A Note on Global Equilibrium Selection in Overlapping Generations Games

Jonathan Levin*
Stanford University

June 2001

Abstract

I consider the possibility of identifying a unique "global" equilibrium in overlapping generations (dynamic) coordination environments. I show this is possible when players believe their action will have no influence on future play. When players directly influence future play, I show a global equilibrium uniqueness result for a class of binary action pure coordination games.

JEL Classification: C72, C73.

*This is based on some earlier notes: "Notes on Dynamic Equilibria in Overlapping Generations Games" (April 2000). I thank Stephen Morris for valuable discussions. Correspondence: Jonathan Levin, Department of Economics, Stanford University, Stanford CA 94305–6072. Email: jdlevin@stanford.edu.

1 Introduction

A familiar feature of strategic settings is that when individuals want to undertake a particular activity only so long as others behave similarly, a variety of different outcomes may be stable. Coordination effects of this sort play a critical role in economic models of financial crises, bank runs, macroeconomic fluctuations, consumer demand under network externalities, and team production. What these models frequently leave unresolved is which outcome *will actually occur*, and *why*.

In this note, I study the problem of equilibrium selection in a class of overlapping generations games with strategic complementarities. In these games, players take actions sequentially, and want to choose higher (or lower) actions when others have done or will do the same. Because the expectation of a certain behavior tends to be self-fulfilling, there is the strong possibility of multiple expectations-driven equilibria. I investigate whether this indeterminacy resolves if there is (potentially small) uncertainty about future payoffs. I show that under a natural form of uncertainty, these games have a unique equilibrium provided that each agent believes her action will have essentially "no influence" over future behavior. When individual agents have influence, uncertainty may significantly reduce the equilibrium set, but will uniquely determine behavior only in particular cases. I characterize one such case, and also provide examples to demonstrate the nature of equilibrium under uncertainty.

These results contribute to a literature initiated by Carlsson and van Damme (1993). Carlsson and van Damme suggested approaching the problem of equilibrium selection by considering a related class of *global games* in which even in equilibrium, agents face considerable strategic uncertainty. Starting with a two-player binary action coordination game, they considered a class of neighboring games with a vanishingly small amount of payoff uncertainty and differential information across agents. Their remarkable result was that in these neighboring games may all have a unique equilibrium despite there multiple Nash equilibrium when uncertainty is eliminated. Recently, Frankel, Morris and Pauzner (2000, henceforth FMP) have generalized this insight to a broader class of supermodular games. Like Carlsson and van Damme, FMP model players as initially uncertain about some underlying payoff parameter. Each player receives a private signal that is informative about this parameter and chooses an action. FMP provide conditions on payoff functions and the information structure under which this global game will have a unique equilibrium, so long as either *(i)* agents have essentially no prior information

on the true payoff parameter, or *(ii)* private signals are almost perfectly informative.[1]

Another series of papers by Burdzy, Frankel and Pauzner (2000), Frankel and Pauzner (2000) and Frankel (2001) develops a seemingly related approach applicable to a class of dynamic games with binary actions. Here, a continuum of far-sighted players select at each point in time a switching rate between two different actions. Payoffs have the property that the returns to an action increase with the fraction of players also engaged in that action. Burdzy-Frankel-Pauzner assume there is some publicly observed payoff-relevant parameter that evolves as a brownian motion. As a result, agents face uncertainty about the fraction of agents who will be engaged in each activity at future dates. They are able to identify conditions under which with uncertainty the entire dynamic path of play is uniquely determined despite the potential presence of multiple steady states and multiple dynamic equilibria with no payoff uncertainty.[2]

This line of research is of considerable interest because it provides a full characterization of how equilibrium play will be determined in situations where indeterminacy seems highly likely. The results are also quite promising from the standpoint of applied modelling (see Morris and Shin, 2000, for an excellent survey). An important open question is whether global arguments for equilibrium uniqueness will extend to broader classes of games, particularly richer dynamic environments. The present results are a step in this direction.

