_1(q_k)$ denote set of states of the world in which $b_k < p^c$, similarly let $\theta_2(q_k)$ correspond to $b_k = p^c$, $\theta_3(q_k)$ to $b_k > p^c > b_{k+1}$, $\theta_4(q_k)$ to $b_{k+1} \geq p^c \wedge \textit{no tie at } b_{k+1}$, and $\theta_5(q_k)$ to $b_{k+1} = p^c \wedge \textit{tie at } b_k$. Further let $\omega_2(q')$ denote the set of states transferred from $\theta_2(q_k)$ to $\theta_3(q')$ (as $q' < q_k$ market clearing price can only fall), $\omega_4(q')$ set of states transferred from $\theta_2(q_k)$ to $\theta_4(q') \cup \theta_5(q')$ and finally $\omega_3(q')$ states transferred from $\theta_3(q_k)$ to $\theta_4(q') \cup \theta_5(q')$.

Notice that

$$\begin{aligned}
\theta_1(q') &= \theta_1(q_k) \\
\theta_2(q') &= \theta_2(q_k) - \omega_2(q') - \omega_4(q') \\
\theta_3(q') &= \theta_3(q_k) + \omega_2(q') - \omega_3(q') \\
\theta_4(q') \cup \theta_5(q') &= \theta_4(q_k) \cup \theta_5(q_k) + \omega_3(q') + \omega_4(q')
\end{aligned}$$

Consider first the expected gross utility of a type s_i:

$$\begin{aligned}
EV(s_i|q_k) &= \Pr(\theta_1(q_k))\, E[V(q_i^c(Q,\mathbf{s},\mathbf{y}(\cdot|s)),s_i)|\theta_1] + \Pr(\theta_2(q_k))\, E[V(q_i^c(Q,\mathbf{s},\mathbf{y}(\cdot|s)),s_i)|\theta_2] \\
&\quad + \Pr(\theta_3(q_k))\, V(q_k,s_i) + \Pr(\theta_4(q_k)\cup\theta_5(q_k))\, E[V(q_i^c(Q,\mathbf{s},\mathbf{y}(\cdot|s)),s_i)|\theta_4\cup\theta_5]
\end{aligned}$$

Perturbing q_k to q' and taking the difference, we obtain (dropping q_k from the argument of θ's):

$$\begin{aligned}
EV(s_i|q_k) - EV(s_i|q') &= \Pr(\theta_2)\{E[V(q_i^c(Q,\mathbf{s},\mathbf{y}(\cdot|s)),s_i)|\theta_2] - E[V(q_i^c(Q,\mathbf{s},\mathbf{y}'(\cdot|s)),s_i)|\theta_2]\} \\
&\quad + \Pr(\theta_3)\{V(q_k,s_i) - V(q',s_i)\} \\
&\quad + \Pr(\theta_4\cup\theta_5)\left\{\begin{array}{c} E[V(q_i^c(Q,\mathbf{s},\mathbf{y}(\cdot|s)),s_i)|\theta_4\cup\theta_5] \\ -E[V(q_i^c(Q,\mathbf{s},\mathbf{y}'(\cdot|s)),s_i)|\theta_4\cup\theta_5] \end{array}\right\} \\
&\quad + \Pr(\omega_2)\{E[V(q_i^c(Q,\mathbf{s},\mathbf{y}'(\cdot|s)),s_i)|\omega_2,\theta_2] - V(q',s_i)\} \\
&\quad + \Pr(\omega_3)\{V(q',s_i) - E[V(q_i^c(Q,\mathbf{s},\mathbf{y}'(\cdot|s)),s_i)|\omega_3,\theta_4\cup\theta_5]\} \\
&\quad + \Pr(\omega_4)\left\{\begin{array}{c} E[V(q_i^c(Q,\mathbf{s},\mathbf{y}'(\cdot|s)),s_i)|\omega_4,\theta_2] \\ -E[V(q_i^c(Q,\mathbf{s},\mathbf{y}'(\cdot|s)),s_i)|\omega_4,\theta_4\cup\theta_5] \end{array}\right\}
\end{aligned}$$

Dividing by $q_k - q'$ and taking the limit we obtain:

$$\begin{aligned}
\frac{\partial EV(s_i|q_k)}{\partial q_k} &= \Pr(\theta_2)\, E\left[v(q_i^c(Q,\mathbf{s},\mathbf{y}(\cdot|s)),s_i)\,\frac{\partial q_i^c(Q,\mathbf{s},\mathbf{y}(\cdot|s))}{\partial q_k}|\theta_2\right] \\
&\quad + \Pr(\theta_3)\, v(q_k,s_i) \\
&\quad + \Pr(\theta_4\cup\theta_5)\, E\left[v(q_i^c(Q,\mathbf{s},\mathbf{y}(\cdot|s)),s_i)\,\frac{\partial q_i^c(Q,\mathbf{s},\mathbf{y}(\cdot|s))}{\partial q_k}|\theta_4\cup\theta_5\right]
\end{aligned} \tag{A-2}$$

where I applied l'Hospital's rule to eliminate all terms involving ω's. Notice that $\lim_{q' \to q_k} \Pr(\omega_i) = 0$ by lemma 3. Moreover, $\lim_{q' \to q_k} [q_i^c(Q, \mathbf{s}, \mathbf{y}(\cdot|s)) | \omega_i, \theta_j] = q_k$ and hence all terms involving ω's disappear.

