Discrete Bids and Empirical Inference in Divisible Good Auctions

Jakub Kastl*
Stanford University and NBER

October 23, 2009

Abstract

I examine a model of a uniform price auction of a perfectly divisible good with private information in which the bidders submit discrete bidpoints rather than continuous downward sloping demand functions. I characterize necessary conditions for equilibrium bidding. The characterization reveals a close relationship between bidding in multi-unit auctions and oligopolistic behavior. I demonstrate that a recently proposed indirect approach to the revenue comparisons of discriminatory and uniform price auctions is not valid if bid functions have steps. In particular, bidders may bid above their marginal valuation in a uniform price auction. In order to demonstrate that discrete bidding can have important consequences for empirical analysis I use my model to examine a dataset consisting of individual bids in uniform price treasury auctions of the Czech government. I propose an alternative method for evaluating the performance of the employed mechanism. My results suggest that the uniform price auction performs well, both in terms of efficiency of the allocation and in terms of revenue maximization. I estimate that the employed mechanism failed to extract at most 3 basis points in terms of the annual yield of T-bills worth of expected surplus while implementing an allocation resulting in almost all of the efficient surplus. Failing to account for discreteness of bids would in my application result in overestimating the unextracted revenue by more than 50%.

Keywords: multiunit auctions, treasury auctions, uniform price auctions, structural estimation, nonparametric identification and estimation

JEL Classification: D44

*This paper is a revised version of Chapter 1 of my PhD dissertation at Northwestern University. I am indebted to my advisors, Robert Porter and Michael Whinston, for their support and guidance throughout this project. I am grateful for the comments from two anonymous referees and the editor, Bruno Biais, that substantially improved the paper. I would also like to thank Susan Athey, Liran Einav, Jeff Ely, Igal Hendel, Ali Hortaçsu, Aviv Nevo, Alessandro Pavan, Salvatore Piccolo, Bill Rogerson, Azeem Shaikh, Tomasz Strzalecki, Elie Tamer, Frank Wolak, Asher Wolinsky, and seminar participants at various universities and conferences for many helpful comments. Special thanks are also due to the Czech National Bank for making the data set available to me. Financial support from the Center for the Study of Industrial Organization at Northwestern University and from the NSF (SES-0752860) is gratefully acknowledged. Any remaining errors are my own responsibility.

1 Introduction

There is a consensus among economists that the most effective way to sell government securities is through an auction. There is not a consensus on the best auction mechanism, however. The theoretical literature on multiunit auctions does not provide a definitive recommendation whether the ultimate goal is either revenue maximization or efficiency of the allocation. In practice, there is a clear preference between the two most widely employed mechanisms. Bartolini and Cottarelli (1997) report that 39 out of the 42 countries surveyed use the discriminatory auction mechanism ("pay your own bid"), and only three countries use a uniform price auction mechanism. In this paper I contribute to the debate on the optimal auction mechanism by providing a method that allows a choice between different auction mechanisms based on data on individual bids. An essential part of my model is that the equilibrium strategies are step functions, which is a restriction that has important implications, but has not been recognized explicitly in the literature.

In both discriminatory and uniform price auctions, bidders may submit multiple price-quantity pairs as their bids. These points trace out a bid function. The auctioneer then aggregates these bid functions. As in any other market, the clearing price is the point at which the aggregate demand (bid) function intersects the supply curve, which in thie case is inelastic at a quantity that is usually preannounced. The securities are then allocated to the bidders for those units for which their bids were higher than the market clearing price. The payments collected from the bidders depend on the auction mechanism. In the discriminatory auction, also known as a pay-your-bid or multiple-price auction, the bidders pay their full bid for all securities that they are allocated. In the uniform price auction, each bidder pays the market clearing price for every unit won. The auctioneer's revenue in the discriminatory auction is therefore the area under the aggregate bid function up to the supply quantity. In the uniform price auction, the revenue is the product of the market clearing price and the quantity supplied. It might be tempting to conclude that the discriminatory auction must therefore lead to a higher revenue, just as a perfectly discriminating monopolist is able to earn more than if she cannot discriminate. This intuition is misleading since the mechanism choice affects bidders' strategic behavior and thus the location and shape of the aggregate bid function. Results from single-unit auction settings are also misleading. For example, one might conclude from the similarity between a second price auction and a uniform price auction, or between the first price auction and a discriminatory auction, that the revenue should be the same if the values are private, bidders are risk neutral and symmetric, and signals are independent. This intuition is misleading since the revenue equivalence theorem requires that the mechanisms be allocationally equivalent, which is typically not the case in a multi-unit environment.

The strategic considerations are quite different in the two auction formats. In a discriminatory auction, a rational bidder would not bid his full marginal valuation for any unit that he might win because he wants to retain some surplus. In a uniform price auction a bidder should shade his bid below his marginal valuation at quantities that might be pivotal and might therefore determine the market clearing price. Hence, in both auction mechanisms, bidders will not always bid their true marginal valuations. Ausubel and Cramton (2002) show that the comparison of the uniform and discriminatory auction formats, in terms of both efficiency and revenue, is an empirical question. Either format can be better than the other, under either criterion, under some circumstances.

Most of the previous empirical literature that compares these two auction mechanisms focuses on "natural experiments" in which different auction formats have been used in different time periods. Those papers examine the difference between the market clearing auction price and the resale or forward price of the security (Umlauf (1993), Simon (1994), Nyborg and Sundaresan (1996)). A drawback of this approach is that the researcher has to maintain strong assumptions on the information structure across the auctions, especially those involving different auction formats. In particular, observed differences that cannot be explained by observable control variables are attributed solely to the auction format.

My paper instead belongs to a small set of recent papers that employ structural econometric modeling to compare the alternative auction mechanisms in a divisible good setting.[1,2] These papers use a bidder's optimality condition to recover structural parameters and, in particular, the distribution of the marginal valuations, as proposed in Guerre, Perrigne and Vuong (2000) in the single unit setting. This approach avoids problems with comparing realizations of different formats, and it is also amenable to answering counterfactual policy questions. My paper differs from these recent papers in two ways. First, some of these papers use parametric assumptions to circumvent the problem of multiple equilibria (for example, by restricting attention to equilibrium strategies that are linear in private signals, as in Fevrier, Preget and Visser (2002)). My approach will instead be nonparametric and does not require explicit solution of equilibrium strategies. Hence I do not need to rely on approximation techniques to obtain these, as in Armantier and Sbaì (2004), who apply constrained strategic equilibrium framework developed in Armantier, Florens and Richard (2002). Second, and more significantly, most of these papers focus on equilibria in strictly downward-sloping continuous bid functions. In the data, however, we instead typically observe step bid functions. This occurs both because bidders are in reality limited in the number of bidpoints they are allowed to submit, and because they choose to submit even fewer bids than the allowed number. For example, in my dataset the bidders are restricted to submit at most 10 bidpoints, yet the average number of submitted bidpoints is less than 3 and the maximum number of submitted bidpoints is 9. I explicitely model this feature of the data and I will show that this seemingly innocuous institutional detail has important implications for empirical analysis. I extend the model and estimation method proposed in Hortaçsu (2002) to explicitely account for this feature and I find that failing to take into account this discreteness in bidding may result in biased estimates of marginal valuations. Hortaçsu rationalized the discrete bids by assuming that they have to lie on a discrete grid of prices and the

[1] A divisible good auction is also known as a share auction.

[2] Leading examples are Armantier and Sbaì (2004), Chapman, McAdams and Paarsch (2006), Hortaçsu (2002) and Wolak (2003).

optimal continuous bid function is thus "constrained" to be defined only on the prices on the grid. My analysis instead focuses on strategic decisions of the bidders where to locate each step, where the location implicitly depends on the location of other steps. Moreover, my analysis shows that in a model with equilibria in step functions rather than continuous downward sloping bid functions, the revenue of the hypothetical uniform price auction in which bidders bid truthfully their values does not constitute an upper bound on the ex post revenue of the uniform price auction. The reason is that bidders might find it optimal to submit bids that are higher than their marginal valuations. In general, the marginal valuation schedule may not be the upper bound on the bid schedule in a uniform price auction, whenever the bidder is not allowed to submit a separate bid for every unit offered for sale and thus submits bids for "bundles." This introduces an additional trade-off between the potential (ex post) loss on the marginal unit (in some states of the world) and the probability of obtaining the inframarginal units in the bundle.

The main contributions of the paper can be classified into two groups. On the theory side, in sections 2 and 3, I introduce a model of a divisible good auction with private information in which the bidders may be restricted in the number of bids they are allowed to submit and thus submit step bid functions. I characterize necessary conditions for equilibrium bidding in this model. These necessary conditions differ from those in the differentiable downward sloping bid functions case.[3] These conditions, which relate the primitives of the model to the observables, serve as the basis for the empirical work later in the paper. They are also useful for understanding equilibrium behavior in multiunit auctions of indivisible goods, in which case the observed bid is a discrete vector, since a model with a divisible good can be viewed as the limiting case of such a class of models. My characterization theorem reveals a close relationship between the optimal behavior of a bidder in a uniform price auction and that of an oligopolist facing uncertain demand.[4] I also demonstrate that when bidders are restricted in the number of bids they can submit, they may submit bids higher than their marginal value for some units. This suggests, for example, that important recent empirical work comparing uniform price and discriminatory auctions by Hortaçsu (2002) may provide an underestimate of the potential (ex post) revenue arising from the uniform price auction.

Sections 4 and 5 turn to the empirical side of the paper. In Section 4, I provide conditions under which the primitives of the model can be identified nonparametrically and propose an estimation method using a generalization of the resampling approach introduced into the literature by Hortaçsu (2002). In two recent related empirical papers, Wolak (2003, 2007) examines Australian electricity auctions taking into account that the bid functions are step functions.[5] He develops an

[3] In a related working paper (Kastl (2008)) I show that these necessary conditions for equilibrium converge to their counterpart in the model with differentiable downward sloping bid functions as the number of submitted bidpoints goes to infinity.

[4] Similarity between auction theory in the single unit environment and a monopolist engaging in third-degree price discrimination has been pointed out in Bulow and Roberts (1989) and further similarities between oligopoly theory and auctions have been described in Bulow and Klemperer (1996).

[5] There is also theoretical literature on electricity auctions which explicitely incorporates indivisibility of the supply functions dues to generating capacities into the model. See e.g., von der Fehr and Harbord (1993) or Fabra, von der

alternative econometric technique to estimate parameters of parametrically specified cost functions from data on individual bids which is based on approximation of the non-differentiable ex-post profit functions by smooth functions. Using the moment conditions implied by each bidder bidding optimally taking into account the uncertainty due to other bidders' bidding behavior and uncertain demand for electricity, he applies GMM to recover the parameters of interest. As my method, Wolak's approach is based on using the best-response hypothesis to provide a link between the observed bids and the primitives of the model. However, unlike my approach, his method does not explain the possibility of bidding above one's value. On the other hand, one of the advantages of Wolak's approach is that since he estimates the model using GMM, he is able to easily obtain standard errors for his estimates without employing costly resampling procedures.

In section 5, I describe my data and apply my estimation method to obtain information about bidders' marginal valuations in uniform price treasury auctions of the Czech government. I show that in a non-negligible share of these auctions, the actual realized revenue exceeds the revenue that would have been obtained had the bidders bid their true marginal valuation schedules in a hypothetical uniform price auction. I propose a new method for evaluating the performance of the auction mechanism using these estimates and find that in my application the uniform price auction performs quite well, both in terms of efficiency and revenue. On average, the employed mechanism implements an allocation that achieves almost all of the efficient surplus. Moreover, the estimated maximum total expected surplus (in terms of the annual yield of T-bills) left to the bidders does not exceed 3 basis points.

In section 5.2 I use my estimates to bound the expected surplus that the bidders forego by using fewer bid points than allowed. Similar to the recent empirical work by Chapman, McAdams and Paarsch (2006), I find that this loss of surplus is very small relative to the magnitudes involved. Chapman, McAdams and Paarsch (2006) study discriminatory auctions of Canadian Receiver-General building on partial identification results from McAdams (2008), who investigates bounds on marginal valuations that are consistent with observed bids by considering many possible deviations from the observed bids and requiring that these be unprofitable given the true marginal valuation schedule. Unlike this paper, they are not interested in evaluating the performance of the auction mechanism, but rather in investigating whether bidders' behavior is consistent with the best-response hypothesis. While they find evidence of frequent departures from best-responses, they argue that the deviations are very small (similar to the findings about implied costs in this paper) and the equilibrium hypothesis might thus be a good approximation.

Finally, section 6 concludes the paper. All proofs are relegated to the appendix.

Fehr and Harbord (2006).

2 Model

I will start with the basic uniform price share auction framework of Wilson (1979) with private information, in which both quantity and price are assumed to be continuous. There are N (potential) bidders, who are bidding for a share of a perfectly divisible good. Each bidder receives a private real-valued signal, s_i, which is the only private information about the underlying value of the auctioned goods. The joint distribution of the signals will be denoted by $F(\mathbf{s})$. The one-dimensionality of private information is essential neither for any of the theoretical results, nor for the estimation technique.

Assumption 1 *Bidder i's signal s_i is drawn from a common support $[0,1]$ according to an atomless marginal d.f. $F_i(s_i)$ with strictly positive density $f_i(s_i)$.*

Winning q units of the security is valued according to a marginal valuation function $v_i(q, s_i, s_{-i})$. For most of this paper, I will deal with the special case of independent private values (IPV). I will discuss the appropriateness of the private value paradigm in the context of my application later. In the case of private values, bidders' valuations do not depend on private information of other bidders, i.e., $v_i(q, s_i, s_{-i}) = v_i(q, s_i)$. At the estimation stage I will not impose symmetry, since I will allow for different groups, within which the bidders share the same marginal valuation function and the same distribution of private signals. I will impose the following assumptions on the marginal valuation function $v_i(\cdot,\cdot)$:

Assumption 2 *$v_i(q, s_i)$ is bounded, strictly increasing in s_i $\forall q$ and weakly decreasing and continuous in q $\forall s_i$.*

I will denote by $V(q, s_i)$ the gross utility: $V(q, s_i) = \int_0^q v_i(u, s_i)\, du$.

Bidders' pure strategies are mappings from private signals to bid functions: $\sigma_i : S_i \to \mathcal{Y}$, where the set $\mathcal{Y}$ includes all possible functions $y : \mathbb{R}^+ \to [0,1]$. A bid function for type s_i can thus be summarized by a function, $y_i(\cdot|s_i)$, which specifies for each price p, how big a share $y_i(p|s_i)$ of the securities offered in the auction (type s_i of) bidder i demands. Q will denote the amount of T-bills for sale, i.e., the good to be divided between the bidders. Q might itself be a random variable if it is not announced by the auctioneer ex ante, or if the auctioneer has the right to augment or restrict the supply after he collects the bids. In either case, I will assume that the distribution of Q and the way it depends on the bids is common knowledge among the bidders.[6] Furthermore, the number of potential bidders participating in an auction, denoted by N, is also commonly known. The natural solution concept to apply in this setting is Bayesian Nash Equilibrium. The expected utility of type s_i of bidder i who employs a strategy $y_i(\cdot|s_i)$ in a uniform price auction given that other bidders

[6] I will provide further details later in the text.

are using $\{y_j(\cdot|\cdot)\}_{j\neq i}$ can be written as:

$$\begin{aligned} EU_i(s_i) &= \mathbb{E}_{Q,s_{-i}|s_i} u(s_i, s_{-i}) \\ &= \mathbb{E}_{Q,s_{-i}|s_i}\left[\int_0^{q_i^c(Q,\mathbf{s},\mathbf{y}(\cdot|s))} v_i(u,s_i)\,du - p^c(Q,\mathbf{s},\mathbf{y}(\cdot|s))\, q_i^c(Q,\mathbf{s},\mathbf{y}(\cdot|s))\right] \end{aligned}$$

where $q_i^c(Q,\mathbf{s},\mathbf{y}(\cdot|\mathbf{s}))$ is the (market clearing) quantity bidder i obtains if the state (bidders' private information and the supply quantity) is $(Q,\mathbf{s})$ and bidders bid according to strategies specified in the vector $\mathbf{y}(\cdot|\mathbf{s}) = [y_1(\cdot|s_1), ..., y_N(\cdot|s_N)]$, and similarly $p^c(Q,\mathbf{s},\mathbf{y}(\cdot|\mathbf{s}))$ is the market clearing price associated with state $(Q,\mathbf{s})$. A Bayesian Nash Equilibrium in this setting is thus a collection of functions such that almost every type s_i of bidder i is choosing his bid function so as to maximize his expected utility: $y_i(\cdot|s_i) \in \arg\max EU_i(s_i)$ for a.e. s_i and all bidders i.

In most of the previous literature, starting with Wilson (1979), the set $\mathcal{Y}$ of admissible strategies is restricted to continuously differentiable functions so that calculus of variations techniques can be applied. These techniques enable us to show that in a symmetric IPV model, and within this restricted class of strategies, a symmetric BNE $y(\cdot|\cdot)$ has to satisfy the following necessary condition for all (p, s_i):

$$v(y(p|s_i), s_i) = p - y(p|s_i)\frac{H_y(p, y(p|s_i))}{H_p(p, y(p|s_i))} \tag{1}$$

where $H(p,x)$ is the probability distribution of the market clearing price when x units are demanded by bidder i and all other bidders $j \neq i$ submit the equilibrium bid functions, i.e., $H(p,x) \equiv \Pr(p^c \leq p|x) = \Pr\left(x \leq Q - \sum_{j\neq i} y(p, s_j)\right)$ (H_p and H_y are the derivatives of $H(\cdot,\cdot)$ with respect to the first and second argument respectively). As Wilson points out, the auction game might have multiple equilibria, some of which lead to low revenue for the auctioneer.[7] Such equilibria, while achieved in a non-cooperative way, are usually called "seemingly collusive" and several authors (e.g., LiCalzi and Pavan (2005) and McAdams (2007)) show how the auctioneer would eliminate at least some of these undesirable equilibria by slightly modifying the uniform price auction. In related papers, Biais and Faugeron-Crouzt (2002) and Biais, Bossaerts and Rochet (2002) analyze the different auction mechanisms, their optimality and susceptibility to tacit collusion in the context of IPO auctions under pure common values paradigm. Their analysis includes the characterization of an optimal mechanism in the presence of better-informed dealers (investment banks) and retail investors.[8]

Because of the restricted set of strategies, it is an essential feature of a candidate equilibrium that the equilibrium strategies are strictly downward-sloping differentiable functions. One implication of this fact is that the rationing rule does not matter for equilibrium behavior, since rationing does

[7] See also Back and Zender (1993 and 2001), Kremer and Nyborg (2004b) and Wang and Zender (2002).

[8] Other papers that analyze equilibria across different mechanisms explicitely include Rostek, Weretka and Pycia (2009) who compare a particular class of equilibria - those in linear strategies - across different mechanisms and Brenner, Galai and Sade (2009) who compare the auction mechanisms in an experimental setting.

not occur in equilibrium.[9] In other words, we always have $q_i^c\left(Q, \mathbf{s}, \mathbf{y}\left(p|\mathbf{s}\right)\right) = y_i\left(p^c\left(Q, \mathbf{s}, \mathbf{y}\left(\cdot|\mathbf{s}\right)\right)|s_i\right)$. While Wilson's model provides useful insights, and illuminates some of the trade-offs bidders face in share auctions, it cannot account for several features of the data in most actual share auctions.