The next section presents the model. There is a sequence of agents who choose from a potentially large set of actions in some exogenous order. The agents have an incentive to coordinate — each agent's incremental returns are increasing in the actions of the others. At the time they act, agents have some knowledge about past behavior, and have further information about their payoffs (their "type" or the current "state of the world"), but they are unsure of the exact payoff of future agents (future types or states). This uncertainty satisfies two important conditions. First, types are strongly correlated, in the sense that if agent t has high incremental returns, she expects agent $t+1$ also to

[1] In the latter case, by taking a sequence of global games that converges to the original game, one can select a particular equilibrium. FMP's paper goes on to identify conditions under which the selected equilibrium is in fact independent of the particular approximating sequence. This independence always holds in the two-player, two-action world studied by CvD, and the "selected" equilibrium is the risk-dominant one.

[2] As in the static models, it is also possible to consider a sequence of global games converging to the certain payoffs game in order to obtain an equilibrium selection result (this is the focus of Burdzy, Frankel and Pauzner, 2000).

have high incremental returns. Second, agents always put positive probability (though potentially very small) on future agents having a dominant strategy.

In Section 3, I study rationalizable behavior under the "no influence" assumption that each agent believes her choice of action will have essentially no effect on future play. No influence is consistent with the "Chamberlinian" assumption often invoked in large games when a player has a only small effect on some payoff aggregate. No influence also obtains exactly under particular payoff assumptions and can be justified in particular informational conditions. Proposition 1 shows that in no influence environments, the assumptions noted above on the structure of uncertainty ensure a unique rationalizable (and Nash equilibrium) strategy profile.

In Section 4, I study a smaller class of binary action coordination games under the assumption that players have significant influence on future behavior and explicitly account for this. In this case, standard coordination assumptions on payoffs do not imply that the dynamic game will have strategic complementarities. I identify a class of "pure coordination" environments in which a unique global game equilibrium can be identified. I illustrate by example how in other environments, a global version of the game may have a much smaller equilibrium set than the original game, but need not have a unique equilibrium.

2 The Model

Consider a infinite collection of players indexed by $t = 1, 2, ...,$. Each player t has a set of possible actions A_t, a compact subset of $\mathbb{R}$. Let $(X_t)_{t\in\mathbb{N}}$, be a collection of random variables, where for each t, X_t takes values in some interval $\mathcal{X} \subset \mathbb{R}$. Suppose that X_t is generated by a Markov Process, so that $X_{t+1}|(x_1, ..., x_t) \stackrel{d}{=} X_{t+1}|x_t$. The realization of X_t (i.e. x_t) will be referred to as agent t's "type".

Agents choose their actions in sequence. Prior to acting, agent t observes her type x_t, and the vector of past types $z_t = (x_\tau)_{\tau<t}$, as well as a summary of past behavior, denoted h_t. Suppose that $h_t \in \mathcal{H}_t$, a compact subset of $\mathbb{R}^{t+t_0}$ ($t_0 > 0$ to allow for initial conditions), and that $\mathcal{H}_t$ is endowed with a partial order $\succ$. Behavioral summaries are generated recursively. For each t, let $g_t : \mathcal{H}_t \times A_t \to \mathcal{H}_{t+1}$ be some function that maps h_t and a_t into a new summary h_{t+1}.

Agent t's payoff depends on her own action a_t, the behavioral summary h_t, the actions of future agents, denoted $\vec{a}_t = (a_\tau)_{\tau>t}$, and the vector of types $(x_t)_{t\in\mathbb{N}}$. Then agent t's

payoff is $u_t : \mathcal{A}_t \times \mathcal{H}_t \times \vec{\mathcal{A}}_t \times \mathcal{X}^{\mathbb{N}} \to \mathbb{R}$, where $\vec{\mathcal{A}} = \times_{\tau>t} A_\tau$.

A strategy for agent t is a map $s_t : \mathcal{H}_t \times \mathcal{X}^{t-1} \times \mathcal{X} \to A_t$. A strategy s_t is monotone if for any $h'_t \succ h_t$, $z'_t \geq z_t$ and $x'_t \geq x_t$, we have $s_t(h'_t, z'_t, x'_t) \geq s_t(h_t, z_t, x_t)$. A strategy s'_t is greater than a strategy s_t if for all (h_t, z_t, x_t), $s'_t(h_t, z_t, x_t) \geq s_t(h_t, z_t, x_t)$. A strategy profile is a set of strategies $(s_t)_{t\in\mathbb{N}}$, and similar definitions of monotonicity apply.