Now consider the same argument applied to the expected payment for a type s_i:

$$\begin{aligned} EPay(s_i|q_k) &= \Pr(\theta_1) Pay(\theta_1) + \Pr(\theta_2)[E(q_i^c(Q, \mathbf{s}, \mathbf{y}(\cdot|s)) | \theta_2) - q_{k-1}] b_k + \Pr(\theta_3)[q_k - q_{k-1}] b_k \\ &+ \Pr(\theta_4)[E[q_i^c(Q, \mathbf{s}, \mathbf{y}(\cdot|s)) | \theta_4] - q_k] b_{k+1} \\ &+ \Pr(\theta_5)[E[q_i^c(Q, \mathbf{s}, \mathbf{y}(\cdot|s)) | \theta_4] - q_k] b_{k+1} \\ &+ \Pr(\theta_4 \cup \theta_5)(q_k - q_{k-1}) b_k + Pay(ind't\ of\ q_k) \end{aligned}$$

Perturbing q_k to q' and taking the difference we obtain:

$$\begin{aligned} EPay(s_i|q_k) - EPay(s_i|q') &= \Pr(\theta_2)[E(q_i^c(Q, \mathbf{s}, \mathbf{y}(\cdot|s)) | \theta_2) - E(q_i^c(Q, \mathbf{s}, \mathbf{y}'(\cdot|s)) | \theta_2)] b_k \\ &+ \Pr(\theta_3)[q_k - q'] b_k \\ &+ \Pr(\theta_4 \cup \theta_5)(q_k - q')(b_k - b_{k+1}) \\ &+ \Pr(\theta_4)[E[q_i^c(Q, \mathbf{s}, \mathbf{y}(\cdot|s)) | \theta_4] - E[q_i^c(Q, \mathbf{s}, \mathbf{y}'(\cdot|s)) | \theta_4]] b_{k+1} \\ &+ \Pr(\theta_5)[E[q_i^c(Q, \mathbf{s}, \mathbf{y}(\cdot|s)) | \theta_5] - E[q_i^c(Q, \mathbf{s}, \mathbf{y}'(\cdot|s)) | \theta_5]] b_{k+1} \\ &+ Terms\ involving\ \omega\text{'s} \end{aligned}$$

Dividing by $q_k - q'$ and taking the limit we get:

$$\begin{aligned} \frac{\partial EPay(s_i|q_k)}{\partial q_k} &= \Pr(\theta_2) b_k E\left[\frac{\partial q_i^c(Q, \mathbf{s}, \mathbf{y}(\cdot|s))}{\partial q_k} | \theta_2\right] \\ &+ \Pr(\theta_3) b_k \\ &+ \Pr(\theta_4 \cup \theta_5)(b_k - b_{k+1}) \\ &+ \Pr(\theta_4) b_{k+1} E\left[\frac{\partial q_i^c(Q, \mathbf{s}, \mathbf{y}(\cdot|s))}{\partial q_k} | \theta_4\right] \\ &+ \Pr(\theta_5) b_{k+1} E\left[\frac{\partial q_i^c(Q, \mathbf{s}, \mathbf{y}(\cdot|s))}{\partial q_k} | \theta_5\right] \end{aligned} \tag{A-3}$$

where again all terms involving ω's disappear after applying l'Hospital's rule as above. Finally, combining (A-2) and (A-3), we can use lemma 1 to eliminate terms involving the derivative of the rationed quantity, as either (i) a tie is a positive probability event at b_{k+1} and then $E[v(q_i^c(Q, \mathbf{s}, \mathbf{y}(\cdot|s)), s_i) | \theta_5] = b_{k+1}$ or (ii) a tie is a zero probability event. Collecting terms and rewriting θ's as the corresponding states of the market clearing price we obtain:

$$\Pr(b_k > p^c > b_{k+1})[v(q_k, s_i) - b_k] = \Pr(b_{k+1} \geq p^c)(b_k - b_{k+1})$$

as desired.

Finally, at the last step K, if $v(q, s_i)$ is locally continuous in q, then since there is no next step, we have $v(\overline{q}, s_i) = b_K$ by lemma 1. QED

A.3 Proof of Proposition 3

Let us start with a discriminatory auction, the claim is that

$$\int_{b_{k+1}}^{b_k} E^{DA}(p, q|s)\, dH(p, q) = E^{DA}_{K,q_k}\left(\vec{b}, \vec{q}|s\right)$$

where the average is over prices (bids) $b \in (b_{k+1}, b_k]$ since at the subsequent step, b_{k+1}, the demand is already q_{k+1}. Hence we need to evaluate

$$\int_{b_{k+1}}^{b_k} \left[v(q_k, s_i) - p - \frac{H(p, q_k)}{H_p(p, q_k)}\right] dH(p, q_k) - H_p(b_k, q_k)(v(q_k, s_i) - b_k) + H(b_k, q_k) +$$
$$\int_{q_{k-1}}^{q_k} \left[(v(u, s_i) - b_k)\frac{\partial u}{\partial q_k} - \frac{H(b_k, q_k)}{\Pr(p^c = b_k)}\right] dF(u) =$$
$$\int_{b_{k+1}}^{b_k} \left[v(q_k, s_i) - p - \frac{H(p, q_k)}{H_p(p, q_k)}\right] dH(p, q_k) - H_p(b_k, q_k)(v(q_k, s_i) - b_k) \tag{A-4}$$

where $F(u)$ is the distribution of the market clearing quantity and H_p is the density of the market clearing price defined on the open interval (b_{k+1}, b_k). The last line follows since

$$\int_{q_{k-1}}^{q_k} \frac{H(b_k, q_k)}{\Pr(p^c = b_k)} dF(u) = \frac{H(b_k, q_k)}{\Pr(p^c = b_k)} \Pr(q^c \in (q_{k-1}, q_k)) = H(b_k, q_k).$$