In the next section, I introduce a concept of a K-step equilibrium, in which I address directly a central feature of most real-world share auctions: bid functions are step functions, and hence not continuously differentiable. I will argue that accounting for these features has important implications for both the theoretical model and empirical inference in these auctions.

3 K-step equilibrium

Why do bidders submit step functions in these auctions? One reason is institutional. In the vast majority of actual share auctions, the auctioneer imposes an upper bound, $\overline{K}$, on the number of bidpoints that the bidders can submit, which restricts the bidders' strategy space and makes submitting a continuous function impossible. Yet, since this institutional contraint is often not binding, it does not seem to be the main cause. Building on Wilson's (1993) results about approximate optimality of multipart tariffs, Kastl (2008) shows that the presence of just a small cost of submitting a step would make it strictly optimal for bidders to use only a few steps to characterize their bids.[10] In this section I develop a model that incorporates these features, characterize its equilibrium and demonstrate that bidders may submit bids that are higher than their marginal values. The most important features of the model are (i) bidders submit only finitely many bidpoints (due to endogenous restrictions such as cost of submitting a step or exogenous restrictions on strategies), and (ii) the quantity in each bidpoint is a continuous choice variable.

While most of the previous literature restricts bid functions to be continuously differentiable, my goal is for them to be step functions. With this possibility, I cannot apply the calculus of variations to characterize equilibrium strategies. Since for a finite $\overline{K}$ bidders submit left-continuous step functions, I can summarize bidder i's action as a K_i-dimensional vector of bidpoints (b_i, q_i), where the k^{th} point denotes the price (the height of current step) and quantity (strictly speaking, the share of total quantity) at which this step ends (its length). I will also assume that there is an upper bound on the maximal bid, which for example in the case of treasury bills could be the face value. In general, any bid above the value of the first infinitesimal unit is weakly dominated by bidding this value, and thus this upper bound is $v\left(0, \bar{s}\right)$ where $\bar{s}$ is the highest possible signal. To summarize:

[9]Because individual bid functions are strictly downward sloping, residual supply is always strictly upward sloping and thus the market always clears exactly.

[10]Formally, the loss from using step functions rather than continuous bid functions vanishes at a quadratic rate in the number of steps.

Assumption 3 *Each player $i = 1, ..., N$ has an action set:*

$$A_i = \left\{ \begin{array}{c} \left(\vec{b}_i, \vec{q}_i, K_i\right) : \dim\left(\vec{b}_i\right) = \dim\left(\vec{q}_i\right) = K_i \in \left\{0, ..., \overline{K}\right\}, \\ b_{ik} \in B = \left[0, \bar{b}\right], \; q_{ik} \in Q = [0,1], \; b_{ik} \geq b_{ik+1}, q_{ik} \leq q_{ik+1} \end{array} \right\}$$

In what follows when more convenient I use the shorthand vector notation (b_i, q_i) to describe the step function $y(\cdot|s_i)$ of type s_i of bidder i.

It is also apparent that because each bidder's bid function is a step function, the residual supply will be a step function, and therefore but for knife-edge cases any equilibrium will involve rationing with probability one. Rationing occurs whenever there is excess demand at the market clearing price, while at all higher prices there is excess supply. On such occasions the auctioneer will determine a *rationing coefficient*, by which demand is adjusted to equal supply. While the theoretical literature has considered a few alternative rationing rules, in my analysis I will consider only the rationing rule that is employed in all uniform price auctions in practice, rationing pro-rata on-the-margin.

Assumption 4 *The rationing rule is pro-rata on-the-margin, under which the rationing coefficient, $R(p^c)$, satisfies*

$$R(p^c) = \frac{Q - TD_+(p^c)}{TD(p^c) - TD_+(p^c)}$$

where $TD(p^c)$ denotes total demand at price p^c, and $TD_+(p^c) = \lim_{p \downarrow p^c} TD(p)$. Only the bids exactly at the market clearing price are adjusted.

Under this rule all bids above the market clearing price are given priority, and only after all such bids are satisfied, the remaining marginal demands at exactly price p^c are reduced proportionally by the rationing coefficient so that their sum exactly equals the remaining supply. An alternative rationing rule would, for example, not give bids at higher prices priority. Kremer and Nyborg (2004a) show that, in a complete information framework, this alternative rationing rule encourages competition and may thus be preferred. Notice, however, that this alternative rationing rule may have an adverse effect on allocative efficiency. Assumptions 1-4 are assumed throughout the analysis.

Definition 1 *A K-step equilibrium is a collection of functions such that for each bidder i and almost every type s_i, $y_i(\cdot|s_i)$ solves*

$$y_i(\cdot|s_i) \in \arg \max_{y_i(\cdot|s_i) \in A_i} EU(s_i)$$

Characterization of equilibrium

Even though the current problem involves many difficulties due to the lack of differentiability, I can provide the equivalent of a first-order necessary condition by working directly with limit arguments. The heart of the proof of the subsequent main characterization result is to consider separately the effects of the demand at the k^{th} step on s_i's expected payoff in different possible states of the residual supply. Shading the demand has a cost when the demand is fully satisfied, and also when there is a tie at k^{th} step. A tie at k^{th} step occurs when there is rationing and at least one other bidder is rationed so that the marginal demand actually matters for allocation due to rationing pro-rata on-the-margin. There is a benefit, however, when the shading leads to a lower expected market clearing price and also when there is a tie (and thus rationing) at $k+1^{st}$ step (as by shading demand at the k^{th} step, ceteris paribus, the marginal demand at $k+1^{st}$ step is higher). These effects need to be traded off against each other.

The next proposition characterizes a necessary condition for a K-step equilibrium in a private values model. This result can also be viewed as a characterization of an equilibrium of a limit of a multiunit auction as the units become arbitrarily small, and it reveals the close relationship between the behavior of a bidder in a uniform price auction and that of an oligopolist facing uncertain demand (as in Klemperer and Meyer (1989)).

Proposition 1 *(Characterization) Under assumptions 1-4, in any Bayesian Nash Equilibrium, for almost every s_i, every step k in the K_i-step function $y_i(\cdot|s_i)$ in the support of i's equilibrium strategy has to satisfy*

$$\begin{aligned}
& \Pr\left(b_k > p^c > b_{k+1}\right)\left[v_i\left(q_k, s_i\right) - \mathbb{E}_{Q,s_{-i}|s_i}\left(p^c | b_k > p^c > b_{k+1}\right)\right] \\
& + \Pr\left(b_k = p^c \ \wedge \ Tie\right)\mathbb{E}_{Q,s_{-i}|s_i}\left[\left(v_i\left(q_i^c\left(Q,\mathbf{s},\mathbf{y}\left(\cdot|\mathbf{s}\right)\right),s_i\right) - b_k\right)\frac{\partial q_i^c\left(Q,\mathbf{s},\mathbf{y}\left(\cdot|\mathbf{s}\right)\right)}{\partial q_k}|p^c = b_k \ \wedge \ Tie\right] \\
& + \Pr\left(b_{k+1} = p^c \ \wedge \ Tie\right)\mathbb{E}_{Q,s_{-i}|s_i}\left[\left(v_i\left(q_i^c\left(Q,\mathbf{s},\mathbf{y}\left(\cdot|\mathbf{s}\right)\right),s_i\right) - b_{k+1}\right)\frac{\partial q_i^c\left(Q,\mathbf{s},\mathbf{y}\left(\cdot|\mathbf{s}\right)\right)}{\partial q_k}|p^c = b_{k+1} \ \wedge \ Tie\right] \\
= \ & q_k \frac{\partial \mathbb{E}_{Q,s_{-i}|s_i}\left(p^c; b_k \geq p^c \geq b_{k+1}\right)}{\partial q_k}
\end{aligned} \tag{2}$$

where $\mathbb{E}_{Q,s_{-i}|s_i}\left(p^c; b_k \geq p^c \geq b_{k+1}\right) \equiv \mathbb{E}_{Q,s_{-i}|s_i}\left(p^c \mathbb{I}\left(b_k \geq p^c \geq b_{k+1}\right)\right)$.[11]

The intuition for the result is more easily provided by abstracting away from ties, in which case the core of the trade-off can be illustrated graphically[12] as in Figure 5 and we get the following condition:

$$\Pr\left(b_k > p^c > b_{k+1}\right)\left[v_i\left(q_k, s_i\right) - \mathbb{E}_{Q,s_{-i}|s_i}\left(p^c | b_k > p^c > b_{k+1}\right)\right] = q_k \frac{\partial \mathbb{E}_{Q,s_{-i}|s_i}\left(p^c; b_k \geq p^c \geq b_{k+1}\right)}{\partial q_k} \tag{3}$$

This condition, which rules out profitable local perturbations of q_k, reveals the parallel between the behavior of a bidder in a multiunit uniform-price auction and an oligopolist facing uncertain demand. When ties do not occur with positive probability, the only states at which a bidder can affect his payoff by varying the quantity demanded, q_k, are those in which the residual supply cuts the vertical piece of his bid function, i.e., between his adjacent bids $b_k > p^c > b_{k+1}$. In all

[11] The continuity and differentiability of this object are examined in Lemmas 2-4 in the Appendix.

[12] The figure, however, does not take into account the effect of shading the demand on the probability of the market clearing price being strictly between the two adjacent bids, hence the need for the unconditional expectation.

states such that the market clearing price is between the two steps of bidder i, he obtains his full quantity request, and the expected marginal cost of quantity shading captured on the LHS is thus the difference between his marginal utility and the expected price. Since in all states that he is rationed he is the only marginal bidder with probability one, there is no cost of quantity shading in those states. On the other hand, the marginal benefit of quantity shading is saving money on the inframarginal units, and this is captured on the RHS. Therefore, the bidder facing random residual supply acts in the same way as a monopolist facing random demand. Notice that (3) can also be rewritten as

$$v_i(q_k, s_i) = \mathbb{E}(p|b_k > p^c > b_{k+1}) + \frac{q_k}{\Pr(b_k > p^c > b_{k+1})} \frac{\partial \mathbb{E}(p^c; b_k \geq p^c \geq b_{k+1})}{\partial q_k} \qquad (4)$$

which is very close to $MC = E[P(Q)] + E[P'(Q)]Q$, i.e., to an oligopolist's optimality condition in a setting where the oligopolist faces uncertain demand in the spirit of Klemperer and Meyer (1989).

In the appendix, I also present a second set of necessary conditions given by (A-3), which ensure that a local perturbation of b_k is not optimal.[13] Bidder i has to balance the change in the expected prices in the steps above and below the k^{th} one. He also needs to take into account the payoff effect of the perturbation if he is rationed at b_k, which includes the indirect effect on the expected quantity received after rationing. It is this condition that we would regularly obtain in a multiunit auction with discrete units, but continuous bid space. Equation (3) would become a system of inequalities in that setting.

Bidding above marginal values

Equation (3) immediately gives us the following important corollary, which states that bids above marginal values may be optimal in a uniform price auction with restricted strategy sets. The intuition (developed in more detail in the subsequent text) has to do with the restriction on the strategies requiring the bidders to "bundle" bids for several units together and thus to trade-off potential (ex post) loss on the last unit in the bundle against the probability of obtaining the high-valued infra-marginal units in the bundle.

Corollary 2 *Under the hypotheses of Proposition 1, when bidders are restricted to submit step functions, they may optimally bid above their marginal valuation schedules in a uniform price auction.*

To see why this corollary holds, it is sufficient to consider one very small bidder, so that he is a "price taker," a non-degenerate distribution of the market clearing price with continuous and strictly positive density over a compact support and let $K_i = 1$.[14] In this case, (2) collapses to (3) as ties

[13] Since these conditions would be applicable only of ties did not occur with positive probability (as otherwise the distribution function of the market clearing price would not be differentiable) and I will not be using them in the empirical exercise, I do not present them here in the sake of brevity.

[14] A bidder is a price taker if no small change in his bid has any effect on the distribution of the market clearing price.

at any bid have zero probability (due to the assumed properties of the distribution of the market clearing price), and the RHS of (3) vanishes because of the bidder being a price taker. This bidder thus optimally asks for a quantity such that his marginal valuation at that quantity is equal to the expected price conditional on this price being lower than his bid, $v_i(q_k, s_i) = \mathbb{E}_{s_{-i}}(p^c | b_k > p^c)$. Therefore, whenever there is a positive probability of the market clearing price being below his bid, his bid will be higher than his marginal valuation for that quantity. This important result indicates that the ex post revenue in a uniform price auction is not necessarily bounded by the revenue of the "best case" Vickrey auction, in which each bidder submits his marginal valuation schedule as his bid without getting any transfer from the auctioneer. Note that this "best case" upper bound is valid for revenues from equilibria in continuously differentiable bid functions since in that setting a bidder never submits a bid above his marginal value such that this bid is in the support of the distribution of the market clearing price. This result is important for empirical work, since calculating counterfactual equilibria and the associated revenues under alternative auction regimes is often an intractable task. The researcher is thus forced to report estimated revenue losses from the realized auction relative to this "best case" Vickrey auction (also sometimes called the "truthful bidding" auction). Corollary 2 reveals, however, that even a uniform price auction can lead to a higher ex post revenue than the "best case" Vickrey auction. As we will see later in the empirical section, this point is not purely theoretical, since in a non-negligible share of auctions in my dataset the realized ex post revenue is higher than the revenue in an auction in which the bidders submit bids equal to the estimated upper bound of their marginal valuation schedules. This result also suggests that using the model with continuously differentiable bid functions might not be a good approximation, at least in situations in which bidders submit demand functions consisting of just few steps.

The trade-off that results in the possibility of bidding above one's value is that between a bidder not willing to give up the probability of winning high-value infra-marginal units versus potential loss of surplus on the last unit in some states of the world. It arises since a bidder is forced to bundle bids for several units together (rather than being allowed to submit a separate bid for every unit for sale). To illuminate this trade-off further, consider the following example: there are two units for sale and a single risk neutral consumer with values 10 and 5 for units 1 and 2, respectively. The market price is uncertain, either \$6 or \$1, where the state corresponding to \$6 occurs with probability ω. Any demand at or above the realized price is satisfied, while no units demanded below that price are sold. The consumer must submit his demand using one step: a price and the number of units he is willing to buy at any weakly lower price, before the uncertainty is resolved. It is easy to see that when $\omega \leq \frac{4}{5}$, the consumer will prefer to submit a demand for two units at a price of 6. The gains from the first unit in both states of the world and the gains on the second unit when the price is low more than offset the losses on the second unit when the price is high. The consumer would also not want to cap his demand at a price of \$1 since then he would lose $3\omega > 0$ relative to demanding both units at prices up to \$6. The gain on the inframarginal first unit in the high price state is simply too valuable and, at the same time, the loss on the second unit in the high price state is outweighed by the surplus enjoyed from this additional unit when the price is low. Notice that this example illustrates that the rationale for bidding above one's marginal value in the current model comes purely from the restrictions on the strategy sets and not from rationing.

One more point worth noticing and one that will be important in my empirical exercise is that the results of Proposition 1 remain valid for all models of the (possibly random) supply Q as long as the appropriate stochastic model is common knowledge among the bidders. In particular, Q can be both purely random and thus independent of bids, or the auctioneer can employ some deterministic rule which maps the actual bids into Q. Bidders will simply take this into account when forming the expectations involved in Proposition 1. The exact rule that governs Q, however, can affect estimation and inference.

The existence of an equilibrium in a model of a uniform price auction with restricted strategy sets is an open question. Kastl (2008) shows that as the restriction on the strategy sets is removed, we obtain existence of a pure strategy equilibrium in the uniform price auction. He also shows that an equilibrium in distributional strategies exists in a discriminatory auction both with and without restriction on strategy sets whenever either marginal valuations are strictly decreasing in quantity or signals are independent. One of the building blocks for those results is that ties with positive probability are not compatible with equilibrium. With restricted strategy sets and uniform price auction, however, two bidders who bid above their marginal valuations for the last unit, might be happy to tie with positive probability and receive only those units for which their marginal valuation weakly exceeds the bid even if the prices were assumed to be continuous. The empirical analysis thus has to be performed conditional on equilibrium existence, or alternatively, assuming that price space is discrete, so that the necessary conditions for quantity choice derived above hold and an equilibrium (at least in distributional strategies) exists.

4 Econometric Model and Identification

In the previous section we analyzed a model with an equilibrium in step functions. But the fact that bidders are restricted to use a finite number of bidpoints is not the whole story. In most auctions the bidders do not attain this institutionally-set upper bound and the number of steps employed by bidders varies both within and across auctions. Moreover, the number of bidpoints bidders submit is usually very low. One way to rationalize this variance is that there is some cost of bid submission that might differ across bidders and/or time and which leads them to submit different number of bids. The presence of such costs would constitute an endogenous, economic restriction on the number of bidpoints and therefore I will now introduce this concept more explicitely. Let us suppose that the cost of submitting K_i steps is private information summarized by a cost function $c(K_i, t_i)$ where the parameter t_i is private information of bidder i.

Assumption 5 *A bidder submitting K_i bidpoints incurs non-negative cost $c(K_i, t_i)$ where t_i is private information of bidder i which is drawn from a distribution function $G_i(t|s_i)$ with support* $[0, 1]$.

Our assumption on t_i guarantees that t_i is independent across bidders whenever s_i's are independent, and allows for arbitrary correlation between s_i and t_i. Notice that this formulation nests the original model as a special case in which $c(K_i, t_i) \equiv 0 \; \forall (K_i, t_i)$. It also includes the case in which there is an exogenous upper bound $\bar{K}$ on the allowed bidpoints, in which case $c(K_i, t_i) = \infty$ for $K_i > \bar{K}$ and any t_i. The expected utility of a bidder of a type (s_i, t_i) in a uniform price auction now becomes:

$$\begin{aligned} EU_i(s_i, t_i) &= \mathbb{E}_{Q, s_{-i}, t_{-i}|s_i, t_i} u(s_i, s_{-i}, t_i, t_{-i}) \\ &= \mathbb{E}_{Q, s_{-i}, t_{-i}|s_i, t_i} \left[\begin{array}{c} \int_0^{q_i^c(Q, \mathbf{s}, \mathbf{t}, \mathbf{y}(\cdot|s,t))} v_i(u, s_i, s_{-i})\, du \\ -p^c(Q, \mathbf{s}, \mathbf{t}, \mathbf{y}(\cdot|\mathbf{s}, \mathbf{t}))\, q_i^c(Q, \mathbf{s}, \mathbf{t}, \mathbf{y}(\cdot|\mathbf{s}, \mathbf{t})) - c(K_i, t_i) \end{array} \right] \end{aligned}$$

where K_i is the number of steps of $y_i(\cdot|s_i, t_i)$. Inspecting the expression for the expected utility, we should note that specifying the cost function as above allows us to obtain the same equilibrium relationship between the bid and the marginal value at every step k as derived in the previous section. The type t_i affects only the extensive margin, the number of steps K_i, whereas the location of the steps (conditional on K_i) is determined by local optimality conditions given by proposition 1.[15] The reason for allowing for more general cost functions is simply to rationalize the data as the number of submitted bidpoints varies both across bidders and even for a given bidder across auctions. Under fairly mild assumptions (such as indepedent uncertainty about Q or BNE in strictly increasing strategies and independent types) and without the cost of an additional step or some other friction bidders should always use as many steps as allowed. Yet in the data the upper bound on the number of steps allowed is never attained.