The model is naturally interpreted as a dynamic game of *overlapping generations*. Agent t is "born" at time t and takes an action, having observed some payoff-relevant historical information, as well as the value of some evolving economic variable X_t. A natural assumption, though not necessary, might be that X_t evolves as a random walk with independent increments, so that $x_t = x_{t-1} + \varepsilon_t$, where ε_t is independent noise. Alternatively, X_t might correspond to a process of social learning. Suppose there is some evolving unknown parameter $\theta_t \sim N(\theta_{t-1}, \sigma_\theta^2)$, with initial condition θ_0, and suppose that a time t a signal $\xi_t \sim N(\theta_t, \sigma^2)$ is publicly observed. Then define X_t to be the conditional expectation of θ at time t, i.e. so $x_t = \mathbb{E}[\theta|\xi_1, ..., \xi_t]$. The properties of Bayesian updating imply that the process $X_t|x_{t-1}$, will follow a driftless random walk with normally distributed increments (of decreasing size).

It is convenient to define the each agent's incremental return function as follows:

$$\Delta_t(a'_t, a_t, h_t, \vec{a}_t, x) \equiv u_t\left(a'_t, h_t, \vec{a}_t, x\right) - u_t\left(a_t, h_t, \vec{a}_t, x\right).$$

Here, $\Delta_t(a', a, h_t, \vec{a}_t, x)$ expresses the incremental returns to agent t of choosing action a' rather than a given history h_t, future actions $\vec{a}_t$, and type vector x.

Two basic assumptions are made on payoffs.

(A1) (Concavity) For all t, u_t is continuous and concave in a_t.[3]

(A2) (Action Complementarity) For all t, $a'_t > a_t$,x, Δ_t is increasing in $h_t, \vec{a}_t$, and g_t is increasing in a_t,h_t, and for some $M_a > 0$,

$$\Delta_t(a'_t, a_t, h_t, (\max A_\tau)_{\tau>t}, x) - \Delta_t(a'_t, a_t, h_t, (\min A_\tau)_{\tau>t}, x) \leq M_a \left|a'_t - a_t\right|$$

A consequence of (A2) is that there is a strategic complementarity between the agents' actions. Agent t has an incentive to choose a high action if past agents have done so and *if she expects future agents to do so as well*. This last feature creates the possibility

[3]This assumption is convenient, but not crucial. Specifically, by ensuring that best-responses are unique, it permits a simple constructive proof of Proposition 1.

of multiple self-fulfilling equilibria: if agents *believe* their successors will choose high (or low) actions, they *will* choose high (or low) actions. In an environment where payoffs are known with certainty (for example, if the vector of types x is generated in advance and revealed to all agents), this multiplicity will tend to be quite robust — agents can easily have a *strict* incentive to choose a high action if they anticipate future actions will be high. The surprising consequence of introducing even small uncertainty about future payoffs is that it eliminates this expectational indeterminacy, and uniquely pins down behavior.

I now introduce assumptions on the stochastic process $(X_t)_{t\in\mathbb{N}}$ and on how types affect agents' payoff. Here, I let $\vec{X}_t = (x_\tau)_{\tau>t}$ denote the vector of types following t.

(A3) (Uncertainty) For any x_t, X_{t+1} has full support $\mathcal{X}$, and there is some $M_\varepsilon > 0$ such that for all t,x_t, x_t' and any measurable set $C \subset \mathcal{X}$,

$$\left|\Pr\left(X_{t+1} \in C \mid x_t'\right) - \Pr\left(X_{t+1} \in C \mid x_t\right)\right| \leq M_\varepsilon \left|x_t' - x_t\right|.$$

(A4) (Positive Shifting) For all t, an increase in x_t leads to larger stochastic increase in X_{t+1}. If $\delta \geq 0$ then

$$X_{t+1} - \delta \mid x_t + \delta \text{ first order stochastically dominates } X_{t+1} \mid x_t.$$

(A5) (Limit Dominance) There exist $\underline{x}, \overline{x} \in \mathcal{X}$ such that for all t, if $x_t < \underline{x}$, then $\min A_t$ is dominant for agent t, and if $x_t > \overline{x}$, then $\max A_t$ is dominant.