The subtracted and subsequently added terms appear since at b_k the Euler condition is different (and H_p is not well-defined), because $q^c(b_k)$ is a random variable and hence $\left.\frac{\partial V(q,s_i)}{\partial q_k}\right|_{p=b_k} = v(q^c(b_k), s_i)\frac{\partial q^c(b_k)}{\partial q_k}$. Since by lemma 1 either there are no ties with positive probability or $v(q^c(b_k), s_i) = b_k$, we have $(v(q^c(b_k), s_i) - b_k)\frac{\partial q^c(b_k)}{\partial q_k} = 0$. Notice that after integrating by parts the term $pH_p(p, q_k)$, the integral expression in (A-4) simplifies to:

$$\int_{b_{k+1}}^{b_k} v(q_k, s_i)\, dH(p, q_k) - [pH(p, q_k)]|_{b_{k+1}}^{b_k} =$$
$$= \Pr(b_k \geq p^c > b_{k+1})\, v(q_k, s_i) - b_k \Pr(b_k \geq p^c) + b_{k+1} \Pr(b_{k+1} \geq p^c)$$

Subtracting the last term in (A-4) and re-arranging, we get

$$v(q_k, s_i) - b_k - \frac{\Pr(b_{k+1} \geq p^c)}{\Pr(b_k > p^c > b_{k+1})}(b_k - b_{k+1})$$

which is equivalent to our equation (3).

Similarly, for a uniform price auction for $b \in (b_{k+1}, b_k]$ we need to evaluate:

$$\int_{b_{k+1}}^{b_k} \left[(v(q_k(p), s_i) - p) - q_k \frac{H_q(p, q_k)}{H_p(p, q_k)} \right] dH(p, q_k) - H_p(b_k, q_k)(v(q_k, s_i) - b_k) \tag{A-5}$$

Let us look at the last term under the integral separately:

$$\begin{aligned} \int_{b_{k+1}}^{b_k} q_k \frac{H_q(p, q_k)}{H_p(p, q_k)} dH(p, q_k) &= \\ = q_k \frac{\partial}{\partial q_k} \int_{b_{k+1}}^{b_k} H(p, q_k)\, dp &= \\ = q_k \frac{\partial}{\partial q_k} \left[H(p, q_k)\, p|_{b_{k+1}}^{b_k} - \int_{b_{k+1}}^{b_k} p H_p(p, q_k)\, dp \right] &= \\ = -q_k \frac{\partial E[p^c; b_k \geq p^c \geq b_{k+1}]}{\partial q_k} \end{aligned}$$

where the first equality follows after interchanging the derivative and integral sign. The second equality follows after integrating by parts and the last equality follows from the following lemma:

Lemma 4 *Suppose $H(p, q_k)$ is the probability distribution of the market clearing price in a K-step equilibrium. Then $\frac{\partial}{\partial q_k} H(b_k, q_k) = 0$.*

Proof. Recall that $H(b_k, q_k) = \Pr\left(q_k \leq Q - \sum_{-i} q_{-i}(b_k)\right)$. Consider first a perturbation of q_k down: $q_k' = q_k - \varepsilon$. In this case, $\forall Q, \mathbf{s_{-i}} : p'^c(Q, s_i, \mathbf{s_{-i}}) \leq p^c(Q, s_i, \mathbf{s_{-i}})$. In other words, $\forall \varepsilon < q_k - q_{k-1}$, in all states that led to market clearing at or below b_k before the perturbation, the market cannot clear at any other step than k of demand of bidder i, and hence the highest possible price is b_k, and therefore $H(b_k, \cdot)$ remains the same.

Now consider the perturbation $q_k' = q_k + \varepsilon$. In this case, $\forall Q, \mathbf{s_{-i}} : p'^c(Q, s_i, \mathbf{s_{-i}}) \geq p^c(Q, s_i, \mathbf{s_{-i}})$. But the only strict increase in the market clearing price can be for $p^c(Q, s_i, \mathbf{s_{-i}}) < b_k$ and at most to b_k, therefore $H(b_k, \cdot)$ remains the same. Consequently, $H(\cdot, \cdot)$ is differentiable at q_k in the second argument, and the derivative vanishes. ■

Notice that $\int_{b_{k+1}}^{b_k} pH_p(p,q_k)\,dp = E[p^c; b_k \geq p^c > b_{k+1}]$, but adding the remaining term $b_{k+1}H(b_{k+1},q_k)$ yields $E[p^c; b_k \geq p^c \geq b_{k+1}]$.

Evaluating the first two terms under the integral in (A-5), and recalling that $\frac{\partial E(q^c(p)|p^c=b_k \vee b_{k+1})}{\partial q_k} = 0$ and plugging in our last result yields:

$$\Pr(b_k > p^c > b_{k+1})\left(v(q_k,s_i) - E[p^c|b_k > p^c > b_{k+1}]\right) + q_k \frac{\partial E[p^c; b_k \geq p^c \geq b_{k+1}]}{\partial q_k}$$

which is our equation (5).

The proof for the Euler conditions for price bids is analogous. Note that both results can be also seen from comparing the rewritten expected utility in a DA given by (2) and its counterpart for continuously differentiable bids:

$$EU^{DA}(s_i) = \int_0^\infty \left\{\int_0^{y(p^c,s_i)} \left[v(q,s_i) - y^{-1}(q,s_i)\right] dq\right\} dH(p^c, y(p^c,s_i))$$

The continuous counterpart simply requires point-wise optimality of each $((b,q)$ pair in the bidding function against the distribution of the market clearing price, whereas the step function version requires the "average optimality" at each step. QED

A.4 Proof of Corollary 1

I show that as the number of bidpoints increases, the optimality condition limits to the one derived from the calculus of variations in Wilson (1979) for a uniform price auction and Hortaçsu (2002) for a discriminatory auction.