Suppose we have data on all bids from T auctions. I will impose the following assumption on the data generation process.

Assumption 6 *Bidders have private values and can be split into G groups within which the marginal valuation function is symmetric. Private information is identically distributed within groups and independent across bidders and auctions. The data $\left\{ \{b_{it}, q_{it}\}_{i=1}^{N_t} \right\}_{t=1}^{T}$ are generated by K-step equilibrium behavior, where N_t is the number of (potential) bidders in auction t.*

[15] The only difference is that the strategy depends on type (s_i, t_i), all expectations are over rivals' types and uncertain supply (Q, s_{-i}, t_{-i}) conditionally on own type (s_i, t_i) and similarly, the market clearing price and quantity are functions of the whole vector of random variables $(Q, \mathbf{s}, \mathbf{t})$.

4.1 Identification

The identification argument follows the first-order condition approach proposed in Laffont and Vuong (1996) and Guerre, Perrigne and Vuong (2000).[16] In particular, the price-quantity pair submitted as the k^{th} out of K_i total bidpoints has to satisfy the necessary conditions (2). However, inspecting (2) reveals that in case of equilibria in which ties happen with positive probability we cannot invert for $v(q_k, s_i)$ unless we know $\mathbb{E}\left(v\left(q_i^c\left(Q, \mathbf{s}, \mathbf{y}\left(\cdot|\mathbf{s}\right)\right), s_i\right) \frac{\partial q_i^c(Q,\mathbf{s},\mathbf{y}(\cdot|\mathbf{s}))}{\partial q_k} | p^c = b_k \ \wedge \ Tie\right)$. In this section I will discuss identification of the marginal valuation using the identification equation (4). This optimality equation is valid whenever ties occur with zero probability or if bidders ignore the effect of their quantity demand on the quantity they are allocated in the event they tie, which seems to be the case according to my discussions with participants in the Czech treasury auctions.[17,18] Since in the 28 auctions in my data set the average supply satisfied at bids above the market clearing price is about 80% (in 16 auctions it is more than 90%) and the mean rationing coefficient is 60%, the data does not suggest that bidders should be too concerned with the ties when placing their bids. When rationing indeed occurs, on average only 4 bidders are rationed and their identities vary across auctions - hence (also given that bidders submit on average slightly more than 2 steps) from the perspective of an individual bidder a tie at a given step is ex ante fairly unlikely.[19,20] Notice that even when ties are ignored, there is still a fundamental difference between the identification condition implied by the model with continuous downward-sloping bids and model with discrete bids. Suppose for the moment that all (uncertain) residual supplies are vertical translations of each other (i.e., have the same slope at every q) and all the uncertainty is only about their location. In this case one can show that the shading factors (implied by a bidder's market power) coincide in both models and the difference becomes that the model with continuous downward-sloping bids implies bidding such that $v(q_k, s) = b_k + Shading\ Factor$ whereas the model with discrete bidding requires $v(q_k, s) = \mathbb{E}(p^c | b_k > p^c > b_{k+1}) + Shading\ Factor$. The model with continuous bids would thus overestimate marginal values implied by bid data by $Bias = b_k - \mathbb{E}(p^c | b_k > p^c > b_{k+1})$.

The general problem of multiunit auctions is that the full marginal valuation function might still not be identified because there could be many functions $v(\cdot, \cdot)$ that: (i) go through the estimated points , (ii) are everywhere non-increasing in the first argument, (iii) are everywhere strictly increasing in the second. We could potentially also use a second set of necessary conditions which are implied by the bid, b_k, being chosen optimally. But as equation (A-3) in the appendix shows,

[16] For a survey of recent results on non-parametric identification of auction models see Athey and Haile (2007).

[17] Hortacsu (2002) also reports that Turkish bidders seem to ignore the effect of their demand on the rationed quantity.

[18] In Appendix B I describe an alternative approach which allows for identification taking into account the ties under an additional assumption on the marginal valuation curve.

[19] The (ex ante) likelihood of tying at a step is roughly 9.7%(=4/18*1/2.3) and conditional on tying one should expect the marginal demand to be shortened by 40%.

[20] The practical problem when allowing for ties can be seen in appendix B: apart from the necessity to impose additional assumptions on $v(\cdot, s)$, the optimality condition involves a ratio of two probabilites, which have to be estimated and thus small errors in the estimates of these probabilities lead to potentially large errors in the estimates of the marginal values.

using these conditions would still not achieve unique identification, because they put restrictions on the area below the marginal valuation function between each two bidpoints (by relating the average surplus and the bid), but there is still not enough information to pin down the curvature. In Kastl (2006) I showed that more information about marginal valuation function can be obtained by using bidders who submit multiple bidpoints if we are willing to make stronger assumptions on the data generation process. Alternatively, we may approach the identification problem via set identification instead. We may be able to use both necessary conditions for bidding (3) and (A-3) and inequalities implied by assumptions on the primitives and by the data to further restrict the set of possible marginal valuation functions that would rationalize the data, in a similar way to Haile and Tamer (2003). McAdams (2008) makes a step in this direction by making use of a large number of potential deviations to tighten the identified set. The difficulty with this approach in my setting, however, is that both the necessary condition (A-3) and inequalities implied by choosing q_k rather than $q_k - \Delta$ involve the gross utility $V(\cdot, s_i)$, which is an integral of the object of interest, $v(\cdot, s_i)$. This research direction is currently left for the future.

While we cannot identify the cost function, $c(K,t)$, we can identify bounds on the incremental cost of an additional step. In equilibrium, the additional cost of submitting one more bidpoint $\Delta c(K_i+1, t_i) = c(K_i+1, t_i) - c(K_i, t_i)$ must be weakly higher than the expected benefit. Similarly, $\Delta c(K_i, t_i) = c(K_i, t_i) - c(K_i - 1, t_i)$ must be weakly less than the expected benefit of going from $K_i - 1$ to K_i bidpoints. This allows us to compute bounds on the implied incremental cost of bidding for a given bid (i.e., for a fixed t_i). In order to obtain valid bounds in our setting of partial identification of the valuation function, we have to also take into account the effect of the marginal valuation function on these costs. In particular, let $\mathbb{V}_i = \{v(q) : v_i(q_k) = \hat{v}_{ik} \, \forall k \leq K_i, \, v_q(q) \leq 0\}$ be the set of all non-increasing level curves (at a particular signal s_i) of marginal valuation functions that are consistent with our estimates $\hat{v}_{ik}$ at all steps $k \leq K_i$. Then

$$\Delta c(K_i+1, t_i) \geq \inf_{v(q)\in\mathbb{V}_i} \left[EU(s_i|\sigma^*(K_i+1)) - EU(s_i|\sigma^*(K_i))\right] \tag{5}$$

denotes the lower bound on cost of going from K_i to $K_i + 1$ steps for bidder i and

$$\Delta c(K_i, t_i) \leq \sup_{v(q)\in\mathbb{V}_i} \left[EU(s_i|\sigma^*(K_i)) - EU(s_i|\sigma^*(K_i-1))\right] \tag{6}$$

the upper bound on going from $K_i - 1$ to K_i steps for bidder i, where $\sigma^*(K_i)$ denotes the optimal bidding strategy conditional on using K_i steps.

4.2 Estimation

Let us now discuss the method for obtaining the point estimates of marginal valuations at the submitted quantity-bids non-parametrically. Notice that all objects on the RHS of (4) that are not directly observed, $\mathbb{E}(p^c|b_k > p^c > b_{k+1})$ and $\frac{\partial\mathbb{E}(p^c; b_k \geq p^c \geq b_{k+1})}{\partial q_k}$, are some functionals of the distribution of the market clearing price. Hortaçsu (2002) shows that this distribution can be consistently estimated from the bidding data using a resampling method, which closely follows the usual bootstrapping approach and which I will now describe and adapt to my application.

Estimating $\mathbb{E}\left(p^c|b_k > p^c > b_{k+1}\right)$

Under assumption 6, we can perform the following resampling procedure:

1) Fix bidder i from group $g \in G$ among N_{tg} bidders in auction t who belong to group g.

2) From the sample of N_{tg} bid vectors in the data set, draw a random sample of $N_{tg} - 1$ bid vectors (i's bid does not need to be excluded because of the assumed symmetry and independence)

3) From all groups h other than g draw N_{th} bid vectors for $h \in G \backslash \{g\}$ with replacement, giving equal probability of $\frac{1}{N_{tg}}$ (or $\frac{1}{N_{th}}$ respectively) to each bid vector in the original sample.

4) Construct the residual supply function generated by these resampled bid vectors.

5) Intersect this residual supply curve with bidder i's bid function to find the market clearing price.

6) Repeat steps 1-5 B (a large number) times for each bidder and for all bidders in the data set.

This procedure generates B market clearing prices conditional on the bid vector $\left(\vec{b}_i, \vec{q}_i\right)$ and one can estimate $\mathbb{E}\left(p^c|b_k > p^c > b_{k+1}\right)$ by looking at the conditional distribution of the market clearing prices which fall in the required interval.

For this method to perform reliably we would like to have a large number of bidders in each group in every auction, so that we observe bid vectors reflecting a large number of signal realizations from the group distribution function of signals. If that is not the case, but we are willing to assume that we have several auctions with no observed or unobserved heterogeneity, we can pool together bid vectors from different auctions. Alternatively, if we have auction-level observables, we can conduct conditional resampling where the resampling weights are not uniform $(\frac{1}{N_{th}})$, as in the case above, but rather a function of the observables (see Hortaçsu and Kastl (2008)). In either case, if we call the estimator obtained by the above procedure the resampling estimator $\mathbb{E}_T^R\left(p^c|b_k > p^c > b_{k+1}\right)$, it can be shown (Hortaçsu (2002), Hortaçsu and Kastl (2008)) that it is consistent for $\mathbb{E}\left(p^c|b_k > p^c > b_{k+1}\right)$ (it converges almost surely) as the number of auctions goes to infinity, $T \to \infty$.[21]

Estimating $\frac{\partial \mathbb{E}(p^c; b_k \geq p^c \geq b_{k+1})}{\partial q_k}$

To obtain this piece of equation (4), we can use the same resampling approach described earlier when estimating $\mathbb{E}\left(p^c|b_k > p^c > b_{k+1}\right)$ to estimate $\mathbb{E}\left(p^c|b_k \geq p^c \geq b_{k+1}\right)$, which together with an estimate of $\Pr\left(b_k \geq p^c \geq b_{k+1}\right)$ and Bayes' rule yields an estimate of $\mathbb{E}\left(p^c; b_k \geq p^c \geq b_{k+1}\right)$. Call this estimate $\mathbb{E}_T^R\left(p^c; b_k \geq p^c \geq b_{k+1}\right)$. Notice that while obtaining this estimate, we condition on the submitted vector of bidpoints. The natural way to estimate the derivative of this expectation with respect to quantity bid at step k is to perturb q_k in the submitted bid vector to some $q_k - \varepsilon_n$

[21] It may also be consistent under other conditions - see Hortaçsu (2002).

and obtain an estimate of $\mathbb{E}_T^R(p^c; b_k \geq p^c \geq b_{k+1})$ conditional on the perturbed bid vector. We can then construct the estimator of the derivative:

$$\frac{\partial \mathbb{E}_T^R(p^c; b_k \geq p^c \geq b_{k+1})}{\partial q_k} = \frac{\mathbb{E}_T^R(p^c; b_k \geq p^c \geq b_{k+1}, q_k) - \mathbb{E}_T^R(p^c; b_k \geq p^c \geq b_{k+1}, q_k - \varepsilon_n)}{\varepsilon_n}$$

where $\{\varepsilon_n\}_{n=1}^{\infty}$ is a sequence converging to zero. One difficulty when estimating the slope of this expectation w.r.t. q_k is choosing the appropriate neighborhood ε_n so that the numerical derivative is a consistent estimate. Loosely speaking, this neighborhood should shrink to zero as the sample size increases. Pakes and Pollard (1989) establish that with a regularity condition (on uniformity), such an estimator is consistent whenever $n^{-\frac{1}{2}}\varepsilon^{-1} = O_p(1)$, i.e., whenever ε does not decrease too fast as the sample size increases.

Proposition 3 *(Consistency of the resampling estimator)*
Under assumptions 1-6:
(i) If $\Pr(b_k > p^c > b_{k+1}) > 0$, *then* $\mathbb{E}_T^R(p^c | b_k > p^c > b_{k+1}) \rightarrow^{a.s.} \mathbb{E}(p^c | b_k > p^c > b_{k+1})$ *as* $T \rightarrow \infty$
(ii) If $\Pr(b_k \geq p^c \geq b_{k+1}) > 0$ *and* $T^{-\frac{1}{2}}\varepsilon^{-1} = O_p(1)$, *then* $\frac{\partial \mathbb{E}_T^R(p^c; b_k \geq p^c \geq b_{k+1})}{\partial q_k} \rightarrow^{a.s.} \frac{\partial \mathbb{E}(p^c; b_k \geq p^c \geq b_{k+1})}{\partial q_k}$ *as* $T \rightarrow \infty$.

Given consistent estimates of all the pieces of the right hand side of (4), we can obtain the point estimates of the marginal valuations at the submitted bids, conditional on the fixed unobserved private signal.

5 Data and Results

5.1 Description of the Data

My dataset consists of 28 auctions of Treasury bills of the Czech government. The sample period is 11/25/1999 until 12/14/2000. The auctions were conducted by the Czech National Bank. The payment by each bidder whose order was accepted was determined according to the uniform price rule; each bidder paid the market clearing price for all units for which his bid was at least the market clearing price. These auctions of T-bills were conducted weekly, with the auction plan being published quarterly. The T-bills that were sold in different auctions differed in maturities. I will consider only auctions of 3-month T-bills, since they were auctioned most often - usually bi-weekly. In the quarterly published auction plan, the Bank announces the intentions of the Ministry of Treasury as to how many securities will be sold on a given week and of which maturity. The main purpose of the T-bills is to smooth out the difference between tax revenue and expenditures by the government.

The bidders who wished to participate in an auction of T-bills had to be preregistered by the Czech National Bank. The only requirement for the registration was that the bidder possesses either a banking license or a broker license in the Czech Republic or other EU member country. The list of registered bidders was publicly available and hence the number of potential bidders is known in every auction. Furthermore, there were limits with which each registered bidder had to comply. Each bidder was obliged to buy at least 3% of the securities offered within a calendar year, and his demand in a given auction could not exceed 50% of the securities offered for sale. The first restriction was usually met by each bidder early in the calendar year. Moreover, since bidders were not given any information about rivals' allocations after any auction, we can safely ignore this restriction in our model, since it is not likely to affect the strategic behavior. The second restriction is virtually never binding in the data (except for 1 occurence).

Let us now briefly discuss the assumption of private values which is necessary for the identification of bidders marginal values outlined in the previous sections. The main motive for the bidders to purchase the treasury bills in the Czech auctions was for their investment portfolios, since T-bills do not carry any risk premium and thus unlike other investments do not have to be outweighed by any cash (or other no-risk) reserves. Moreover, many of the banks involved in these auctions are subject to investment risk regulation for various reasons, and T-bills are one of the few ways to profit from their cash reserves. Most banks thus hold the T-bills in their portfolios until the maturity. It is for these reasons that the secondary market for T-bills in the Czech Republic is virtually nonexistent. The absence of active trading on the secondary market suggests that we may not have to worry about an unknown common resale value component in the auctions.[22] Finally, Hortaçsu and Kastl (2008) develop a formal test for private versus common values in the context of Canadian T-bill auctions and they fail to reject private values in the case of 3-months T-bills, while they reject private values for 12-months T-bills.[23] Taking all of these findings together leads me to believe that the private values model might be appropriate for the auction of Czech 3-months T-bills.

Table 1 describes the summary statistics of the important data components. The face value of all T-bills is 1,000,000 Czech Korunas (approximately $26,300). The range of bids in annual yield is 66 basis points, while the range of the market clearing yield is 32 basis points. Notice that our unit of observation is the whole bid function (characterized by price-quantity pairs), and hence the relatively low variation in the bids does not imply low variation in the data. Indeed, the variation in the quantity demands is much higher. This point also highlights why using a share auction model, in which identification of marginal valuations comes from the necessary conditions for the choice of quantity, seems to be more appropriate than an alternative model, in which identification is based on the optimality of price-bids.

Bidders submitted bids for as little as 0.01% of total quantity supplied and for as much as 50% which is the maximal amount they can demand in an auction. Bidders are allowed to submit up

[22] Another important point to note is that an active resale market is also usually accompanied by an active when-issued (forward) market and hence any private information about the resale value should be already reflected in the prices on the when-issued market. Therefore the variation in the bids should rather be ascribed to other private information than that related to the resale value.

[23] This test is unfortunately not applicable in my current data since it relies on a particular feature of the Canadian auctions which is here absent.

to 10 bidpoints (price-quantity pairs) in any given auction. Yet the average number of bidpoints submitted by a bidder in an auction is less than 3 and the maximal number of submitted bidpoints is 9.[24] For each auction I observe all individual bids (including the noncompetitive ones placed on behalf of the government, which will be described below), the preannounced supply quantity, and the market clearing price. I also observe the final allocation. My dataset includes 16 unique bidder identities. 7 of these bidders can be classified as belonging to the "small bidder" group, since they request less than 5% of the total quantity in any given auction and also submit fewer bidpoints on average than their larger opponents. The remaining 9 bidders will be treated as belonging to the "large bidder" group. The classification of bidders into groups applies across all auctions. Table 2 offers a split of summary statistics between these groups.[25]

An important feature of many treasury auctions of government securities is the possibility of "noncompetitive bids." These bids specify a quantity which the bidder would like to obtain at the market clearing price no matter what this price will be. Therefore, in terms of modeling, these bids simply decrease the available supply of T-bills in a given auction. While the rules of the auction allow for such bids to be submitted by regular bidders, they rarely use this possibility. In my dataset, none of the bidders submits a noncompetitive bid in any auction. On the other hand, the auctioneer himself, as instructed by the Treasury, can submit such a bid even after observing the bids of regular bidders. In fact, in each announcement about an upcoming auction, which includes the details such as the number of T-bills to be auctioned off, there is a disclaimer that, "The issuer of the security reserves the right to include part or all of the emission in his own portfolio." This possibility then serves as an insurance device against low market clearing prices.[26] In my data set, the treasury kept as much as 77% of the supply in its portfolio. However, most of this supply withdrawal was not unexpected - a big part of the supply reduction has been previously communicated to the bidders, since the Treasury keeps from time to time part of the emission in its portfolio in order to be able to conduct various financial repo operations later. Table 1 shows that the unexpected supply reduction amounted to less than 0.2%. In only two auctions did the unexpected supply withdrawal exceed 2% of the preannounced issue and no supply was withdrawn in 6 auctions. There is thus some uncertainty with respect to the actual quantity for sale on the part of the bidders.[27] Further notice that the reference interest rate that the banks use for transactions among themselves has all

[24] In the auctions in my data there is no detectable time trend in the number of bid points used.