(A6) (Parameter Complementarity) For all t, a_{-t}, a_t', a_t, Δ_t is increasing in x. Moreover, for some $M_x > 0$, and all $x_t', x_t \in [\underline{x}, \overline{x}]$ with $x_t' > x_t$

$$\Delta_t(a_t', a_t, h_t, \vec{a}_t, x_t', x_{-t}) - \Delta_t(a_t', a_t, h_t, \vec{a}_t, x_t, x_{-t}) \geq M_x \left|x_t' - x_t\right| \left|a_t' - a_t\right|$$

and for some $M_{-x} > 0$, and all $x_{-t}' \geq x_{-t}$

$$\Delta_t(a_t', a_t, h_t, \vec{a}_t, x_t, x_{-t}') - \Delta_t(a_t', a_t, h_t, \vec{a}_t, x_t, x_{-t}) \leq M_{-x} \left|a_t' - a_t\right|$$

These assumptions have two important implications. First, Positive Shifting (A4) and Parameter Complementarity (A6) generate a complementarity between an agent's type and her behavior. An agent with a high type has a direct reason to choose a high action (by (A6)) and also (by (A4)) believes future agents will have high types. Second,

Uncertainty (A3) and Limit Dominance (A5) imply that there is always a chance that expectations will play no role in the strategic choices of future agents (in particular, there is always positive probability — potentially quite small — that future agents will have a dominant strategy).

Perhaps the most demanding assumption is Positive Shifting (A4). It requires not only that types be positively correlated, in the sense that observing a higher realization of x_t leads agent t to believe the distribution of unobserved types is higher, but that a higher realization of x_t leads agent t to believe that the distribution of these types *relative to her own* is higher, i.e. that the distribution of $X_\tau - x_t$ is higher, for $\tau > t$.

For instance, suppose $x_t = x_{t-1} + \varepsilon_t$ with $\varepsilon_t \sim f_t(\cdot|x_{t-1})$. Positive Shifting requires that an increase in x_{t-1} increases the distribution of ε_t in the sense of first-order stochastic dominance. Thus, it means the process generating types may not be mean-reverting. Of course, one natural case that satisfies the condition trivially is if ε_t is independent of x_{t-1}.

3 Games with No Influence

I first consider games where agent t acts under the assumption that she has no direct influence over the actions of those who follow her. To motivate this assumption, suppose that agents can observe past actions, but that these actions affect payoffs only through some evolving summary statistic, say an dynamic average of past actions: $h_t = \lambda h_{t-1} + (1-\lambda)a_t$. If $\lambda \approx 1$, one might naturally think of agent t disregarding her own effect on h_{t+1} (e.g. assuming $h_{t+1} = h_t$ rather than just $h_{t+1} \approx h_t$).

To formulate this precisely, observe that to calculate expected payoffs, agent t must form a probability distribution over future actions. This involves calculating a probability distribution over $\vec{X}_t$, given x_t, and then, for any realization $\vec{x}_t$ computing future actions using opponent strategies s_{-t} and the behavioral summary $h_{t+1} = g_t(h_t, a_t)$. Let us say that agent t *presumes to have no influence* if, when she makes this calculation, rather than computing $h_{t+1} = g_t(h_t, a_t)$, she assigns a value $h_{t+1} = \phi_t(h_t)$ that is *independent of her own action.* Call ϕ_t an assignment function for agent t — these functions are assumed to be monotone. Suppose that some set of assignment functions $(\phi_t)_{t\in\mathbb{N}}$ is given.

Define agent t's expected utility:

$$U_t(a_t, h_t, z_t, x_t; s_{-t}) \equiv \mathbb{E}\left[u(a_t, h_t, \vec{a}_t, x) \mid h_t, z_t, x_t, s_{-t}\right].$$

Here, I suppress the dependence on ϕ_t with the understanding that agent t uses $h_{t+1} = \phi_t(h_t)$ to compute future actions given a realization $\vec{x}_t$.

A profile s is a Nash equilibrium of the no influence game if for all t, h_t, x_t,

$$s_t(h_t, z_t, x_t) \in \arg\max_{a \in A_t} U_t(a, h_t, z_t, x_t; s_{-t}).$$

Proposition 1 *Under Assumptions (A1)–(A6), there is a unique equilibrium of the no influence game and the game is dominance solvable.*

Proof. I prove the result in a series of steps.