Define $H(p,q) = \Pr(p^c \leq p|q) = \Pr\left(\sum_{j\neq i} q_j(p) + q \leq Q\right)$. Similarly, define $G(p,q) = \Pr(p^c < p|q)$. Recall that equation (5) can be written as:

$$\Pr(b_k > p^c > b_{k+1})\, v(q_k,s_i) = E_{s_{-i}}(p^c(s_{-i},q_k); b_k > p^c > b_{k+1}) + q_k \frac{\partial E_{s_{-i}}(p^c(s_{-i},q_k); b_k \geq p^c \geq b_{k+1})}{\partial q_k}$$

Consider the last term:

$$\begin{aligned}
\frac{\partial E(p; b_k \geq p^c \geq b_{k+1})}{\partial q_k} &= \frac{\partial}{\partial q_k}\left[\int_0^{b_k} pH_p(p,q_k)\,dp - \int_0^{b_{k+1}} pG_p(p,q_k)\,dp\right] \\
&= pH_q(p,q_k)\big|_0^{b_k} - \int_0^{b_k} H_q(p,q_k)\,dp - pG_q(p,q_k)\big|_0^{b_{k+1}} + \int_0^{b_{k+1}} G_q(p,q_k)\,dp \\
&= -\int_0^{b_k} H_q(p,q_k)\,dp + \int_0^{b_{k+1}} G_q(p,q_k)\,dp
\end{aligned}$$

where the first equality follows by definition of the expectation, the second after interchanging the derivative and integral sign and integration by parts. Finally the third equation follows from Lemma 4. Hence we can rewrite the first order condition as:

$$\frac{[G(b_k, q_k) - H(b_{k+1}, q_k)]}{b_k - b_{k+1}} v(q_k, s_i)$$
$$= \frac{\int_0^{b_k} pG_p(p, q_k)\,dp - \int_0^{b_{k+1}} pH_p(p, q_k)\,dp + q_k\left[-\int_0^{b_k} H_q(p, q_k)\,dp + \int_0^{b_{k+1}} G_q(p, q_k)\,dp\right]}{b_k - b_{k+1}}$$

and take the limit as $b_{k+1} \to b_k$. Now $E_{s_{-i}}(p; b_k > p > b_{k+1}) \to b_k H_p(b_k, q_k)$ after applying l'Hospital's rule, which can be applied if $G \to H$. This limit holds as in the limit the distribution of the market clearing price does not have any masspoints as the probability of ties for any finite K is zero at all bids at which $b_k < v(q_k, s_i)$ and without restrictions on K, no bidder would submit a bid exceeding the marginal value. We have:

$$\lim_{b_{k+1}\to b_k} \frac{\int_0^{b_k} pG_p(p, q_k)\,dp - \int_0^{b_{k+1}} pH_p(p, q_k)\,dp}{b_k - b_{k+1}} = \frac{-b_k H_p(b_k, q_k)}{-1}$$

Furthermore:

$$\lim_{b_{k+1}\to b_k} \frac{\left[-\int_0^{b_k} H_q(p, q_k)\,dp + \int_0^{b_{k+1}} G_q(p, q_k)\,dp\right]}{b_k - b_{k+1}} = \frac{H_q(b_k, q_k)}{-1}$$

since $G \to H$. Finally:

$$\lim_{b_{k+1}\to b_k} \frac{[G(b_k, q_k) - H(b_{k+1}, q_k)]}{b_k - b_{k+1}} = H_b(b_k, q_k)$$

Collecting terms, replacing b with p as bid will be submitted at all possible prices and thus omitting subscript k, we obtain:

$$v(q, s_i) = p - q\frac{H_q(p, q)}{H_p(p, q)}$$

Since $q = y(p|s_i)$, this equation is the same as in (4).

Now consider the necessary condition for a discriminatory auction and rewrite it using

G and H defined as above:

$$\left[G\left(b_k, q_k\right) - H\left(b_{k+1}, q_k\right)\right]\left[v\left(q_k, s_i\right) - b_k\right] = H\left(b_{k+1}, q_k\right)\left(b_k - b_{k+1}\right)$$

Dividing both sides by $b_k - b_{k+1}$ and take the limit as $b_{k+1} \to b_k$.

$$\lim_{b_{k+1}\to b_k} \frac{\left[G\left(b_k, q_k\right) - H\left(b_{k+1}, q_k\right)\right]\left[v\left(q_k, s_i\right) - b_k\right]}{b_k - b_{k+1}} = H\left(b_{k+1}, q_k\right)$$

As above

$$\lim_{b_{k+1}\to b_k} \frac{\left[G\left(b_k, q_k\right) - H\left(b_{k+1}, q_k\right)\right]}{b_k - b_{k+1}} = H_p\left(b_k, q_k\right)$$

Using $q_k = y\left(p|s_i\right)$ and $b_k = p$ we can rewrite the necessary condition as

$$v\left(y\left(p|s_i\right), s_i\right) = p + \frac{H\left(p, y\left(p|s_i\right)\right)}{H_p\left(p, y\left(p|s_i\right)\right)}$$

which is the same equation as (1). QED

A.5 Proof of Proposition 4

Since the behavior of rivals is fixed, we have a single agent problem. I follow the strategy of Chao and Wilson (1987) of applying the results from numerical analysis to approximations of integrals by finite sums. Let $G\left(Y\right) = \Pr\left(q_i^c \geq Y\right)$, i.e., (one minus) the probability distribution of the market clearing quantity (given s_i's bid) - in other words this is the probability bidder i gets allocated at least Y. The potential surplus (expected utility) of bidder of type s_i is:

$$U_\infty = \int_0^Q \left[v\left(q, s_i\right) - y^{-1}\left(q|s_i\right)\right] G\left(q\right) dq$$

With submitting an optimal bid under the restriction to K steps, let $\Delta_k = q_{k+1} - q_k$ be the difference in adjacent demands and let $q_0 = 0$ and $q_K \leq Q$ where Q solves $v\left(Q, s_i\right) = 0$. The realized surplus can be written as:

$$U_K = \sum_{k=1}^{K} \overline{v}_k \overline{G}_k \Delta_k$$

where for step k the average (expected) net surplus, $\overline{v}$, and the (average) probability of demand at k^{th} step being (at least partially) satisfied, $\overline{G}_k$, are:

$$\overline{v}_k = \frac{\int_{q_k}^{q_k+\Delta_k} \left[v\left(q, s_i\right) - y^{-1}\left(q|s_i\right)\right] dq}{\Delta_k}$$

and

$$\overline{G}_k = \frac{\int_{q_k}^{q_k+\Delta_k} G\left(q\right) dq}{\Delta_k}$$