[25] Note that assuming multiple bidder groups in case that all bidders are symmetric does not affect the consistency of the estimates, but only results in efficiency loss.

[26] One can also view this part of the mechanism as allowing for the possibility of "shill-bidding" (or a secret reserve price). Vincent (1995) showed that under common values, keeping the reserve price secret might encourage participation and thus alleviate the winner's curse. Izmalkov (2003) establishes that allowing shill-bidding in an English auction with asymmetric independent private values leads to an equilibrium of such an English auction which coincides with the equlibrium of Myerson's (1981) optimal auction. While it is not clear that keeping the reserve price secret might benefit the auctioneer through informational externalities, it might, however, eliminate the low-price equilibria of a uniform price auction that Wilson (1979) and others identified.

[27] This also suggests that a model which focuses on the optimality of the choice of quantity-bid rather than price-bid might be more appropriate for treasury bills.

descriptive statistics only slightly higher than the corresponding statistics of the market clearing yield of T-bills, which suggests that it might be a factor in the auctioneer's decision how much supply to withdraw. From my discussion with the insiders it is apparent that the Treasury is using the noncompetitive bids to keep the market clearing yield within some fairly narrow band around the reference interest rate (especially when the market clearing yield in the auction would exceed this reference rate). In terms of empirical implementation, I consider two alternative models of the noncompetitive bids of the government:

In model M1, I treat government as a separate bidder group and thus resample from the observed (noncompetitive) bids in the same way as I resampled from the other two groups. In particular, I resample the government bid independently of the resampled bids of regular bidders. The idea why this approach might yield a good approximation is that for estimating a bidder's marginal value at step k, the distribution of the market clearing price matters only in the interval $[b_{k+1}, b_k]$. Therefore, if the independent draw of the government bid would cause a big change in the market clearing price (e.g., due to too big a withdrawal of supply), this realization would not matter for the estimate of $\hat{v}(q_k, s_i)$.

In model M2, I postulate the following rule for government bid which is motivated by my discussion with insiders: Withdraw supply if market would not clear or if the clearing rate would exceed the reference interest rate by more than 1 basis point. In terms of estimation, I first resample the residual supply, and if the resulting market clearing price were to exceed the reference interest rate (in that auction week) by more than 1 basis point, I adjust the supply. We will see in the results I report below that using M1 or M2 results in very similar estimates of marginal values and bidders' interim profits. This rule prevents a situation with an undersubscribed T-bill emission and thus a zero market clearing price. I conjecture that having such rule in place (i.e., having such an option) may also eliminate the "collusive-seeming" equilibria of a uniform price auction, in which bidders non-cooperatively "split" the market and achieve a low market clearing price (see e.g., LiCalzi and Pavan (2005) or McAdams (2007)).

5.2 Results

5.2.1 Estimating marginal valuations

I first illustrate the resampling procedure, described in Section 4, that I use to estimate the distribution of the market clearing price, and thus the conditional expectation and its derivative. Consider a particular auction labeled as Auction 52 in my data. There are 13 bidders (8 large and 5 small) who actually submitted a bid and there are 15 potential bidders (8 large and 7 small) that were registered with the auctioneer before the auction. For the purposes of resampling, this is not a large number and therefore I pool 4 neighboring auctions, in which T-bills of the same maturity were offered, and assume that no observed or unobserved heterogeneity is present.[28] I split my sample of 28 auctions into 7 groups with 4 auctions in each. Since the number of preregistered bidders virtually does not change across auctions, I assume that the classification of bidders into bidder groups remains the same and also that the number of potential bidders is the same with one exception. In particular, I assume that there are 7 potential small bidders and 8 potential large bidders.[29] The reason for assuming there are 8 potential large bidders even though there are 9 bidder identities that I classify as large is that one large bidder starts bidding first in auctions later in the sample and another large bidder at that point stops bidding and never submits a bid again during the sample period. Bids of those two bidders overlap only in two auctions, and therefore for the group of four auctions in which these two particular auctions belong I assume that there are 9 potential large bidders rather than 8. I assume that any bidder for whom I do not observe a bid in a given auction submitted a losing bid (a bid of zero for any quantity) and I include such a bid function in the sample from which I resample.

Grouping four auctions together might be problematic, since as I argued above the private information driving the marginal valuation of each bidder is assumed to come from the current state of its cash reserves and alternative investment opportunities, both of which could be affected by the outcome of previous auctions, or be correlated across auctions. In Kastl (2006) I included a robustness check against this possibility by testing whether winning larger quantities in earlier auctions results in lower levels of private signals for the later auctions. I decided to pool 4 neighboring auctions for two reasons. For resampling, I want to include bid functions from auctions from as short a time-span as possible in order to be more confident about the economic environment not changing. On the other hand, I need a larger number of bid functions so that resampling generates enough variation, because the heart of the consistency argument is that the observed data should in the limit as the number of auctions goes to infinity include the equilibrium bid for every type and with the appropriate population frequency of that type. Given that there are 15 potential bidders in an auction pooling 4 auctions together yields 60 bid functions for the purposes of resampling which should contain enough variation in bids. Each four neighboring auctions I pool together were conducted in a time frame of two months, and the macroeconomic variables such as the consumer price index or the interest rate were stable across this period. In principle, with richer data one could modify the resampling method in order to allow for some auction covariates Z. Instead of resampling with replacement with equal probability $\frac{1}{N}$ on all bid functions, we could instead use a probability distribution $\Gamma(Z, N)$, for example using a normal kernel. Such procedure has been developed in detail and implemented in Hortaçsu and Kastl (2008).

[28] The reference interest rate, an obvious source of observed heterogeneity, varies very little within each group of 4 auctions. In principle, with enough data one could also perform conditional resampling by introducing covariates which would control for the economic environment. One possible implementation is described in the subsequent paragraph.

[29] I also estimated the model assuming that the number of potential bidders differs across the groups of auctions and is equal to the largest number of active bidders within an auction in that group. The results were similar.

In the first three auctions, there are 5 active small bidders and 8 active large bidders. In the fourth auction, there are 6 active small bidders and 8 active large bidders. Under the assumptions of full symmetry and constant number of potential bidders (7 small and 8 large bidders), pooling these four auctions results in 60 ex ante symmetric bidders, who differ ex post because of their private information. Alternatively, with two groups, this results in 28 ex ante symmetric small bidders and 32 large bidders. Let us fix bidder 1's bid function and generate the different residual supply curves he might face by the above described resampling procedure. Figure 1 shows the procedure with 15 different realizations of the residual supply curves using M1 as the model of government's supply withdrawal. This process generates a distribution of market clearing prices. The distribution generated by 5000 residual supply draws under M1 is depicted in Figure 2.

With the distribution of the market clearing price, we can recover the marginal valuations for the bidder by using our optimality equation (4). Figure 3 shows using squares point estimates of marginal valuation of bidder 1 at quantities for which he submitted a bid. Open circles depict the conditional expectation of the market clearing price $\mathbb{E}\left[p^c|b_k > p^c > b_{k+1}\right]$. The distance between these two points is the amount of shading that the bidder executes, which is a direct measure of bidder's market power. Notice that, as a possibility suggested in Corollary 2, the actual bid of bidder 1 is above the estimated marginal valuation for the first bidpoint. The fact that it occurs at the first bidpoint is not a coincidence, since the incentives to shade increase in the quantity demanded. Thus, it is more likely that for smaller quantities the marginal valuation will be closer (given market power) to the conditional expectation of the market clearing price and thus below the actual bid. However, it may also occur at larger quantities - when in the states that lead to the market clearing price being even lower than the bid for those quantities this bidder's demand is not pivotal and thus marginally shading this demand does not affect the market clearing price. Similarly, Figure 4 shows the results of the estimation for bidder 6. At many quantities, the bid again exceeds the estimated marginal value.

Repeating the same procedure for each bidder in the auction, we obtain point estimates of the marginal valuation function $v(q, s)$ at the (observed) quantities that the bidders request and at the (unobserved) signal levels s. As described in the working paper version of this paper, we could use information from bidders who submit at least two bidpoints to estimate $v(q, s)$ nonparametrically, as long as in the limit, as the number of data points increases, the whole domain of $v(.,.)$ would be covered. Even if the latter condition were satisfied, however, this exercise would not be useful for empirical estimation with little data, since it involves a three-dimensional kernel regression.

5.2.2 Standard Errors

Obtaining the asymptotic variance of the estimated marginal valuations is cumbersome, since our marginal valuation estimator is a nonlinear function of the distribution of the market clearing price, which is also estimated. For this reason, I employ bootstrap methods[30] to compute the standard errors of my estimates. The reported standard errors are from the sample of 500 estimates generated by repetitions of the estimation procedure with a new bootstrap sample of bid functions at each round. The argument that the bootstrap can be used for estimates based on the distribution of the market clearing price is based on Theorem 2 of Bickel and Freedman (1981), which proves validity of the bootstrap for U-statistic.[31]

5.2.3 Step functions versus continuous downward sloping bids

One might wonder what difference it makes to assume that bidders submit step functions strategically, rather than treating the observed bidpoints as some selection from a downward sloping continuous function. Equation (1) reveals that in the continuous bid functions setting the observed bids should equal marginal valuations less a markup associated with that bidder's market power. In other words, it is necessarily the case that within such a model the marginal valuations are strictly above the observed bids (as long as these bids are within the support of the distribution of the market clearing price), unless the bidder is a price-taker, in which case the two values coincide. To illustrate the difference between using the optimality conditions for the model with step functions from the one with continuously differentiable bids, we can approximatethe marginal values in the latter model by adding the estimated shading factor[32] from the model with step functions to the observed bid rather than to the conditional expectation of price.[33] The difference is statistically significant whenever $b_k - \mathbb{E}\left[p^c | b_k > p^c > b_{k+1}\right]$ is statistically different from zero, which has to hold simply by definition of the conditional expectation. Since using the necessary conditions from the model with continuously differentiable bid functions would overestimate marginal valuations, using these biased estimates for counterfactual exercises would result in upward biased counterfactual revenues from a discriminatory auction. Figures 3 and 4 show that the model used for estimation can matter, especially when estimating marginal valuations at quantities, at which the two adjacent bids are substantially different. As I will present later, in my data using the model with continuously differentiable bids for estimation would result in approximately 60% overestimate of bidder's surplus.[34]

I conjecture that in applications with more uncertainty about the market clearing price the difference between estimates from a model that takes into account the discreteness of bids and

[30]For an introduction to bootstrap see Efron and Tibshiranim (1993).

[31]The resampling estimator is basically a V-statistic and by Lehmann (1999, Theorem 6.2.2, p.388) the asymptotic distribution of this V-statistic is identical to that of the U-statistic. See Hortaçsu and Kastl (2008) for more detailed discussion of the asymptotics.

[32]Recall that the shading factor is the difference between the conditional expectation of price and the estimated marginal value.

[33]As mentioned in section 4 this approximation would be exact if the residual supplies were just vertical translations of each other and the uncertainty would thus be only over their location.

[34]The estimated unextracted revenue using the model with discrete bidding is approx 3 basis points per T-bill and it is about 5 basis points per T-bill when using the continuous model.

those from a model that ignores the discreteness will be even more pronounced. In my application this uncertainty is somewhat reduced by allowing the ex-post supply adjustment which eliminates the uncertainty about the market clearing interest rate in the upward direction (the auctioneer does not want this rate to exceed the reference interest rat), as, rather than allowing a higher yield, the auctioneer would potentially withdraw supply. In absence of this institution, bidders might have to submit "steeper" bid functions, i.e., with steps further apart, and the difference between the bid and the expectation of the market clearing price conditional on it being between the two steps, $b_k - \mathbb{E}\left[p^c | b_k > p^c > b_{k+1}\right]$, might thus be larger.

Furthermore, I will now show that in a non-negligible number of the auctions, the actual ex post revenue exceeded the revenue that would have been realized had all bidders bid the upper bound of their estimated marginal valuation functions. As pointed out earlier, this result would not obtain if we ignored the discreteness of bids.

5.2.4 Counterfactual: Truthful Bidding

In my first counterfactual analysis, I compare the actual revenue to the revenue from a "best case" Vickrey auction, i.e., a uniform price auction in which bidders truthfully bid their marginal valuation schedules without actually receiving any payments. To perform this experiment exactly, we need to know the full functional form of $v(q, s)$. Instead, I construct an upper and lower envelope of marginal valuations by using step functions that have steps at the estimated marginal valuations. Unfortunately, we do not have enough information to construct the upper bound on the marginal valuation to the left of the first step. Similarly, we can only bound the marginal valuation to the right of the last step from below by zero and from above by the last estimated marginal value. I therefore assume that the estimated first marginal valuation is also equal to the highest possible marginal valuation. This assumption should not be too influential, since for the important (large) bidders whose demands are essential for market clearing, the market usually clears at one of their "interior" steps, and we use the appropriate bounds for those. Nevertheless, to test the robustness of the results with respect to this assumption, I also tried using the first step plus a mark-up as the maximum marginal valuation for smaller quantities, and obtained qualitatively similar results. While the upper bound on the marginal valuation for larger quantities than the last observed bidpoint is the marginal value estimated at this bidpoint, I cannot use such a bound in my analysis. The reason being that there can be a small bidder who demands just a negligible share of the total supply with a high marginal value at his last step, and by bidding such an upper bound for all larger quantities she might win the full supply. I will therefore assume that the marginal value for larger quantities than the one demanded at the last bidpoint is zero. Using these upper and lower envelopes of marginal valuations, I obtain the market clearing price given the same ex post realization of noncompetitive bids as in the actual auction. Tables 3 and 4 report the results in terms of the market clearing price. The first column reports the actual realized market clearing price and the second and third column the market clearing price under bidding truthfully the lower or upper envelope respectively.

These tables reveal that the actual market clearing prices are not far from those that would be obtained under truthful bidding. This suggests that bidders do not have enough (local) market power around the expected market clearing price to affect adversely auction's revenue.

In 6 of the 28 auctions, which are highlighted by an asterisk in the table, the actual ex post revenue exceeds the revenue from bidding the upper bound of the marginal valuation schedules, which suggests that the point raised in Corollary 2 is not purely theoretical. These results may cast some doubt on the conclusions that Hortaçsu (2002) reaches in his empirical study of Turkish treasury auctions, which have a discriminatory format. In particular, he concludes that since the revenue generated in a uniform price auction in which bidders submit the upper bound of the estimated marginal valuations (which is constructed in the same way as in this paper) as their bids is lower than the actual revenue, the discriminatory auction performs better ex post. From the ex ante perspective, when he draws the bid functions randomly before the auction, he cannot reject the revenue equivalence hypothesis. My results suggest that using a model with continuously differentiable bid functions as an approximation to the true model of discrete bidding to conduct any counterfactual exercises will most likely lead to results that are biased towards the discriminatory auction because of overestimated marginal values.

5.2.5 Effectiveness of value extraction

How effective a mechanism are these uniform price auctions? Could the Czech government do better by using a discriminatory auction? One way to get a handle on these questions is to compare the performance of the employed mechanism to the ideal mechanism, which would implement an efficient allocation and extract full surplus. We can use the upper envelope of the estimated marginal valuations together with the estimated distribution of the market clearing price to obtain estimates of (an upper bound for) bidders' expected (interim) utility per T-bill sold in the auction. If this expected utility is close to zero for every bidder, and the allocation is efficient, then the auction mechanism would perform well even from an ex ante perspective. Under the equilibrium hypothesis, the observed bid function of each bidder should be a best response of his type to the equilibrium strategies of other bidders. Using the estimated distribution of the market clearing price conditional on bidder i's bid and setting $c(K_i, t_i) \equiv 0$, I can evaluate i's expected utility given the submitted bid function, i.e., conditional on his type. In equilibrium, this submitted bid function should deliver the highest utility this bidder can obtain (given his type). Therefore this exercise indeed delivers an estimate of the maximal interim utility of each bidder. The results are reported in Tables 5 and 6.

The minimal estimated interim utility is close to zero, which suggests that submitted bid functions are individually rational. It also suggests that using the upper envelope of the marginal valuations may be close to the true valuation functions. This should hold at least for bidders with interim utility very close to zero, since a lower marginal valuation curve would result in negative interim utility, in which case the observed bid function would not be individually rational. Allocations in all auctions appear to be fairly efficient, since the loss of surplus due to misallocation amounts to about 2.5 basis points. Moreover, the sum of expected surpluses across all bidders (strictly speaking, it is the sum across their actual realized types) reported in the columns labeled "Total 1" and "Total 2" of Tables 5 and 6 is close to zero. I conclude that the uniform price auction mechanism performed well, in terms of both efficiency and value extraction. The columns labelled "Total 1" and "Total 2" reveal that in 19 auctions we cannot reject the hypothesis that full expected surplus has been extracted. On average the mechanism failed to extract about 3 basis points worth of bidders' surplus. Performing the same exercise with marginal values implied by the model with continuous bids results in unextracted revenue corresponding to about 5 basis points[35] or in relative terms, in overestimation of bidders' surplus by about 60%.

Because the estimated average total expected bidders' surplus is a consistent estimate of the part of the surplus that the mechanism fails to extract ex ante, and because the allocation is nearly efficient, I conclude that the uniform price auction exhibits excellent performance. Value extraction might be even better, since I considered the upper bound on the marginal valuation functions of each realized type when performing the computations. Because the uniform price auction mechanism performed well in terms of both value extraction and allocative efficiency, switching to an alternative auction mechanism is unlikely to result in economically significant improvements in either aspect.

5.2.6 Bidding costs

As discussed in Section 4.1, we can identify bounds on the incremental cost of submitting one more (or one less) step by using equations (5) and (6). Unfortunately, searching over all possible marginal valuation functions in the set $\mathbb{V}_i$ and for each $v \in \mathbb{V}_i$ searching for the optimal bid with K or $K+1$ steps is extremely computationally demanding and rather infeasible given current computational constraints. Instead I assume that bidder i's marginal valuation is the upper envelope of my point estimates as I did when evaluating the performance of the mechanism[36] and compute the lower bound on the incremental cost of the second bidstep for bidders submitting one bidpoint by searching the whole space of bid functions with two steps for the optimal one given the distribution of residual supplies obtained in the resampling procedure. The estimates suggest that the costs of going from 1 to 2 bidpoints can total as little as \$2 and as much as \$150. I use the same procedure for bidders who submitted two bidpoints to obtain an upper bound on cost of the second bidpoint given this marginal valuation curve. I estimate that the upper bound on costs of the second bidpoint is as low as \$13 and as high as \$360. These figures are a negligible fraction of the expected surplus

[35]See the line "Mean Cont" in table 6 and notice that one basis point (in terms of annual yield) translates for 3-months T-bills into approximately 25 price units (when face value is 1 million).

[36]Notice that since the minimal interim utility was slightly negative in virtually all auctions, any marginal valuation that would depart a lot from the upper envelope I consider would imply that the observed bid violates individual rationality.

that bidders enjoy. Therefore these computations suggest that the extra benefit of fine-tuning the bid function a little more may not be that high, at least for the assumed shape of the marginal valuation function. In a slightly different auction setting and using a different approach based on McAdams (2008), Chapman, McAdams and Paarsch (2006) also found that the additional benefit of finer bids is very small.