Using Hermite interpolation formula (Ralston and Rabinowitz (1978), Chapter 3.7):

$$\overline{v}_k = v\left(q_k, s_i\right) - y^{-1}\left(q_k|s_i\right) + \frac{1}{2}\Delta_k \left(v'\left(q_k, s_i\right) - y^{-1'}\left(q_k|s_i\right)\right) + 0\left(\Delta_k^2\right)$$
$$= v_k + \frac{1}{2}\Delta_k v_k' + 0\left(\Delta_k^2\right)$$

where v_k denotes the net surplus at q_k, and

$$\overline{G}_k = \frac{\int_{q_k-\Delta_k}^{q_k} G\left(q\right) dq}{\Delta_k} = G_k + \frac{1}{2}\Delta_k G'\left(q_k\right) + 0\left(\Delta_k^2\right)$$

where $G_k = G\left(q_k\right)$. Combining these, we can rewrite the expected surplus as:

$$U_K = \sum_{k=1}^{K} \left(v_k + \frac{1}{2}\Delta_k v_k'\right)\left(G_k + \frac{1}{2}\Delta_k G_k'\right)\Delta_k + 0\left(\Delta_k^3\right)$$
$$= \sum_{k=1}^{K} v_k G_k \Delta_k + \frac{1}{2}\left(v_k G_k\right)' \Delta_k^2 + 0\left(\Delta_k^3\right)$$

Since the step sizes, Δ_k's, were chosen optimally, considering an evenly spaced grid, with steps $\Delta = \frac{Q}{K}$ apart, must result in surplus that is weakly lower:

$$U_K \geq U_{\hat{K}} = \left[\sum_{k=1}^{K} v_k G_k \Delta\right] + \frac{1}{2}\Delta \left[\sum_{k=1}^{K} \left(v_k G_k\right)' \Delta\right] + 0\left(\Delta^3\right) \tag{A-6}$$

Applying the trapezoid rule for numerical integration (Gautschi (1997), Chapter 3.2.1), we get that:

$$U_\infty = \int_0^Q \left[v\left(q, s_i\right) - y^{-1}\left(q|s_i\right)\right] G\left(q\right) dq = \sum_{k=1}^{K} v_k G_k \Delta - \frac{1}{2}\Delta \left[v_0 G_0 + v_K G_K\right] + 0\left(\Delta^2\right)$$

or

$$\sum_{k=1}^{K} v_k G_k \Delta = U_\infty + \frac{1}{2}\Delta \left[v_0 G_0 + v_K G_K\right] - 0\left(\Delta^2\right)$$

and by a similar argument

$$\begin{aligned}\sum_{k=1}^{K} \left(v_k G_k\right)' \Delta &= \int_0^Q \left(vG\right)' dq + 0\left(\Delta\right) \\ &= v_Q G_Q - v_0 G_0 + 0\left(\Delta\right)\end{aligned}$$

Note that by definition of Q, $v_Q = v_K = 0$, as $v_K = v\left(Q, s_i\right) - y^{-1}\left(Q|s_i\right)$ and hence plugging these expressions into (A-6), the expressions $\frac{1}{2}\Delta v_0 G_0$ conveniently cancel and we get

$$U_K \geq U_{\hat{K}} = U_\infty - 0\left(\Delta^2\right)$$

which was to be shown. QED

A.6 Proof of Proposition 6

I will follow the strategy of Reny and Zamir (2004) of discretizing the action space to a fine grid and applying existence results from such settings. Let the discretized quantity space be $Q^n = \{0, ..., 1\}$ where $|Q^n| = M^n$ and the discretized bid space be $B_i^n = \{l\} \cup \{\varepsilon_{in}, ..., \bar{p}_{in}\}$. Notice that I require that the discretized bid space is potentially individual specific, while the discretized quantity space be the same. In each discretized game G^n we thus have a uniform price auction of M^n identical units.

The set of non-increasing bid schedules for M^n units forms a finite lattice. Moreover, we will require that the price-bid grids of no two players have any common point other than l, which is essential for proving the modularity of payoffs. If this property would be violated, then it would no longer be true that changing one's bid for k^{th} unit does not alter the probability of winning some other unit. For example if there is a chance that two players will tie for k^{th} unit at some $\hat{b}$, then with rationing pro-rata on-the-margin one bidder can increase his chances of winning this unit by increasing his bid for $(k+1)^{th}$ unit to $\hat{b}$.

In order to apply McAdams' (2003) result on existence of an isotone equilibrium, we need to verify two conditions - single-crossing of interim payoffs in own action and type and quasisupermodularity of interim payoffs in own actions. In fact similarly to McAdams (2006a) we will prove two stronger conditions for our setting.

Let us start with the first simple lemma.

Lemma 5 *If types are independent, then interim payoff satisfies non-decreasing differences in own bid and type for all actions of other bidders.*

Proof. Ex post payment does not depend on own type, but only on own bid, so we need to examine only ex post valuations. By submitting a higher price-bid or increasing the quantity requested at any step of the current bid function or both always leads to winning a weakly higher quantity. Since $v(q, s_i) \geq 0 \; \forall q$ and this marginal utility is strictly increasing in own type by assumption (and hence the incremental value is non-negative), ex post payoff must exhibit non-decreasing differences in own actions and types. Since non-decreasing differences are preserved under integration, interim payoff satisfies non-decreasing differences also. ■

Notice that non-decreasing differences is a stronger condition than single-crossing, and also that the previous lemma would hold under more general structure of values, since only independence of types is crucial.

Lemma 6 *Interim payoff is modular in own action for all $s_i \in S_i$ and all actions by other bidders.*

Proof. McAdams (2006a) proves that in an $(M^n)^{th}$ price auction the payoff is modular in own bid in the case that all bidders have the same finite bid space and with a particular (and not very realistic) rationing rule under quite general structure of values. Since ties are impossible in our discretized game, the rationing rule does not matter as long as it allocates only the "marginal" units demanded exactly at the market clearing price, and hence the payoff in our discretized game is modular by the same argument as in McAdams for the case of rationing pro-rata on-the-margin. The key property - that the set of allocations and market clearing prices under any two bid schedules is the same as the respective set under the upper and lower envelope of those bids - is still valid. The reason being that the allocation ex post depends only on the total demand by each bidder at the market clearing price, and not on any other steps. Therefore the allocation and market clearing price under the upper envelope in any state, must correspond to one of the allocations and market clearing prices under one of the original bids, and same of course holds for the lower envelope. ■

Having the two lemmas and the action spaces being finite lattices, Theorem 1 of McAdams (2003) applies, and we can be assured of the existence of an isotone pure strategy equilibrium in any such discretized game G^n. The crucial step of the argument is now to show that as we take the limit of a sequence of these discretized games, the limiting strategy and the associated limiting payoff form indeed an equilibrium of the limiting game with the original unrestricted strategy spaces.

With rationing pro-rata on-the-margin, we have a problem with the continuity of ex post payoff in own bid in an event of a tie. As a second step we will now consider a sequence of equilibria of the discretized games described above, in which we know a pure strategy isotone equilibrium exists, as the grid becomes finer and finer and show that the limit of such a sequence is indeed an equilibrium of the limiting game.

For $n = 1, 2, ...$, let G^n denote the modified auction in which bidder i's finite set of possible bids is denoted by B_i^n and common quantity grid Q^n. Further, suppose that $B_i^n \supseteq B_i^{n-1}$ and that $B_i^\infty = \cup_n B_i^n$ is dense in B_i and similarly for Q^n. Let $(\mathbf{b}^n \times \mathbf{q}^n)$ denote a monotone pure-strategy equilibrium of G^n where $\mathbf{q}^n$ is simply a vector of quantities that are a "gridstep" apart. Since b_{ik}^n is bounded below by l and above by assumption and $q_{ik}^n \in [0, 1]$, by Helley's selection theorem the sequence of equilibrium strategies (taking a subsequence if necessary) converges (pointwise) to a limit $\left(\hat{\mathbf{b}} \times \hat{\mathbf{q}}\right)$ for a.e. s_i, where each $\left(\hat{b}_{ik}, \hat{q}_{ik}\right)$ is nondecreasing on the support of signals. Now we need to argue that $\left(\hat{\mathbf{b}}, \hat{\mathbf{q}}\right)$ is an equilibrium of the unrestricted uniform price auction game. The problem in this argument arises because of the possible discontinuity in ex post payoff due to ties. In order to establish existence in the unrestricted game, we need to show the following:

$$\begin{aligned} \sup_{b \times q \in (B_i \times Q)^K} EU_i\left(b, q, \hat{\mathbf{b}}_{-i}, \hat{\mathbf{q}}_{-i} | s_i\right) &\leq \lim_n EU_i\left(b_i^n(s_i), q_i^n(s_i), \mathbf{b}_{-i}^n, \mathbf{q}_{-i}^n | s_i\right) \\ &= EU_i\left(\hat{b}_i(s_i), \hat{q}_i(s_i), \hat{\mathbf{b}}_{-i}, \hat{\mathbf{q}}_{-i} | s_i\right) \end{aligned} \tag{A-7}$$

Consider a type s_i and an arbitrary bid vector (b, q) such that the bid does not strictly exceed the marginal valuation for any unit. This is without loss as a bid schedule in which the bid exceeds the marginal value for some units is weakly dominated by an alternative bid schedule where the bids above marginal value are lowered. Then we have

$$EU_i\left(b, q, \hat{\mathbf{b}}_{-i}, \hat{\mathbf{q}}_{-i} | s_i\right) \leq \lim_{b' \downarrow b} \lim_{q' \to q} EU_i\left(b', q', \hat{\mathbf{b}}_{-i}, \hat{\mathbf{q}}_{-i} | s_i\right) \tag{A-8}$$

where the inequality follows since the only case in which we do not obtain an equality is that some component of b equals somebody else's bid and this bid becomes a market clearing price with positive probability. In that case as we take the limit as $b' \downarrow b$ the payoff to i would be strictly higher, since he would be breaking the tie in his favor at any point along the sequence. Since i can submit as many steps as he desires, he will never submit a bid that is strictly higher than his marginal valuation for any unit, and hence preventing a tie is desirable.

As B_i^n becomes dense in B_i, for every $b' \in B_i$, every $\varepsilon > 0$, every step k and a.e. s_i, there exists $\bar{n} \geq 1$ and $\bar{b} \in (B_i^{\bar{n}})^K, \bar{q} \in \left(\bar{Q}^{\bar{n}}\right)^K$ such that

$$\begin{aligned} \lim_{b' \downarrow b} \lim_{q' \to q} EU_i\left(b', q', \hat{\mathbf{b}}_{-i}, \hat{\mathbf{q}}_{-i} | s_i\right) &\leq EU_i\left(\bar{b}, \bar{q}, \hat{\mathbf{b}}_{-i}, \hat{\mathbf{q}}_{-i} | s_i\right) + \varepsilon \\ &\leq EU_i\left(\bar{b}, \bar{q}, \mathbf{b}_{-i}^n, \mathbf{q}_{-i}^n | s_i\right) + 2\varepsilon, \text{ for } n \geq \bar{n} \\ &\leq EU_i\left(b_i^n(s_i), q_i^n(s_i), \mathbf{b}_{-i}^n, \mathbf{q}_{-i}^n | s_i\right) + 2\varepsilon, \text{ for } n \geq \bar{n} \end{aligned}$$