6 Conclusion

In this paper, I analyze a model of a uniform price auction of a perfectly divisible good with private information. I show that the fact that bidders submit step functions has important implications for equilibrium. I characterize equilibrium strategies in a model in which bidders submit step functions. There is a close relationship between the optimal behavior of an oligopolist facing uncertain demand and that of a bidder in a multiunit auction with private information. My results suggest that it is difficult to make an indirect comparison between a uniform price and discriminatory auction as, for example, is done in Hortaçsu (2002). In the uniform price auction, bidders may submit bids above their marginal valuation schedule when bid functions have finite number of steps. This point is not purely theoretical. In many of the auctions in my empirical analysis, actual revenue exceeds the revenue that would have been achieved had the bidders bid their marginal valuation schedules.

I propose a new method to evaluate the performance of the employed mechanism, based on estimating the effectiveness of values extraction and the efficiency of the allocation. In the empirical analysis of Czech treasury auctions, I examine the performance of the uniform price auction. I conclude that the auction performed well. The allocation was nearly efficient, and the mechanism extracted almost all of bidders' values.The excellent performance of the mechanism studied in this application may be attributable in part to the flexibility of the auctioneer to adjust supply ex post. My estimation method also allows me to obtain an estimate of the implicit bidding costs faced by bidders in these auctions. I find that the bidders may not benefit much from submitting a finer bid function. I conjecture that in situations in which these estimated costs are low, discrete bidding leads to approximately same outcome in expectation (in terms of revenue extraction and allocative efficiency) as continuous bids. The important contribution of my paper is, however, that when trying to obtain estimates of bidders' valuations corresponding to the submitted bids in order to conduct counterfactuals, one has to take into account the discreteness. In my application, if one were to use the values implied by the optimality condition for continuous bids to conduct counterfactuals in alternative auction mechanisms, the results might be qualitatively and quantitatively quite different.[37] In particular, the model with continuously differentiable bid functions might not be a good approximation, since the results may be biased towards the discriminatory auction. In my application using this latter model would result in overestimation of the bidders' surplus by

[37] I have not performed such counterfactuals, because we currently lack the tools for computing (even numerically) equilibria of share auctions, but for a few very special parametric cases.

60%.

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A Appendix

A.1 Proof of Proposition 1

With a slight abuse of notation, I will summarize a state (Q, s_{-i}) by s_{-i} and denote by S_{-i} the set of all states. In what follows I will evaluate the following limit:

$$\lim_{q' \to q_k} \frac{\mathbb{E}_{s_{-i}} u(s_i|q_k) - \mathbb{E}_{s_{-i}} u(s_i|q')}{q_k - q'}$$

Define the following sets given a vector of bidpoints $\left(\vec{b}, \vec{q}\right)$ which consitute a partition of S_{-i}.:

$$\begin{aligned}
\theta_{1k}(q_k) &= \left\{s_{-i} : \exists p : b_{k+1} < p \leq b_k : q_k \in S^R(p, s_{-i}) \wedge \nexists q < q_k : q \in S^R(b_k, s_{-i})\right\} \\
\theta_{2k}(q_k) &= \left\{s_{-i} : \exists q \in S^R(b_k, s_{-i}) : q_{k-1} < q < q_k\right\} \\
\theta_{3k}(q_k) &= \left\{s_{-i} : \exists q \in S^R(b_{k+1}, s_{-i}) : q_k < q < q_{k+1} \ \wedge q_k \notin S^R(b_k, s_{-i})\right\} \\
\theta_{4k}(q_k) &= \left\{s_{-i} : S^R(b_k, s_{-i}) \leq q_{k-1}\right\} \\
\theta_{5k}(q_k) &= \left\{s_{-i} : S^R(b_{k+1}, s_{-i}) \geq q_{k+1}\right\}
\end{aligned}$$

The first set includes all vectors s_{-i} such that there is a market clearing price, which is in the interval $(b_{k+1}, b_k]$ and bidder i gets his full demand. The second set includes all vectors s_{-i} such that the market clearing price will be b_k and player i will be rationed. The third set includes all s_{-i} such that the market clearing price will be b_{k+1} and player i will be rationed, in which case his payoff might be affected by perturbation of q_k in case of rationing on-the-margin, since his share depends on his marginal demand $q_{k+1} - q_k$. The fourth set includes all s_{-i} such that the market clearing price will be strictly above b_k and perturbing q_k does not affect the payoff. The last set includes all s_{-i} such that the market clearing price is weakly less than b_{k+1}, and perturbing q_k will not affect the payoff. Further denote S_{-i} as the set of all possible realizations of the vector of random variables including the signals of all players other than player i and Q.

To economize on space I will write $\Pr(\theta_{jk}(q_k))$ for $\Pr(s_{-i} \in \theta_{jk}(q_k))$. By the law of total probability, we can rewrite $\mathbb{E}_{s_{-i}} u(s_i)$ as:

$$\begin{aligned}
\mathbb{E}_{s_{-i}} u(s_i) &= \sum_{j=1}^{5} \Pr(\theta_{jk}(q_k)) \mathbb{E}_{s_{-i}} [u(s_i) | \theta_{jk}(q_k)] \qquad \text{(A-1)} \\
&= \Pr\left(\cup_{j=1}^{3} \theta_{jk}(q_k)\right) \mathbb{E}_{s_{-i}} \left[u(s_i) \mid \cup_{j=1}^{3} \theta_{jk}(q_k)\right] + \\
&\quad + \sum_{j=4}^{5} \Pr(\theta_{jk}(q_k)) \mathbb{E}_{s_{-i}} [u(s_i) | \theta_{jk}(q_k)]
\end{aligned}$$

Notice that $\Pr\left(\cup_{j=1}^{3}\theta_{jk}\left(q_k\right)\right)$ is constant for any local perturbation of q_k, since any such perturbation only causes some reshuffling of states s_{-i} between θ_{1k},θ_{2k}, and θ_{3k}. Since in states in θ_{4k} and θ_{5k} bidder i actually obtains at most q_{k-1} or at least q_{k+1} respectively, perturbing q_k will not result in any change in (conditional) expected utility in these states.

The main point of the following long derivation is to show that the terms obtained by direct differentiation of the expected payment $b_k\frac{\partial \mathbb{E}_{s_{-i}}(q(s_{-i});p=b_k)}{\partial q_k}$, $b_{k+1}\frac{\partial \mathbb{E}_{s_{-i}}(q(s_{-i});p=b_{k+1})}{\partial q_k}$ and $q_k\frac{\partial \mathbb{E}_{s_{-i}}(p^c(s_{-i});b_{k+1}<p<b_k)}{\partial q_k}$ can be combined into one term: $q_k\frac{\partial \mathbb{E}_{s_{-i}}(p^c(s_{-i});b_{k+1}\leq p\leq b_k)}{\partial q_k}$ and that this object exists in equilibrium for a.e. type s_i.

Consider a perturbation of q_k to $q'=q_k-\varepsilon$. Let $\vec{q'}$ be the perturbed quantity-bid vector, i.e., $q'_m=q_m\forall m\neq k$ and $q'_k\neq q_k$. Define the following subsets of θ_{2k} and θ_{3k}:

$$\begin{aligned}
\omega_{1k}\left(q'\right) &= \left\{s_{-i}:s_{-i}\in\theta_{2k}\left(q_k\right)\cap\theta_{1k}\left(q'\right)\right\}\\
\omega_{2k}\left(q'\right) &= \left\{s_{-i}:s_{-i}\in\theta_{2k}\left(q_k\right)\cap\theta_{3k}\left(q'\right)\right\}\\
\omega_{3k}\left(q'\right) &= \left\{s_{-i}:s_{-i}\in\theta_{1k}\left(q_k\right)\cap\theta_{3k}\left(q'\right)\right\}
\end{aligned}$$

The set ω_{1k} includes states in which bidder i was rationed at price b_k originally, and after perturbing q_k down to q' he gets his full demand. Set ω_{3k} includes states in which he originally got q_k, but after perturbation the market is going to clear at b_{k+1} and bidder i will thus be rationed and obtains a higher quantity. Finally set ω_{2k} includes states in which he was rationed at b_k and after perturbing his demand q_k, he will be rationed at b_{k+1} instead.

Notice that with these sets we can now express the probabilities of sets $\theta_{jk}\left(q'\right)$ as follows:

$$\begin{aligned}
\Pr\left(\theta_{1k}\left(q'\right)\right) &= \Pr\left(\theta_{1k}\left(q_k\right)\right)+\Pr\left(\omega_{1k}\left(q'\right)\right)-\Pr\left(\omega_{3k}\left(q'\right)\right)\\
\Pr\left(\theta_{2k}\left(q'\right)\right) &= \Pr\left(\theta_{2k}\left(q_k\right)\right)-\Pr\left(\omega_{1k}\left(q'\right)\right)-\Pr\left(\omega_{2k}\left(q'\right)\right)\\
\Pr\left(\theta_{3k}\left(q'\right)\right) &= \Pr\left(\theta_{3k}\left(q_k\right)\right)+\Pr\left(\omega_{3k}\left(q'\right)\right)+\Pr\left(\omega_{2k}\left(q'\right)\right)
\end{aligned}$$

Let us first state some preliminary results: We will show that in equilibrium for a.e. type s_i for every step k (i) $\Pr\left(\theta_{jk}\left(q_k\right)\right)$ is continuous at q_k and (ii) $\mathbb{E}_{s_{-i}}\left[p|\theta_{1k}\left(q_k\right)\right]$ is continuous at q_k, and hence $\mathbb{E}_{s_{-i}}\left[p;b_k\geq p\geq b_{k+1}\right]$ is continuous at q_k, thus locally differentiable a.e.

Lemma 1 *In equilibrium,*

$$\Pr\left\{s_{-i}:\exists p_L,p_U \textit{ such that } p_L\neq p_U, p_U\leq b_k\left(s_i\right) \textit{ and } \forall p\in\left[p_L,p_U\right]\ S^R\left(p,s_{-i}\right)=\hat{q}\right\}=0$$

for all bidpoints $(b_k,q_k=\hat{q})$ that are submitted with positive probability by type s_i of bidder i, for a.e. s_i and every step k.

Proof. Suppose for contradiction that in equilibrium residual supply can be vertical at q_k with probability π. Recall that p_L is the lowest price such that $S^R(p, s_{-i}) = q_k$ and $p_L < b_k(s_i)$.

Consider a deviation to $q'_k = q_k - \varepsilon$ and $b_k = b_k$. This deviation results in a decrease in the market clearing price to p_L in all states in which the residual supply and i's bid overlapped under the original strategy and thus in an increase in surplus from the inframarginal $q_k - \varepsilon$ units by $(p_U - p_L)$ on every such unit with positive probability π. Deviating bidder is also losing surplus $\mathbb{E}\left[v\left(\hat{q}^{RAT}, s_i\right) - b_k | p^c = b_k\right]$ due to being allocated slightly less $\hat{q}^{RAT} < q^{RAT}$ in the event of possible rationing at b_k due to slightly lower marginal demand at b_k and also potentially not winning units in $(q_k - \varepsilon, q_k)$. Notice that the expected payoff in the event of rationing at b_k is continuous in the demand q_k: the expected gross utility is an integral of the marginal valuation function, which is bounded by Assumption 2, and since the product of q_k and the rationing coefficient is continuous, we get continuity by applying the dominated convergence theorem to $\int I_{u \in [0, q_k R(s_i, s_i)]} v(u, s_i)\, du$ where $R(s_i, s_{-i})$ is the rationing coefficient in the state of the world (Q, s_i, s_{-i}). Therefore the loss of surplus resulting from the lower allocation in the event of rationing can be made arbitrarily small. Because the residual supply can be vertical only at finitely many quantities with positive probability, there must exist an ε small enough, such that the loss of expected surplus from not winning the units in $(q_k - \varepsilon, q_k)$ is also arbitrarily small. On the other hand the lower bound on the expected gain from this deviation is $\pi(q_k - \varepsilon)(p_U - p_L)$. Therefore for small enough ε such a deviation would be strictly profitable. Hence only zero measure of such types of bidder i can have a profitable deviation. ■

For the following lemmas, we will make use of the fact that $\lim_{q' \to q_k} \omega_{jk}(q') = 0\ \forall j, k$ which is a direct corollary to the last lemma.

Lemma 2 *In equilibrium,* $\Pr(\theta_{jk}(q_k))$ *is continuous at* q_k *(*k^{th} *component of* $\vec{q}$*)* $\forall k, j$ *for a.e.* s_i.

Proof. Consider θ_{1k}. Pick $\varepsilon > 0$ and consider $q_k - \delta$. Using notation defined above, $\Pr(\theta_{1k}(q_k - \delta)) = \Pr(\theta_{1k}(q_k)) + \Pr(\omega_{1k}(q_k - \delta)) - \Pr(\omega_{3k}(q_k - \delta))$. Therefore to prove continuity we need to show that $\max\{\Pr(\omega_{1k}(q_k - \delta)), \Pr(\omega_{3k}(q_k - \delta))\} \leq \varepsilon$.

Consider first $\Pr(\omega_{1k}(q_k - \delta))$ and consider a decreasing sequence $\{\delta_n\}$, such that $\lim \delta_n = 0$. By the elementary theorem from probability theory, the limit of the probabilities of the sets along the sequence is equal to probability of the limiting set. The limiting set has zero measure by definition of θ's and by Lemma 1, and hence $\Pr(\omega_{1k}(q_k - \delta_n)) \to 0$ which implies $\exists m : \forall n \geq m :$ $\Pr(\omega_{1k}(q_k - \delta_n)) - 0 \leq \varepsilon$.

Analogous argument implies existence of $m\prime$ such that $\Pr(\omega_{3k}(q_k - \delta_{m'})) \leq \varepsilon$. Choosing $\delta = \min\{\delta_m, \delta_{m'}\}$ concludes the proof since the case $q_k + \delta$ is analogous. A similar argument establishes continuity of $\Pr(\theta_{jk}(q_k))$ for $j \in \{2, 3\}$, and since $\Pr(\theta_{jk}(q_k)) = \Pr(\theta_{jk}(q'))$ for $j \in \{4, 5\}$ and $\forall q' \in (q_{k-1}, q_{k+1})$ continuity is satisfied for these states as well. ■

Lemma 3 *In equilibrium,* $\mathbb{E}_{s_{-i}}[p^c(s_{-i}, q_k) | \theta_{1k}(q_k)]$ *is continuous at* q_k $\forall k$ *for a.e.* s_i.

Proof. By Lemma 2, $\Pr(\theta_{1k}(q_k))$ is continuous in q_k. Recall that the conditional expectation we are interested in is defined as:

$$\mathbb{E}(p^c|b_k > p^c > b_{k+1}, q_k) = \int_{s_{-i}\in\theta_{1k}(q_k)} p^c(s_{-i}, q_k) \frac{dF(s_{-i})}{\Pr(\theta_{1k}(q_k))}$$

where $p^c(s_{-i}, q_k)$ solves: $\sup_p p$ s.t. $q_k \in 1 - \sum_{j\neq i} q_j(s_j, p)$. Let's fix $\varepsilon > 0$. Now we want to show that there is $\delta > 0$, s.t. $\forall q \in B(q_k, \delta) : |\mathbb{E}(p^c|b_k > p^c > b_{k+1}, q_k) - \mathbb{E}(p^c|b_k > p^c > b_{k+1}, q)| \leq \varepsilon$. Perturbing q will have two effects on the conditional expectation: a direct effect through changing $p^c(s_{-i}, q_k)$ and an indirect effect through changing the set $\theta_{1k}(q_k)$. We want to pick δ such that neither of these effects is larger than $\frac{\varepsilon}{2}$.

Consider first the direct effect. The change in the market clearing price for a state s_{-i} can happen only in the case that the residual supply corresponding to this state has at least one vertical piece between q' and q_k, call the set of such states $\eta_1(q', q_k)$. But under the BNE hypothesis the probability measure of a set of states s_{-i} that lead to a vertical residual supply exactly at q_k between prices b_k and b_{k+1} and must be zero by Lemma 1. $\eta_1(q', q_k)$ is therefore continuous by the same argument as in Lemma 2 and in a neighborhood sufficiently close to q_k the probability measure of this set is arbitrarily small. Moreover, since the new market clearing price still has to fall between $b_k(s_i)$ and $b_{k+1}(s_i)$, the induced direct change is bounded by $|b_k(s_i) - b_{k+1}(s_i)|$, and therefore we can pick δ_1 such that:

$$|b_k(s_i) - b_{k+1}(s_i)| \max[\Pr(\eta_1(q_k - \delta_1, q_k)), \Pr(\eta_1(q_k + \delta_1, q_k))] \leq \frac{\varepsilon}{2}$$

Now consider the indirect effect. Changing q_k to q' can result in some states s_{-i} that originally led to market clearing price between $b_k(s_i)$ and $b_{k+1}(s_i)$ to no longer satisfy this restriction. Call the set of such states $\eta_2(q', q_k)$. On the other hand there might be other states s_{-i} which originally did not lead to prices between $b_k(s_i)$ and $b_{k+1}(s_i)$, which now do; call this set $\eta_3(q', q_k)$. Again by the same argument as in Lemma 2, as q' becomes arbitrarily close to q_k the probability measure of either of these sets is arbitrarily close to zero, and it is continuous and limiting to 0 as $\delta \to 0$ on $[q_k - \delta, q_k]$ and on $[q_k + \delta, q_k]$. Since the change in expectation cannot exceed $|b_k(s_i) - b_{k+1}(s_i)|$, we can pick δ_2 and δ_3 such that

$$\begin{aligned}
|b_k(s_i) - b_{k+1}(s_i)| \max[\Pr(\eta_2(q - \delta_2, q_k)), \eta_2(q + \delta_2, q_k)] &\leq \frac{\varepsilon}{4} \\
|b_k(s_i) - b_{k+1}(s_i)| \max[\Pr(\eta_3(q - \delta_3, q_k)), \eta_3(q + \delta_3, q_k)] &\leq \frac{\varepsilon}{4}
\end{aligned}$$

Therefore we can pick $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ concluding the proof. ■

Lemma 4 *In equilibrium,* $\mathbb{E}_{s_{-i}}[p^c(s_{-i}, q_k); \theta_{1k}, \theta_{2k}, \theta_{3k}] = \mathbb{E}_{s_{-i}}[p^c(s_{-i}, q_k); b_k \geq p^c \geq b_{k+1}]$ *is continuous at* q_k $\forall k$ *and thus locally differentiable a.e. for a.e. type* s_i.

Proof. We have:

$$\mathbb{E}_{s_{-i}}\left[p^c(s_{-i},q_k);\cup_{j=1}^3\theta_{jk}(q_k)\right]=\Pr(\theta_{1k}(q_k))\,\mathbb{E}_{s_{-i}}\left[p^c(s_{-i},q_k)\,|\theta_{1k}(q_k)\right]+\Pr(\theta_{2k}(q_k))\,b_k+\Pr(\theta_{3k}(q_k))\,b_{k+1}$$

By Lemma 2, $\Pr(\theta_{jk}(q_k))$ is continuous in q_k and by Lemma 3 $\mathbb{E}_{s_{-i}}\left[p^c(s_{-i},q_k)\,|\theta_{1k}(q_k)\right]$ is also continuous. Therefore the object of interest is a sum and product of continuous functions, and hence is itself continuous. ■

With the preliminaries in hand, we are now ready for the main derivation.