where the first and second line follow because the bid vector $\bar{b}$ can be chosen so that the probability that any component of $\hat{\mathbf{b}}_j(s_j)$ equals any component of $\bar{b}$ is arbitrarily small for any bidder j, and hence the probability of type s_i tying is arbitrarily small. The third line follows since the profile $\left(\bar{b}, \bar{q}\right) \in (B_i^{\bar{n}} \times Q^{\bar{n}})^K$ is feasible in G^n for every $n \geq \bar{n}$ and (b_i^n, q_i^n) is a best reply to $\left(\mathbf{b}_{-i}^n, \mathbf{q}_{-i}^n\right)$. Because $\varepsilon > 0$ is arbitrary, and so is b in (A-8), we obtain the first inequality in (A-7) for all i and a.e. s_i.

To obtain the last equality in (A-7) we need to establish that no two price-bids under the limiting strategy can be the same and at the same time equal to the market clearing price with positive probability, since the interim payoff would then be continuous at the limiting bid strategy.

Fix a type s_i of bidder i. Suppose that there is a price-bid under the limiting strategy that is the same as some price bid by a rival and can at the same time clear the market with positive probability π. Since bidder i submits a finite number of steps, there must exist a positive measure subset of types $\hat{S}_{-i} \in [0,1]^{N-1}$ such that for some bidder j, some profile of types $s_{-i} \in \hat{S}_{-i}$ and some steps k and l we have $\hat{b}_{ik}(s_i) = \hat{b}_{jl}(s_j) = p^c\left((s_i, s_{-i}), \hat{\mathbf{b}}, \hat{\mathbf{q}}\right) = \bar{p}$ with an allocation $\hat{q}_i^c$ and $\hat{q}_j^c$ to the players that tie at the market clearing price. Now we will show that this would contradict having an equilibrium in a nearby discretized game by showing that with a sufficiently fine grid, one of these two players would have a profitable deviation from bidding according to (b^n, q^n) at least for a subset of his types. Suppose without loss that $b_{ik}^n(s_i) < b_{jl}^n(s_j)$. This implies that the tie that arises in the limit is broken in j's favor in the discretized game, and thus $(q_i^c)^n < \hat{q}_i^c$. Now consider a deviation to $b' > b_{jl}^n(s_j) > b_{ik}^n(s_i)$ such that $b' = \inf\left\{b \in B_i^n : b > b_{jl}^n(s_j)\right\}$. For a sufficiently fine grid (high enough n) this deviation results in an arbitrarily small increase in expected market clearing price, while resulting in a strict increase in expected allocation to bidder i, thus being strictly profitable whenever $v(\hat{q}_{ik}) > \hat{b}_{ik}$ (type s_i strictly prefers to win all units he demands at step k even if he has to pay his full bid) or $v(\hat{q}_{ik}) = \hat{b}_{ik}$ and $\nexists q' \neq \hat{q}_{ik}$ such that $v(q, s_i) = \hat{b}_{ik}$ for $q \in [q', \hat{q}_{ik}]$ (type s_i strictly prefers to win all units but the last infinitesimal one at step k).

To show that this deviation is strictly profitable formally, consider a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \varepsilon_n = 0$ and $\varepsilon_n > 0\, \forall n$. Then $\forall \varepsilon_n : \exists \bar{n} : \forall n \geq \bar{n}, \left| b' - \max\left\{\hat{b}_{ik}(s_i), b_{ik}^n(s_i)\right\}\right| < \varepsilon_n$ since in the limit i's action set is dense in $B_i = \{l\} \cup [0, \bar{p}]$. Bidding b' instead of $b_{ik}^n(s_i)$ results in a change of market clearing price of at most ε_n (since $\left| b' - \max\left\{\hat{b}_{ik}(s_i), b_{ik}^n(s_i)\right\}\right| < \varepsilon_n$), and hence the change in the expected payment is at most ε_n (since the most a bidder can win is $q = 1$). Increasing i's bid also increases the probability of i winning some units. The lowest bound on the change in expected gross utility is $\pi_n \left[V(q_i^c, s_i) - V((q_i^c)^n, s_i)\right]$ where π_n is the probability of the states in which i now (after the proposed deviation) beats j but in which these bidders tie in the limit, q_i^c is the allocation to i after the deviation (and thus i beats j's bid) and $(q_i^c)^n$ is i's allocation before the deviation (and thus j beats $i's$ bid). π_n must be positive for sufficiently high n since $\lim_{n\to\infty} \pi_n = \pi > 0$. Fix some small $\delta > 0$ such that $\pi > \delta$ and let n_1 solve:

$$\forall n \geq n_1 : |\pi_n - \pi| < \delta$$

Let n_2 solve:

$$\forall n \geq n_2 : \varepsilon_n < (\pi - \delta)\left[V(q_i^c, s_i) - V((q_i^c)^n, s_i)\right] \tag{A-9}$$

The right hand side of (A-9) is strictly positive, since $\pi > \delta$, $q_i^c > (q_i^c)^n$ and $V(\cdot, s_i)$ is non-decreasing in the first argument. Therefore by picking ε_{n^*} where $n^* = \max\{n_1, n_2, \bar{n}\}$, this deviation would be strictly profitable in some discretized game G^{n^*}, a contradiction to $\left(b^{n^*}, q^{n^*}\right)$ being an equilibrium.