Let us focus on $\Pr\left(\cup_{j=1}^3\theta_{jk}(q_k)\right)\mathbb{E}_{s_{-i}}\left[u(s_i)\,|\,\cup_{j=1}^3\theta_{jk}(q_k)\right]$. First, $\mathbb{E}_{s_{-i}}u(s_i,t_i)$ can be further split into two parts: (i) the expected gross utility $\mathbb{E}_{s_{-i}}V(y(s_{-i},q_k),s_i)$ where $y(s_{-i},q_k)$ is either q_k in case of a state in θ_{1k}, the rationed quantity $q^{RAT}(s_{-i},q_k-q_{k-1})$ in case of a state in θ_{2k}, or $q^{RAT}(s_{-i},q_{k+1}-q_k)$ in case of a state in θ_{3k}; and (ii) the expected payment $\mathbb{E}_{s_{-i}}\left[y(s_{-i},q_k)\,p^c(s_{-i},q_k)\right]$ where both $y(s_{-i},q_k)$ and $p^c(s_{-i},q_k)$ depend on the state: e.g., $y(q_k)=q_k$ in θ_{1k}, but $p^c(s_{-i},q_k)$ is random, in θ_{2k} on the other hand $p^c(s_{-i},q_k)=b_k$, but $y(s_{-i},q_k)$ is random due to rationing and similarly for θ_{3k}. Recall that

$$\begin{aligned}
\Pr\left(\theta_{1k}(q')\right) &= \Pr(\theta_{1k}(q_k))+\Pr\left(\omega_{1k}(q')\right)-\Pr\left(\omega_{3k}(q')\right)\\
\Pr\left(\theta_{2k}(q')\right) &= \Pr(\theta_{2k}(q_k))-\Pr\left(\omega_{1k}(q')\right)-\Pr\left(\omega_{2k}(q')\right)\\
\Pr\left(\theta_{3k}(q')\right) &= \Pr(\theta_{3k}(q_k))+\Pr\left(\omega_{3k}(q')\right)+\Pr\left(\omega_{2k}(q')\right)
\end{aligned}$$

The difficulty we are facing is that $y(s_{-i},q_k)$ and $p^c(s_{-i},q_k)$ are not continuous over the cells of our partition - in particular they are different functions at each cell, and hence the usual Leibnitz rule fails. To illustrate this, consider Figure 6. $y(s_{-i},q_k)$ and $p^c(s_{-i},q_k)$ are the same functions on A and A' evaluated at q_k and q' respectively (for example if the set A is our θ_{1k}, then $y(\cdot,x)=x$). But in states falling to set C under q', these functions would be different under q_k. We can, however, always extend the same continuous function f that we are integrating on cell A under q_k onto the cell A under q' and add to it the integral of the same function on cell C under q'. Similarly we can assume that the same function f that we are integrating on B under q_k will hold on B under q' and then subtract the integral of the same function on set C under q'.

[Figure 6 about here.]

Let's consider first the effect that a perturbation in q_k would have on the expected gross utility. Deriving it indirectly using the limit:

$$
\begin{aligned}
&\lim_{q' \to q_k} \frac{\mathbb{E}_{s_{-i}} V\left(y\left(s_{-i}, q'\right), s_i\right) - \mathbb{E}_{s_{-i}} V\left(y\left(s_{-i}, q_k\right), s_i\right)}{q' - q_k} \\
&= \lim_{q' \to q_k} \frac{\sum_{j=1}^{3} \left[\mathbb{E}_{s_{-i}}\left[V\left(y\left(s_{-i}, q'\right), s_i\right); \theta_{jk}\left(q'\right)\right] - \mathbb{E}_{s_{-i}}\left[V\left(y\left(s_{-i}, q_k\right), s_i\right); \theta_{jk}\left(q_k\right)\right]\right]}{q' - q_k} \\
&= \lim_{q' \to q_k} \frac{\mathbb{E}_{s_{-i}}\left[V\left(q', s_i\right) - V\left(q_k, s_i\right); \theta_{1k}\left(q_k\right)\right] + \left[\Pr\left(\omega_{1k}\right) - \Pr\left(\omega_{3k}\right)\right] V\left(q', s_i\right)}{q' - q_k} \\
&+ \lim_{q' \to q_k} \frac{\left[\begin{array}{c} \mathbb{E}_{s_{-i}}\left[V\left(q^{RAT}\left(q' - q_{k-1}, s_{-i}\right), s_i\right) - V\left(q^{RAT}\left(q_k - q_{k-1}, s_{-i}\right), s_i\right); \theta_{2k}\left(q_k\right)\right] - \\ \mathbb{E}_{s_{-i}}\left[V\left(q^{RAT}\left(q' - q_{k-1}, s_{-i}\right), s_i\right); \omega_{1k}\right] - \mathbb{E}_{s_{-i}}\left[V\left(q^{RAT}\left(q' - q_{k-1}, s_{-i}\right), s_i\right); \omega_{2k}\right] \end{array}\right]}{q' - q_k} \\
&+ \lim_{q' \to q_k} \frac{\left[\begin{array}{c} \mathbb{E}_{s_{-i}}\left[V\left(q^{RAT}\left(q_{k+1} - q', s_{-i}\right), s_i\right) - V\left(q^{RAT}\left(q_{k+1} - q_k, s_{-i}\right), s_i\right); \theta_{3k}\left(q_k\right)\right] + \\ \mathbb{E}_{s_{-i}}\left[V\left(q^{RAT}\left(q_{k+1} - q', s_{-i}\right), s_i\right); \omega_{3k}\right] + \mathbb{E}_{s_{-i}}\left[V\left(q^{RAT}\left(q_{k+1} - q', s_{-i}\right), s_i\right); \omega_{2k}\right] \end{array}\right]}{q' - q_k}
\end{aligned}
$$

where the first equality follows by the law of total probability and the fact that on θ_{4k} and θ_{5k} perturbing k^{th}-step q_k to q' does not alter the gross utility and also not their respective probabilities. The second equality results after plugging in the conditional gross utility before and after the perturbation using the approach described above - extending the continuous functions to the partition cells under q_k and collecting terms.

Now invoking the definition of the derivative and noting that $\lim_{q' \to q_k}\left[q^{RAT}(.)\,|\omega_{jk}\right] = q_k$ and $\lim_{q' \to q_k} \Pr\left(\omega_{jk}\left(q', q\right)\right) = 0$ and hence after applying l'Hospital's rule all terms involving ω_{jk} vanish in the limit, we can simplify the last expression above to:

$$
\begin{aligned}
&\Pr\left(\theta_{1k}\left(q_k\right)\right) v\left(q_k, s_i\right) + \\
&+ \mathbb{E}_{s_{-i}}\left[v\left(q^{RAT}\left(s_{-i}, q_k - q_{k-1}\right), s_i\right) \frac{\partial q^{RAT}\left(s_{-i}, q_k - q_{k-1}\right)}{\partial q_k}; \theta_{2k}\left(q_k\right)\right] + \\
&+ \mathbb{E}_{s_{-i}}\left[v\left(q^{RAT}\left(s_{-i}, q_{k+1} - q_k\right), s_i\right) \frac{\partial q^{RAT}\left(s_{-i}, q_{k+1} - q_k\right)}{\partial q_k}; \theta_{3k}\left(q_k\right)\right]
\end{aligned}
$$

Now let us move to the key step in the proof - the effect of the perturbation in q_k on the expected payment. Again using the limit derivation:

$$
\begin{aligned}
& \lim_{q' \to q_k} \frac{\mathbb{E}_{s_{-i}}\left[y\left(s_{-i}, q'\right) p^c\left(s_{-i}, q'\right); \cup_{j=1}^3 \theta_{jk}\left(q'\right)\right] - \mathbb{E}_{s_{-i}}\left[y\left(s_{-i}, q_k\right) p^c\left(s_{-i}, q_k\right); \cup_{j=1}^3 \theta_{jk}\left(q_k\right)\right]}{q' - q_k} \\
= & \lim_{q' \to q_k} \frac{\sum_{j=1}^3 \left[\mathbb{E}_{s_{-i}}\left[y\left(s_{-i}, q'\right) p^c\left(s_{-i}, q'\right); \theta_{jk}\left(q'\right)\right] - \mathbb{E}_{s_{-i}}\left[y\left(s_{-i}, q_k\right) p^c\left(s_{-i}, q_k\right); \theta_{jk}\left(q_k\right)\right]\right]}{q' - q_k} \\
= & \lim_{q' \to q_k} \frac{\left[\begin{array}{c} \mathbb{E}_{s_{-i}}\left[q' p^c\left(s_{-i}, q'\right); \theta_{1k}\left(q_k\right)\right] + \mathbb{E}_{s_{-i}}\left[q' p^c\left(s_{-i}, q'\right); \omega_{1k}\right] \\ -\mathbb{E}_{s_{-i}}\left[q' p^c\left(s_{-i}, q'\right); \omega_{3k}\right] - \mathbb{E}_{s_{-i}}\left[q_k p^c\left(s_{-i}, q_k\right); \theta_{1k}\left(q_k\right)\right] \end{array}\right]}{q' - q_k} + \\
& + \lim_{q' \to q_k} \frac{\left[\begin{array}{c} \mathbb{E}_{s_{-i}}\left[q^{RAT}\left(q' - q_{k-1}, s_{-i}\right) b_k; \theta_{2k}\left(q_k\right)\right] - \mathbb{E}_{s_{-i}}\left[q^{RAT}\left(q' - q_{k-1}, s_{-i}\right) b_k; \omega_{1k}\right] \\ -\mathbb{E}_{s_{-i}}\left[q^{RAT}\left(q' - q_{k-1}, s_{-i}\right) b_k; \omega_{2k}\right] - \mathbb{E}_{s_{-i}}\left[q^{RAT}\left(q_k - q_{k-1}, s_{-i}\right) b_k; \theta_{2k}\left(q_k\right)\right] \end{array}\right]}{q' - q_k} + \\
& + \lim_{q' \to q_k} \frac{\left[\begin{array}{c} \mathbb{E}_{s_{-i}}\left[q^{RAT}\left(q_{k+1} - q', s_{-i}\right) b_{k+1}; \theta_{1k}\left(q_k\right)\right] + \mathbb{E}_{s_{-i}}\left[q^{RAT}\left(q_{k+1} - q', s_{-i}\right) b_{k+1}; \omega_{3k}\right] \\ +\mathbb{E}_{s_{-i}}\left[q^{RAT}\left(q_{k+1} - q', s_{-i}\right) b_{k+1}; \omega_{2k}\right] - \mathbb{E}_{s_{-i}}\left[q^{RAT}\left(q_{k+1} - q_k, s_{-i}\right) b_{k+1}; \theta_{3k}\left(q_k\right)\right] \end{array}\right]}{q' - q_k}
\end{aligned}
$$

where the second equality follows by the law of total probability after substituting in for the probabilities of the different partition cells after perturbation and extending (or reducing) the functions to the old partition cells as described earlier.

By adding and subtracting $\mathbb{E}_{s_{-i}}\left[q_k p^c\left(s_{-i}, q'\right); \theta_{1k}\left(q_k\right)\right]$, collecting terms and using the definition of a derivative, we can rewrite the last expression as:

$$
\begin{aligned}
& \lim_{q' \to q_k} \frac{\mathbb{E}_{s_{-i}}\left[q' p^c\left(s_{-i}, q'\right); \theta_{1k}\left(q_k\right)\right] - \mathbb{E}_{s_{-i}}\left[q_k p^c\left(s_{-i}, q'\right); \theta_{1k}\left(q_k\right)\right]}{q' - q_k} + \\
& + \lim_{q' \to q_k} \frac{\left[\begin{array}{c} \mathbb{E}_{s_{-i}}\left[q_k p^c\left(s_{-i}, q'\right); \theta_{1k}\left(q_k\right)\right] + \mathbb{E}_{s_{-i}}\left[q' p^c\left(s_{-i}, q'\right); \omega_{1k}\right] \\ -\mathbb{E}_{s_{-i}}\left[q' p^c\left(s_{-i}, q'\right); \omega_{3k}\right] - \mathbb{E}_{s_{-i}}\left[q_k p^c\left(s_{-i}, q_k\right); \theta_{1k}\left(q_k\right)\right] \end{array}\right]}{q' - q_k} + \\
& + b_k \mathbb{E}_{s_{-i}}\left[\frac{\partial q^{RAT}\left(s_{-i}, q_k - q_{k-1}\right)}{\partial q_k}; \theta_{2k}\left(q_k\right)\right] + b_{k+1} \mathbb{E}_{s_{-i}}\left[\frac{\partial q^{RAT}\left(s_{-i}, q_{k+1} - q_k\right)}{\partial q_k}; \theta_{3k}\left(q_k\right)\right] + \\
& + \lim_{q' \to q_k} \frac{b_{k+1} \mathbb{E}_{s_{-i}}\left[q^{RAT}\left(q_{k+1} - q', s_{-i}\right); \omega_{3k} \cup \omega_{2k}\right] - b_k \mathbb{E}_{s_{-i}}\left[q^{RAT}\left(q' - q_{k-1}, s_{-i}\right); \omega_{1k} \cup \omega_{2k}\right]}{q' - q_k} \\
= & \mathbb{E}_{s_{-i}}\left[p^c\left(s_{-i}, q_k\right); \theta_{1k}\left(q_k\right)\right] + \\
& + b_k \mathbb{E}_{s_{-i}}\left[\frac{\partial q^{RAT}\left(s_{-i}, q_k - q_{k-1}\right)}{\partial q_k}; \theta_{2k}\left(q_k\right)\right] + b_{k+1} \mathbb{E}_{s_{-i}}\left[\frac{\partial q^{RAT}\left(s_{-i}, q_{k+1} - q_k\right)}{\partial q_k}; \theta_{3k}\left(q_k\right)\right] + \\
& + \lim_{q' \to q_k} \frac{\left[\begin{array}{c} q_k \left[\mathbb{E}_{s_{-i}}\left[p^c\left(s_{-i}, q'\right); \theta_{1k}\left(q_k\right)\right] + \mathbb{E}_{s_{-i}}\left[p^c\left(s_{-i}, q'\right); \theta_{2k}\left(q_k\right)\right] + \mathbb{E}_{s_{-i}}\left[p^c\left(s_{-i}, q'\right); \theta_{3k}\left(q_k\right)\right]\right] \\ -q_k \left[\mathbb{E}_{s_{-i}}\left[p^c\left(s_{-i}, q_k\right); \theta_{1k}\left(q_k\right)\right] + \mathbb{E}_{s_{-i}}\left[p^c\left(s_{-i}, q_k\right); \theta_{2k}\left(q_k\right)\right] + \mathbb{E}_{s_{-i}}\left[p^c\left(s_{-i}, q_k\right); \theta_{3k}\left(q_k\right)\right]\right] \end{array}\right]}{q' - q_k} + \\
& + \lim_{q' \to q_k} \frac{\begin{array}{c} \mathbb{E}_{s_{-i}}\left[q' p^c\left(s_{-i}, q'\right); \omega_{1k}\right] - \mathbb{E}_{s_{-i}}\left[q' p^c\left(s_{-i}, q'\right); \omega_{3k}\right] \\ -q_k \mathbb{E}_{s_{-i}}\left[p^c\left(s_{-i}, q'\right); \omega_1\right] + q_k \mathbb{E}_{s_{-i}}\left[p^c\left(s_{-i}, q'\right); \omega_{3k}\right] \end{array}}{q' - q_k} \\
= & \mathbb{E}_{s_{-i}}\left[p^c\left(s_{-i}, q_k\right); \theta_{1k}\left(q_k\right)\right] + q_k \frac{\partial \mathbb{E}_{s_{-i}}\left[p^c\left(s_{-i}, q'\right); \cup_{j=1}^3 \theta_{jk}\left(q_k\right)\right]}{\partial q_k}
\end{aligned}
$$

where the first equality is the key step: (i) first term is obtained by simplification; and (ii) we add and subtract terms to complete the function $q_k p^c\left(s_{-i}, q'\right)$ to full $\cup_{j=1}^3 \theta_{jk}$. In doing that we make use of the following facts:

$$\begin{aligned}
&\mathbb{E}_{s_{-i}}\left[p^c\left(s_{-i},q'\right);\theta_{2k}\left(q_k\right)\right]-\mathbb{E}_{s_{-i}}\left[p^c\left(s_{-i},q'\right);\theta_{2k}\left(q_k'\right)\right]\\
&-\mathbb{E}_{s_{-i}}\left[p^c\left(s_{-i},q'\right);\omega_{1k}\right]-\mathbb{E}_{s_{-i}}\left[p^c\left(s_{-i},q'\right);\omega_{2k}\right]=0\\
&\mathbb{E}_{s_{-i}}\left[p^c\left(s_{-i},q'\right);\theta_{3k}\left(q_k\right)\right]-\mathbb{E}_{s_{-i}}\left[p^c\left(s_{-i},q'\right);\theta_{3k}\left(q_k'\right)\right]\\
&+\mathbb{E}_{s_{-i}}\left[p^c\left(s_{-i},q'\right);\omega_{3k}\right]+\mathbb{E}_{s_{-i}}\left[p^c\left(s_{-i},q'\right);\omega_{2k}\right]=0
\end{aligned}$$

Therefore we can multiply all terms by q_k and add them to our limit. Final expression following the first equality obtains by rearranging terms. Finally the last equality then follows by definition of the derivative and because

$$\begin{aligned}
&\lim_{q'\to q_k}\frac{\mathbb{E}_{s_{-i}}\left[q'p^c\left(s_{-i},q'\right);\omega_{1k}\right]-\mathbb{E}_{s_{-i}}\left[q'p^c\left(s_{-i},q'\right);\omega_{3k}\right]-q_k\mathbb{E}_{s_{-i}}\left[p^c\left(s_{-i},q'\right);\omega_{1}\right]+q_k\mathbb{E}_{s_{-i}}\left[p^c\left(s_{-i},q'\right);\omega_{3k}\right]}{q'-q_k}\\
&=\lim_{q'\to q_k}\frac{\partial\Pr\left(\omega_{1k}\right)}{\partial q'}\left[q'\mathbb{E}_{s_{-i}}\left[p^c\left(s_{-i},q'\right)|\omega_{1k}\right]-q_k\mathbb{E}_{s_{-i}}\left[p^c\left(s_{-i},q'\right)|\omega_{1k}\right]\right]\\
&+\lim_{q'\to q_k}\frac{\partial\Pr\left(\omega_{3k}\right)}{\partial q'}\left[q'\mathbb{E}_{s_{-i}}\left[p^c\left(s_{-i},q'\right)|\omega_{3k}\right]-q_k\mathbb{E}_{s_{-i}}\left[p^c\left(s_{-i},q'\right)|\omega_{3k}\right]\right]\\
&+\lim_{q'\to q_k}\left[\Pr\left(\omega_{1k}\right)K_1+\Pr\left(\omega_{3k}\right)K_2\right]\\
&=0
\end{aligned}$$

where the first equality follows after first splitting the expectations, which can be done because q' is constant on ω_{jk}.

$$\mathbb{E}_{s_{-i}}\left[q'p^c\left(s_{-i},q'\right);\omega_{jk}\right]=q'\Pr\left(\omega_{jk}\right)\mathbb{E}_{s_{-i}}\left(p^c\left(s_{-i},q'\right)|\omega_{jk}\right)$$

and applying l'Hospital's rule (note that $\Pr\left(\omega_{jk}\right)$ is a function of q'). Finally as we noted earlier $\lim_{q'\to q_k}\mathbb{E}_{s_{-i}}\left[q'|\omega_{jk}\right]=q_k$ and $\lim_{q'\to q_k}\Pr\left(\omega_{jk}\right)=0$, and since both K_1 and K_2 are bounded ($\frac{\partial\Pr\left(\omega_{jk}\right)}{\partial q'}$ is also bounded since roughly speaking this is just an integral of some density of s_{-i} which is bounded by assumption), all terms vanish in the limit.