This deviation may not be strictly profitable if $v(q, s_i) = \hat{b}_{ik}$ for $q \in [q', \hat{q}_{ik}]$ and this bidder would thus be indifferent between winning some of the units close to $\hat{q}_{ik}$ or not. But since $Q = [0, 1]$ can be split only in countably many disjoint intervals of positive measure, this would imply that only countably many types of bidder i (or j) could be indifferent in the sense described above and hence $\bar{p}$ cannot clear the market with positive probability ex ante as $v(\cdot, \cdot)$ is strictly increasing in the second argument. Otherwise there would be a quantity $\bar{q}$ and two types s' and s'' such that $v(\bar{q}, s') = v(\bar{q}, s'')$ which is a contradiction. As only countably many bids by bidders other than i can be submitted with positive probability and at each of those only countably many types of bidder i can be indifferent as described above, for a.e. s_i, we do not have ties. But this implies that the expected payoff $EU_i\left(b, \hat{q}_i(s_i), \hat{\mathbf{b}}_{-i}, \hat{\mathbf{q}}_{-i}|s_i\right)$ is continuous in b (also at l since l is isolated) at the limit $\hat{b}_i(s_i)$ for a.e. s_i. Therefore $\lim_n EU_i\left(b_i^n(s_i), q_i^n(s_i), \mathbf{b}_{-i}^n, \mathbf{q}_{-i}^n|s_i\right) = EU_i\left(\hat{b}_i(s_i), \hat{q}_i(s_i), \hat{\mathbf{b}}_{-i}, \hat{\mathbf{q}}_{-i}|s_i\right)$ and so (A-7) implies that $\left(\hat{\mathbf{b}}, \hat{\mathbf{q}}\right)$ is an equilibrium, since it establishes that $\hat{b}_i(s_i), \hat{q}_i(s_i)$ are best replies of type s_i to $\left(\hat{\mathbf{b}}_{-i}, \hat{\mathbf{q}}_{-i}\right)$ for a.e. s_i. QED

A.7 Proof of Proposition 5

We can use virtually the same logic as in the proof above. In the discretized game where ties are impossible but strategies are restricted to K steps, we cannot guarantee existence of an isotone equilibrium with a discriminatory price auction since payoff functions need not be modular anymore, but we have an equilibrium in distributional strategies σ^n in each G^n by Theorem 2 of Milgrom and Weber (1985). Given a space of all probability distributions on $S \times [Q \times B]^K$ is compact with respect to the weak topology, the sequence (or a subsequence if necessary) of equilibria converges to a limit $\hat{\sigma}$ which is a probability distribution itself. If no ties occur with positive probability under $\hat{\sigma}$ for a.e. s_i, then $\hat{\sigma}$ is indeed an equilibrium of the original game. To rule out ties which could cause a downward jump in expected payoff of some bidder, we will modify the argument from proof of Proposition 6. In particular, if there was a tie under $\hat{\sigma}$ for some type s_i, then for high enough n, one of the bidders tying under $\hat{\sigma}$ would have a profitable deviation from σ^n.

Suppose that there is a tie under $\hat{\sigma}$ and all tying bidders earn strictly positive surplus on some units allocated to them at the step at which they tie. Since one of the bidders has to lose in the discretized game (as ties are impossible), for sufficiently fine grids, he could simply raise his bid slightly to outbid his rival(s) so that the increase in his total payment is lower than the strictly positive gain in surplus (due to a positive probability of tying at the limit) in the same way as in the proof of proposition 6, and hence a contradiction to σ_n being an equilibrium for sufficiently high n.

Therefore a tie could occur only if all, but at most one of the tying bidders earned no positive surplus on any units demanded at the step at which they are tying under $\hat{\sigma}$ with positive probability. If all bidders earn no surplus on these units, then the expected payoffs are continuous at $\hat{\sigma}$ as winning or losing the last units is payoff irrelevant and the tie has to occur at the last steps since otherwise the bidder who wins in the discretized game could decrease his bid slightly whereby strictly increasing his payoff. The remaining step is to argue that the probability of there being one bidder, for whom the tie under $\hat{\sigma}$ at his last step would result in a decrease in payoff since at least one other bidder would tie him from below, vanishes. But since $v(q, s_i)$ is strictly increasing in s_i, when signals are independent there can be only countably many types that tie with bidder i and are indifferent between winning or not winning the last units allocated to them under $\hat{\sigma}$ and therefore the probability of bidder's tying at their last steps and at the same time being indifferent between winning or losing the last units is zero. Therefore, the expected payoffs are continuous and a K-step equilibrium in distributional strategies exists in a discriminatory auction with restricted strategy sets and independent signals. QED

A.8 Proof of Corollary 2

In order to apply Proposition 6, we need to have (quasisuper-)modularity of own payoffs. Therefore, we will first consider existence of an equilibrium in non-decreasing pure strategies in a game, Γ_K, with discretized quantity space to common K-dimensional grid and continuous price-bids. This implies that the finest bid will have at most K steps. In this case, the component-wise maximum of any two bids is still an admissible strategy since it will have at most K steps. Therefore, existence of a pure strategy equilibrium in non-decreasing strategies in this modified game follows as a corollary to Proposition 6. Moreover, no ties can occur with positive probability under these equilibrium strategies by the same arguments as in the proof of Proposition 6 (if there were a tie, some bidder would have a profitable deviation in slightly increasing her bid). Therefore, we can immediately apply Proposition 4 to establish that for every bidder i and for sufficiently large K_i, the equilibrium strategy of bidder i in this modified game leads to payoffs that are arbitrarily close to the payoffs obtained when a continuous bid function (i.e., as we make the grid arbitrarily fine, $K \to \infty$) were used and bidders $-i$'s strategies were fixed (i.e., remained step functions with K steps). But since in the game $\Gamma_{\max K_i}$ this is true for every bidder i, these equilibrium strategies constitute an ε-equilibrium of the limiting game. In particular, fixing $\varepsilon > 0$, Proposition 4 implies that there must exist $\underline{K}$ such that the loss from using a $\underline{K}$-step bid function is less than ε for every bidder i. QED