The last step is to note that the event $\{s_{-i}\in\theta_{1k}\}$ is equivalent to the event $\{b_k>p^c>b_{k+1}\}$, and that whenever the market clearing price is b_k or b_{k+1} the allocation is responsive to s_i's demand only when there is a tie (i.e., at least one other bidder being rationed). After collecting terms our optimality condition becomes:

$$\begin{aligned}
&\Pr\left[b_k > p^c > b_{k+1}\right]\left[v\left(q_k, s_i\right) - \mathbb{E}_{s_{-i}}\left(p^c\left(s_{-i}, q_k\right)|b_k > p^c > b_{k+1}\right)\right] \\
&+\Pr\left[b_k = p^c \ \wedge \ Tie\right]\mathbb{E}_{s_{-i}}\left[\left(v\left(q^{RAT}, s_i\right) - b_k\right)\frac{\partial q^{RAT}}{\partial q_k}|b_k = p^c \ \wedge \ Tie\right] \\
&+\Pr\left[b_{k+1} = p^c \ \wedge \ Tie\right]\mathbb{E}_{s_{-i}}\left[\left(v\left(q^{RAT}, s_i\right) - b_{k+1}\right)\frac{\partial q^{RAT}}{\partial q_k}|b_{k+1} = p^c \ \wedge \ Tie\right] \\
= \quad & q_k \frac{\partial \mathbb{E}_{s_{-i}}\left(p^c\left(s_{-i}, q_k\right); b_k \geq p^c \geq b_{k+1}\right)}{\partial q_k}
\end{aligned}$$

which is equation (2).

For completeness, we can also state the necessary conditions governing the choice of the bid at step k, b_k. However, we cannot guarantee that these necessary conditions hold as equalities in equilibria that involve ties. Notice that expected payment can be written as

$$\begin{aligned}
&\mathbb{E}_{s_{-i}}\left[p^c\left(s_{-i}\right)q^c\left(s_{-i}\right)\right] = \\
&\quad = \Pr\left(b_k < p < b_{k-1}\right)q_{k-1}\mathbb{E}_{s_{-i}}\left[p^c\left(s_{-i}\right)|b_k < p < b_{k-1}\right] + \\
&\quad\quad \Pr\left(p = b_k\right)b_k\mathbb{E}_{s_{-i}}\left[q^c\left(s_{-i}\right)|p = b_k\right] + \\
&\quad\quad \Pr\left(b_{k+1} < p < b_k\right)q_k\mathbb{E}_{s_{-i}}\left[p^c\left(s_{-i}\right)|b_{k+1} < p < b_k\right] + \\
&\quad\quad \Pr\left(p \leq b_{k+1} \cup p \geq b_k\right)\mathbb{E}_{s_{-i}}\left[p^c\left(s_{-i}\right)q\left(s_{-i}\right)|p \leq b_{k+1} \cup p \geq b_k\right] \\
&\quad = q_{k-1}\mathbb{E}_{s_{-i}}\left[p^c\left(s_{-i}\right); b_k < p < b_{k-1}\right] + q_k\mathbb{E}_{s_{-i}}\left[p^c\left(s_{-i}\right); b_{k+1} < p < b_k\right] \\
&\quad\quad + b_k\mathbb{E}_{s_{-i}}\left[q^c\left(s_{-i}\right); p = b_k\right] + \mathbb{E}_{s_{-i}}\left[p^c\left(s_{-i}\right)q^c\left(s_{-i}\right); p \leq b_{k+1} \cup p \geq b_{k-1}\right]
\end{aligned}$$

where the last term does not depend on b_k. Taking the derivative w.r.t. b_k delivers

$$\begin{aligned}
&q_{k-1}\frac{\partial \mathbb{E}_{s_{-i}}\left[p^c\left(s_{-i}\right); b_k < p < b_{k-1}\right]}{\partial b_k} + q_k\frac{\partial \mathbb{E}_{s_{-i}}\left[p^c\left(s_{-i}\right); b_{k+1} < p < b_k\right]}{\partial b_k} + \\
&+\mathbb{E}_{s_{-i}}\left[q\left(s_{-i}\right); p = b_k\right] + b_k\frac{\partial \mathbb{E}_{s_{-i}}\left[q\left(s_{-i}\right); p = b_k\right]}{\partial b_k}
\end{aligned} \tag{A-2}$$

Notice that doing the same simple exercise w.r.t. q_k would not lead directly to our FOC, since the heart of the argument of the proof above involves combining the terms $b_k\frac{\partial \mathbb{E}_{s_{-i}}(q(s_{-i});p=b_k)}{\partial q_k}$, $b_{k+1}\frac{\partial \mathbb{E}_{s_{-i}}(q(s_{-i});p=b_{k+1})}{\partial q_k}$ and $q_k\frac{\partial \mathbb{E}_{s_{-i}}(p^c(s_{-i});b_{k+1}<p<b_k)}{\partial q_k}$ into one term: $q_k\frac{\partial \mathbb{E}_{s_{-i}}(p^c(s_{-i});b_{k+1}\leq p\leq b_k)}{\partial q_k}$. Combining the derivative of the expected payment w.r.t. b_k given by (A-2) with the derivative of the gross utility yields:

$$\frac{\partial \mathbb{E}_{Q,s_{-i}|s_i}\left[V\left(q_i^c\left(Q,\mathbf{s},\mathbf{y}\left(\cdot|\mathbf{s}\right)\right),s_i\right);b_{k-1}>p^c>b_{k+1}\right]}{\partial b_k}= \tag{A-3}$$

$$=\mathbb{E}_{Q,s_{-i}|s_i}\left(q_i^c\left(Q,\mathbf{s},\mathbf{y}\left(\cdot|\mathbf{s}\right)\right);p^c=b_k\right)+b_k\frac{\partial \mathbb{E}_{Q,s_{-i}|s_i}\left(q_i^c\left(Q,\mathbf{s},\mathbf{y}\left(\cdot|\mathbf{s}\right)\right);p^c=b_k\right)}{\partial b_k}$$

$$+q_k\frac{\partial \mathbb{E}_{Q,s_{-i}|s_i}\left(p^c;b_k>p^c>b_{k+1}\right)}{\partial b_k}+q_{k-1}\frac{\partial \mathbb{E}_{Q,s_{-i}|s_i}\left(p^c;b_{k-1}>p^c>b_k\right)}{\partial b_k}$$

Also notice that by similar arguments as in Lemmas 1 and 2 we can establish continuity and local differentiability a.e. of all expectations involved in (A-3) with respect to the bid b_k for types that do not face a positive probability of tying. QED

A.2 Proof of Proposition 3

Proposition 3 is a corollary of Hortaçsu's (2002) Proposition 1 (Part 1).

B Appendix

In this appendix, I discuss how to point identify the marginal valuations at those quantities at which bids were submitted taking into account that ties can have positive probability at bids that are above the marginal value. I will impose the following additional assumption on marginal valuation function that will allow me to obtain identification:

Assumption 7 $\mathbb{E}\left(v\left(q_i^c\left(Q,\mathbf{s},\mathbf{y}\left(\cdot|\mathbf{s}\right)\right),s_i\right)|p^c=b_k\ \wedge\ Tie\right)=v\left(q_k,s_i\right)$

This assumption ensures that the marginal valuation function has sufficiently long flat segments at each step, so that the marginal value of the last unit allocated in the event of rationing at k^{th} step is the same as the marginal value of the last unit demanded at that step. Inspecting (2), two effects are at play when deciding on a bid. First, the bidder would like to equalize his surplus on the last unit he demands weighted by the probability he wins exactly that many units, with the effect of demanding this last unit on the market clearing price, and thus on his total payment. The second effect, which occurs in the event of a tie, forces the bidder to set his bid equal to the marginal value for the unit he expects to win after rationing, because if he is rationed, changing his demand does not have any effect on the market clearing price in those states. How important this second effect is relative to the first one depends on the ratio of the probability of a tie at this step, i.e., probability of multiple bidders submitting a bid at that price and that price actually clearing the market, to the probability of being allocated all units he demands at that step. Let λ_{k1} denote this ratio at k^{th} step, i.e., and $\lambda_{k1}=\frac{\Pr(p^c=b_k\ \wedge\ Tie)}{\Pr(b_k>p^c>b_{k+1})}$ and λ_{k2} at the subsequent step, $\lambda_{k2}=\frac{\Pr(p^c=b_{k+1}\ \wedge\ Tie)}{\Pr(b_k>p^c>b_{k+1})}$. Under Assumption 7, equation (2) can again be inverted to obtain an estimate of the marginal valuation at the last step $v\left(q_K,s_i\right)$ as follows:

$$v(q_K, s_i) = \frac{\mathbb{E}(p; b_K > p^c) + \Pr(b_K = p^c \wedge Tie)\, b_K \mathbb{E}\left(\frac{\partial q_i^c(Q,\mathbf{s},\mathbf{y}(\cdot|\mathbf{s}))}{\partial q_K} | b_K = p^c \wedge Tie\right) + q_K \frac{\partial \mathbb{E}(p^c; b_K \geq p^c)}{\partial q_K}}{\Pr(b_K > p^c) + \Pr(b_K = p^c \wedge Tie)\, \mathbb{E}\left(\frac{\partial q_i^c(Q,\mathbf{s},\mathbf{y}(\cdot|\mathbf{s}))}{\partial q_K} | b_K = p^c \wedge Tie\right)} \tag{B-1}$$

Thus, as before we need to estimate $\mathbb{E}(p^c | b_k > p^c > b_{k+1})$ and the derivative $\frac{\partial \mathbb{E}(p^c; b_k \geq p^c \geq b_{k+1})}{\partial q_k}$. Moreover, we need to estimate $\Pr(b_K = p^c \wedge Tie)$ and the expected rationing coefficient, $\mathbb{E}\left(\frac{\partial q_i^c(Q,\mathbf{s},\mathbf{y}(\cdot|\mathbf{s}))}{\partial q_K} | b_K = p^c \wedge Tie\right)$, both of which can be estimated in a similar way using the resampled residual supplies. Since at all other steps than the last one, the demand q_k potentially has an effect on the allocation at the subsequent $k + 1^{st}$ step (in the event of rationing at b_{k+1}), the marginal valuations at quantities at other steps can be obtained recursively using the following relationship obtained by rearranging (2).

$$v(q_k, s_i) = \frac{\begin{array}{c}\mathbb{E}(p; b_k > p^c > b_{k+1}) + \Pr(b_k = p^c \wedge Tie)\, b_k \mathbb{E}\left(\frac{\partial q_i^c(Q,\mathbf{s},\mathbf{y}(\cdot|\mathbf{s}))}{\partial q_k} | b_k = p^c \wedge Tie\right) + q_k \frac{\partial \mathbb{E}\left(p^c; b_k \geq p^c \geq b_{k+1}\right)}{\partial q_k} \\ - \Pr(b_{k+1} = p^c \wedge Tie)\left[v(q_{k+1}, s_i) - b_{k+1}\right] \mathbb{E}\left(\frac{\partial q_i^c(Q,\mathbf{s},\mathbf{y}(\cdot|\mathbf{s}))}{\partial q_k} | b_{k+1} = p^c \wedge Tie\right)\end{array}}{\Pr(b_k > p^c > b_{k+1}) + \Pr(b_k = p^c \wedge Tie)\, \mathbb{E}\left(\frac{\partial q_i^c(Q,\mathbf{s},\mathbf{y}(\cdot|\mathbf{s}))}{\partial q_k} | b_k = p^c \wedge Tie\right)} \tag{B-2}$$

Notice also that the two terms in (B-2) involving ties also include the rationing coefficient $\frac{\partial q_i^c(Q,\mathbf{s},\mathbf{y}(\cdot|\mathbf{s}))}{\partial q_K}$ and that the effect of demand at step k on the quantity allocated in case of rationing is positive for rationing at b_k and negative for rationing at b_{k+1} as an increase in q_k decreases the marginal demand at step $k + 1$, and thus these two terms are likely to act in the opposite direction, thus reducing the bias involved if these two terms were ignored in the estimation.

Results when allowing for ties

Using the resampling procedure I estimated the likelihood of ties relative to obtaining full demand at a given step, $\lambda_k = \frac{\Pr(b_k = p^c \wedge Tie)}{\Pr(b_k > p^c > b_{k+1})}$, and the expected rationing coefficient, $\mathbb{E}\left[\frac{\partial q_i^c(Q,\mathbf{s},\mathbf{y}(\cdot|\mathbf{s}))}{\partial q_K} | b_k = p^c \wedge Tie\right]$. Since in the data, the average product of the estimates of these two terms is 0.21 (average $\hat{\lambda}$ being 0.34), the terms involving ties in (2) are close to zero irrespective of $\mathbb{E}(v(q^c, s_i) | p = b_k, Tie)$.[38] Because of the small magnitude of $\lambda_k \mathbb{E}\left[\frac{\partial q_i^c(Q,\mathbf{s},\mathbf{y}(\cdot|\mathbf{s}))}{\partial q_K} | b_k = p^c \wedge Tie\right]$ mentioned above, the difference $[\mathbb{E}(v(q^c, s_i) | p^c = b_k, Tie) - b_k]$ would have to be very large in order to have a significant effect on the estimate. I estimated the model assuming that marginal valuation functions are step functions and allowing for ties. I obtained estimates of marginal valuations for each bidder recursively starting with the last submitted step as described above. The results are qualitatively very similar to the ones reported when ties are ignored - the estimated valuations are somewhat larger. However, due to the fact that λ is a ratio of two probabilities, the standard errors of the estimates are rather big. In few cases, the estimates are also too big (for example, exceeding the face value of a T-bill), hence some trimming needs to be implemented in order to make the results comparable.[39]

[38] Notice that $\overline{\lambda} = 0.21$ means that most of the weight is on the states involving no ties.

[39] For example, the outliers can have a huge impact on the efficiency calculations.

Table 1: Data Summary

	Mean	Min	Max	StDev
Active Bidders in an Auction[a]	13	10	16	1.4
Number of Submitted Bidpoints	2.3	1	9	1.55
Price Bids (in CZK)[b]	987,039	986,203	987,776	236.8
Annual yields corresponding to price bids	5.30	4.99[c]	5.65	0.10
Quantity Bids[d]	0.091	0.0001	0.5	0.11
Unannounced Noncompetitive Bid[e]	0.0024	0	0.036	0.008
Market Clearing Price	986,994	986,463	987,173	174.71
Annual yields corresponding to mkt. cl. price	5.32	5.22	5.54	0.08
Reference interest rate	5.39	5.32	5.74	0.10
Auction Revenue (in mil USD)	210	78.9	263.2	62.6

[a] Active bidder is any bidder actually submitting a serious (nonzero) bid.
[b] 1USD is approximately 38CZK over the sample
[c] Lowest yield corresponds to highest bid
[d] As a share of total quantity offered for sale, across all steps
[e] Unexpected supply withdrawal as a share of total quantity offered for sale

Table 2: Data Summary - Large vs Small Bidders

	Large	Small
Active Bidders in an Auction	7.5	5.5
	(0.84)	(0.92)
Number of Submitted Bidpoints	2.95	1.41
	(1.67)	(0.68)
Price Bids[a] (in CZK)	987,041	987,031
	(237.3)	(235.7)
Quantity Bids[a,b]	0.11	0.026
	(0.12)	(0.01)

[a] Average taken across all bidpoints.
[b] As a share of total quantity offered for sale.
[c] Means reported, standard deviations in parentheses

Table 3: Comparison with truthful bidding - part 1

Auction	Actual p	TruthBidMin1[a]	TruthBidMax1[b]	TruthBidMin2[c]	TruthBidMax2[d]
52	986,463	986,451	986,510	986,439	986,512
		(24.10)	(48.51)	(1.70)	(3.18)
55	986,770	986,723	986,815	986,729	986,816
		(14.00)	(36.16)	(1.17)	(44.81)
56	986,723	986,663	986,788	986,677	986,787
		(23.73)	(15.94)	(3.25)	(3.88)
60	987,054	986,891	987,078	986,892	987,101
		(45.16)	(25.00)	(38.84)	(26.34)
61	987,173	987,122	987,173	987,123	987,169
		(11.90)	(28.66)	(12.50)	(34.87)
64	987,078	987,078	987,093	987,054	987,091
		(11.44)	(8.41)	(12.28)	(8.21)
65	987,054	987,056	987,078	987,057	987,083
		(8.02)	(8.01)	(7.51)	(7.38)
67*	987,149	987,094	987,106	987,098	987,102
		(11.69)	(15.48)	(11.87)	(16.20)
69	987,078	987,079	987,121	987,078	987,122
		(12.23)	(8.23)	(11.57)	(9.30)
72	987,102	987,095	987,152	987,096	987,152
		(5.87)	(5.26)	(6.50)	(3.87)
73	987,149	987,121	987,151	987,121	987,158
		(11.15)	(11.19)	(12.00)	(9.75)
75*	987,149	987,108	987,108	987,108	987,108
		(5.95)	(5.13)	(6.51)	(5.34)
76	987,149	987,103	987,157	987,103	987,154
		(7.99)	(9.28)	(7.90)	(9.35)
81	987,102	987,068	987,107	987,067	987,118
		(10.75)	(13.14)	(11.14)	(11.84)
82	987,054	987,051	987,080	987,079	987,083
		(14.99)	(4.11)	(12.77)	(3.51)
85	987,102	987,083	987,103	987,078	987,103
		(7.64)	(7.11)	(7.78)	(6.58)
Mean:[e]	986,994	986,958.71	986,997.11	986,958.64	986,998.54

* Ex post revenue higher than under truthful bidding
[a] Market clearing price when bidding the lower envelope of marginal valuations (Model 1)
[b] Market clearing price when bidding the upper envelope of marginal valuations (Model 1)
[c] Market clearing price when bidding the lower envelope of marginal valuations (Model 2)
[d] Market clearing price when bidding the upper envelope of marginal valuations (Model 2)
[e] Mean across all 28 auctions (52-108).
[f] Bootstrap std. errors in parentheses

Table 4: Comparison with truthful bidding - part 2

Auction	Actual p	TruthBidMin1[a]	TruthBidMax1[b]	TruthBidMin2[c]	TruthBidMax2[d]
86*	987,149	987,101	987,107	987,103	987,107
		(10.44)	(11.18)	(9.98)	(10.58)
87	987,078	987,063	987,063	987,078	987,078
		(11.49)	(11.80)	(10.51)	(11.56)
91	987,078	987,072	987,107	987,073	987,110
		(3.29)	(7.21)	(3.24)	(6.40)
92	987,054	987,071	987,080	987,063	987,080
		(5.44)	(2.24)	(5.38)	(2.21)
94	987,078	986,962	987,024	986,960	987,022
		(38.85)	(47.52)	(35.93)	(48.98)
95*	987,078	986,932	986,932	986,932	986,932
		(25.04)	(25.38)	(21.53)	(22.38)
99	986,888	986,865	986,912	986,865	986,912
		(11.04)	(9.16)	(11.01)	(8.36)
100	986,865	986,865	986,865	986,865	986,865
		(6.41)	(8.35)	(6.39)	(8.43)
103*	986,865	986,796	986,797	986,790	986,790
		(6.37)	(6.39)	(10.10)	(10.10)
104*	986,794	986,767	986,767	986,758	986,758
		(4.14)	(4.03)	(4.79)	(4.79)
107	986,770	986,770	986,800	986,770	986,802
		(6.60)	(8.15)	(0.07)	(13.7)
108	986,794	986,794	986,845	986,786	986,844
		(5.27)	(6.31)	(7.05)	(6.17)
Mean:	986,994	986,958.71	986,997.11	986,958.64	986,998.54

[*] Ex post revenue higher than under truthful bidding
[a] Market clearing price when bidding the lower envelope of marginal valuations (Model 1)
[b] Market clearing price when bidding the upper envelope of marginal valuations (Model 1)
[c] Market clearing price when bidding the lower envelope of marginal valuations (Model 2)
[d] Market clearing price when bidding the upper envelope of marginal valuations (Model 2)
[e] Mean across all 28 auctions (52-108).
[f] Bootstrap std. errors in parentheses

Table 5: Interim profit of bidders per T-bill for sale - part 1

Auction	Expected Surplus[b] and Efficiency						
	Average[c]	Maximal[c]	Minimal[c]	Total 1[c]	Effic 1[c,d]	Total 2[e]	Effic 2[d,e]
52	2.12	10.26	-1.59	27.55	2.09	3.32	2.97
	(1.73)	(8.62)	(1.43)	(22.49)	(0.08)	(2.88)	(0.00)
55	10.88	54.56	-1.47	141.38	3.37	97.21	3.70
	(6.14)	(34.02)	(0.60)	(79.80)	(0.08)	(76.84)	(0.11)
56	12.02	46.59	-0.43	156.26	3.59	79.07	3.47
	(4.73)	(18.12)	(0.23)	(61.54)	(0.05)	(29.45)	(0.03)
60	45.40	246.57	-0.77	635.54	9.43	496.77	9.22
	(11.16)	(130.74)	(0.41)	(156.21)	(0.39)	(141.41)	(0.34)
61	27.72	265.69	-0.37	388.09	11.07	400.08	11.17
	(7.67)	(87.82)	(0.23)	(107.41)	(0.21)	(95.91)	(0.21)
64	5.59	22.54	-0.18	67.09	1.83	73.30	1.63
	(1.68)	(8.93)	(0.20)	(20.11)	(0.08)	(24.10)	(0.04)
65	5.62	38.43	-0.20	67.40	2.08	70.72	2.69
	(3.05)	(32.14)	(0.20)	(36.63)	(0.14)	(37.81)	(0.11)
67	6.87	46.39	-0.79	96.22	3.72	102.93	4.18
	(3.38)	(28.22)	(0.46)	(47.37)	(0.11)	(52.96)	(0.11)
69	8.64	83.90	-0.14	112.28	5.49	122.58	5.43
	(5.20)	(70.90)	(0.06)	(67.61)	(0.11)	(72.88)	(0.11)
72	3.91	26.58	-0.19	62.50	2.87	66.41	2.72
	(1.44)	(13.84)	(0.11)	(22.97)	(0.05)	(22.87)	(0.05)
73	3.71	21.00	-0.33	59.38	1.99	64.95	2.12
	(1.51)	(17.15)	(0.18)	(24.17)	(0.08)	(26.35)	(0.08)
75	2.74	22.47	-16.57	38.35	1.85	42.82	1.54
	(2.15)	(20.71)	(2.30)	(30.06)	(0.06)	(33.24)	(0.06)
76	9.07	42.19	-0.24	117.92	3.13	123.46	2.83
	(2.56)	(12.13)	(0.10)	(33.29)	(0.07)	(31.70)	(0.07)
81	2.35	18.69	-3.95	32.92	1.39	33.67	1.45
	(2.57)	(21.85)	(1.82)	(35.93)	(0.06)	(24.72)	(0.02)
82	1.91	20.13	-0.62	26.74	1.31	29.32	1.23
	(1.63)	(19.93)	(0.34)	(22.85)	(0.03)	(19.08)	(0.02)
85	-0.44	6.95	-15.49	-5.68	0.53	-2.13	0.52
	(0.60)	(4.83)	(3.66)	(7.77)	(0.04)	(10.02)	(0.05)
Mean[f]	6.26	42.21	-5.09	83.67	2.50	77.73	2.54
MeanCont:[h]	10.18	52.62	0.48	133.45	2.50	126.65	2.54

[*] Ex post revenue was higher than under truthful bidding
[a] Standard errors in parentheses
[b] Using the upper envelope of marginal valuations and expressed in CZK (cca 25 CZK amounts to 1 basis point difference in yield)
[c] Using estimates from Model 1 (independent supply withdrawal)
[d] Efficiency loss due to misallocation in basis points
[e] Using estimates from Model 2 (predetermined withdrawal rule based on reference IR)
[f] Across all auctions (52-108)
[g] Across all auctions with statistically non-significant entries set to zero

Table 6: Interim profit of bidders per T-bill for sale - part 2

Auction	Expected Surplus^b and Efficiency						
	Average^c	Maximal^c	Minimal^c	Total 1^c	Effic 1^{c,d}	Total 2^e	Effic 2^{d,e}
86	2.63	21.62	-10.44	34.14	1.30	45.44	1.38
	(1.41)	(9.25)	(6.10)	(18.29)	(0.02)	(25.98)	(0.03)
87	0.21	9.50	-10.45	2.54	0.55	10.44	0.57
	(0.73)	(2.11)	(6.09)	(8.77)	(0.01)	(12.77)	(0.01)
91	1.61	11.84	-0.20	19.38	0.93	28.68	0.84
	(1.26)	(11.81)	(0.17)	(15.13)	(0.01)	(34.74)	(0.02)
92	-0.85	2.30	-12.41	-10.26	0.50	-0.37	0.52
	(0.35)	(1.95)	(2.86)	(4.19)	(0.01)	(11.19)	(0.01)
94	11.7	74.00	-17.00	117.02	3.17	124.09	3.13
	(7.34)	(44.79)	(18.02)	(73.41)	(0.08)	(60.42)	(0.06)
95	-1.30	9.87	-30.89	-12.99	0.48	-5.57	0.48
	(2.43)	(11.97)	(15.89)	(24.34)	(0.01)	(25.85)	(0.01)
99	1.03	13.49	-6.18	14.40	1.63	18.71	1.64
	(2.93)	(34.79)	(3.86)	(40.98)	(0.17)	(24.22)	(0.11)
100	0.18	5.93	-4.38	1.97	0.30	8.66	0.22
	(2.73)	(29.47)	(4.19)	(30.03)	(0.02)	(11.16)	(0.02)
103	7.23	28.25	-0.69	94.02	1.98	92.90	1.83
	(1.62)	(11.09)	(0.71)	(21.08)	(0.02)	(21.64)	(0.02)
104	1.24	9.48	-1.19	14.94	0.68	5.87	0.89
	(1.99)	(14.43)	(1.05)	(23.93)	(0.08)	(20.16)	(0.07)
107	0.48	6.09	-4.06	6.19	1.13	5.32	1.17
	(1.46)	(11.79)	(3.21)	(19.02)	(0.10)	(15.50)	(0.07)
108	2.88	16.43	-1.25	37.43	1.60	37.88	1.62
	(1.66)	(15.61)	(1.49)	(21.62)	(0.05)	(21.72)	(0.05)
Mean^f	6.26	42.21	-5.09	83.67	2.50	77.73	2.54
MeanCont:^{h}	10.18	52.62	0.48	133.45	2.50	126.65	2.54

[*] Ex post revenue was higher than under truthful bidding
[a] Standard errors in parentheses
[b] Using the upper envelope of marginal valuations and expressed in CZK (cca 25 CZK amounts to 1 basis point difference in yield)
[c] Using estimates from Model 1 (independent supply withdrawal)
[d] Efficiency loss due to misallocation in basis points
[e] Using estimates from Model 2 (predetermined supply withdrawal rule based on reference IR)
[f] Across all auctions (52-108)
[g] Across all auctions with statistically non-significant entries set to zero
[h] Means of corresponding estimates implied by a model with continuous bids.

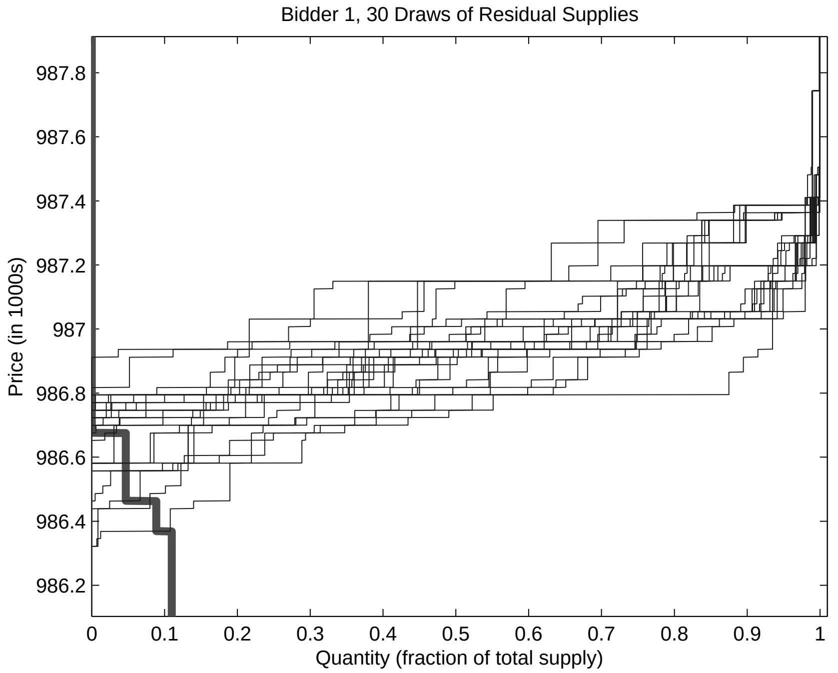

Figure 1: Resampling residual supplies

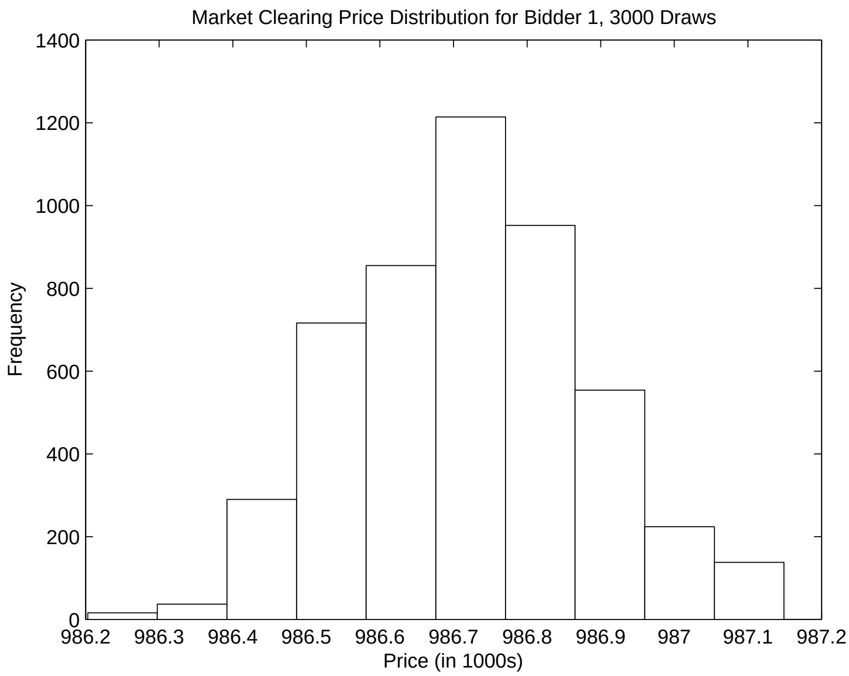

Figure 2: Distribution of market clearing price

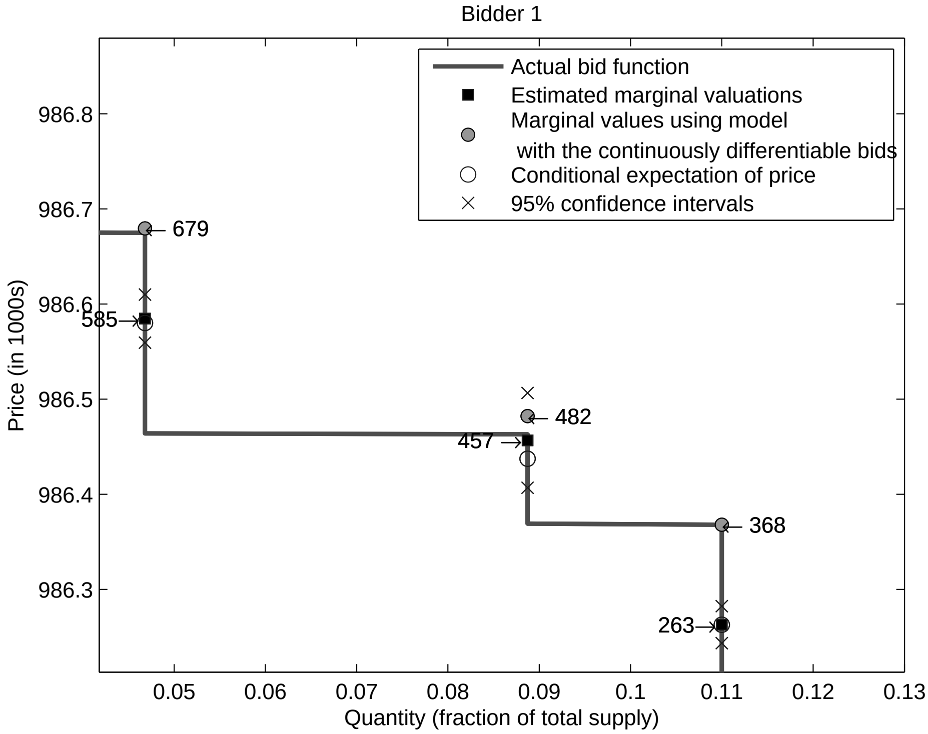

Figure 3: Marginal valuation estimation - bidder 1

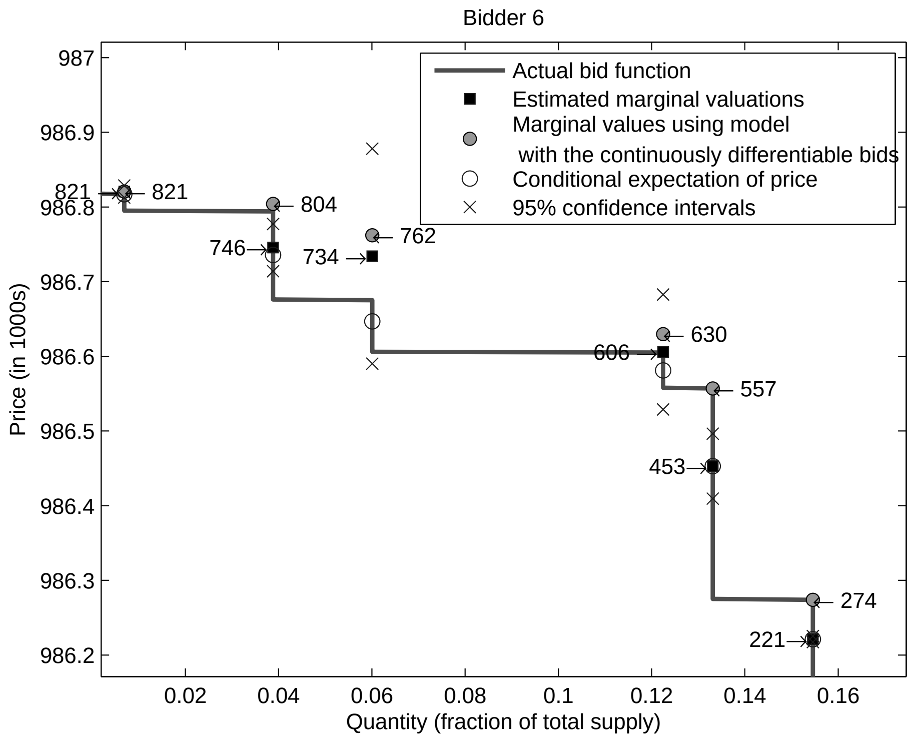

Figure 4: Marginal valuation estimation - bidder 6

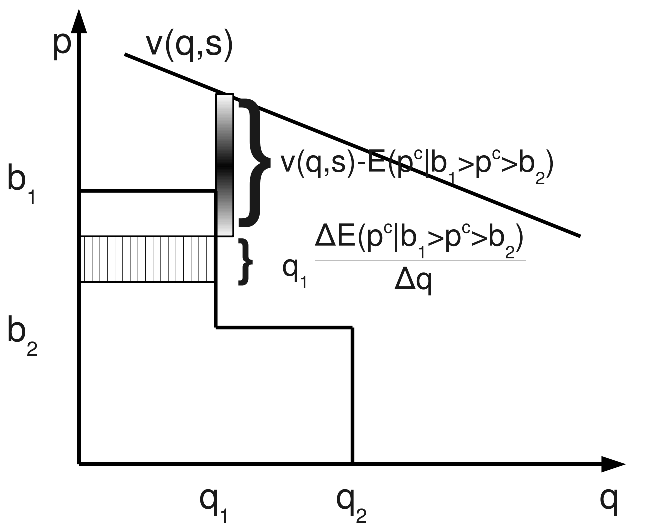

Figure 5: Bidder's problem

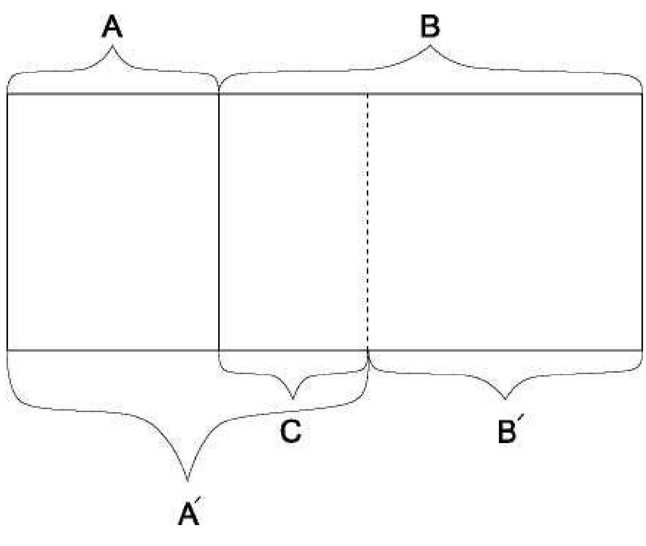

Figure 6: Different Cell Partitions