Step 1. (*Strategic Complementarities*) I first show that when s_{-t} is monotone, U_t has increasing differences in (a_t, h_t), (a_t, z_t), (a_t, x_t) and (a_t, s_{-t}), i.e. that if $a'_t > a_t$, then

$$U_t(a'_t, h_t, z_t, x_t; s_{-t}) - U_t(a_t, h_t, z_t, x_t; s_{-t}) = \mathbb{E}\left[\Delta(a'_t, a_t, h_t, \vec{a}_t, x) \mid h_t, z_t, x_t, s_{-t}\right] \quad (1)$$

is increasing in h_t, z_t, x_t, s_{-t}. Observe that, fixing a realization $\vec{x}_t$ of $\vec{X}_t$, agent t can perfectly forecast $\vec{a}_t$ as implied by $\vec{x}_t, z_t, x_t$, $h_{t+1} = \phi_t(h_t)$ and s_{-t}. Now suppose that $h'_t \succ h_t$, $z'_t \geq z_t$, $x'_t \geq x_t$, $s' \geq s$. Under x'_t, agent t's probability distribution over $\vec{X}_t$ first-order stochastically dominates her probability distribtution over $\vec{X}_t$ under x_t. For any fixed realization of $\vec{X}_t$, action profile $\vec{a}_t$ implied by $(h'_t, z'_t, x'_t, \vec{x}_t, s'_{-t})$ is higher than the profile $\vec{a}_t$ implied by $(h_t, x_t, z_t, \vec{x}_t, s_{-t})$. So (1) is higher fixing $(h'_t, z'_t, x'_t, \vec{x}_t, s'_{-t})$ than fixing $(h_t, z_t, x_t, \vec{x}_t, s_{-t})$. Moreover, (1) is increasing in $\vec{x}_t$ given either $(h'_t, z'_t, x'_t, s'_{-t})$ or (h_t, z_t, x_t, s_{-t}), so it must be that (1) is in fact increasing in $h_t, z_t, x_t,$ and s_{-t}.

Step 2: (*Iterated Dominance*) Let $\mathcal{S}$ be the space of all strategy profiles. Define the best-response operator $B : \mathcal{S} \to \mathcal{S}$ so that

$$(B_t s)(h_t, z_t, x_t) = \arg\max_{a \in A_t} U_t(a, h_t, z_t, x_t; s_{-t}).$$

Define $\mathcal{S}^M$ to be the space of monotone profiles that satisfy limit dominance,

$$\mathcal{S}^M = \left\{ s \in \mathcal{S} : \begin{array}{l} \text{for all } t,\ s_t \text{ is increasing in } h_t, z_t, x_t \\ \text{and } s_t(h_t, z_t, x_t) = \min A_t\ (\max A_t) \text{ for all } h_t, z_t, x_t < \underline{x}\ (\geq \overline{x}) \end{array} \right\}.$$

From step one, note that $B : \mathcal{S}^M \to \mathcal{S}^M$. Moreover, observe that if we start with the highest possible profile $\overline{s} = (\max A_t)_{t \in \mathbb{N}} \in \mathcal{S}^M$, and apply the best-response operator iteratively, we obtain a decreasing sequence of monotone profiles — each of which becomes in turn a new upper bound on serially undominated strategies (this procedure

follows Milgrom and Roberts, 1990). Similarly, if we start with $\underline{s} = (\min A_t)_{t\in\mathbb{N}} \in \mathcal{S}^M$ and apply B iteratively, we obtain an increasing sequence of monotone profiles — each of which becomes in turn a new lower bound on rationalizable profiles. We now show that these processes converge to a unique serially undominated (and Nash equilibrium) profile s^*.

STEP 3: (*Contraction Mapping*)[4] To do this, I show that B is a contraction mapping on $\mathcal{S}^M$ under an appropriate metric. Given a strategy $s_t : \mathcal{H}_t \times \mathcal{Z}_t \times \mathcal{X} \to A_t$, define $s_t \oplus \delta : \mathcal{H}_t \times \mathcal{Z}_t \times \mathcal{X} \to A_t$, such that

$$(s_t \oplus \delta)(h_t, z_t, x_t) = \begin{cases} s_t(h_t, z_t, x_t + \delta) & \text{if } x_t \geq \underline{x} \\ \min A_t & \text{if } x_t < \underline{x}. \end{cases}$$

Given a profile s, define $s \oplus \delta = (s_t \oplus \delta)_{t\in T}$. If $s \in \mathcal{S}^M$, so is $s \oplus \delta$, and moreover $s \oplus \delta \geq s$. Now define $\rho : \mathcal{S}^M \times \mathcal{S}^M \to \mathbb{R}$ as follows:

$$\rho(s, s') = \inf_{\delta} \left\{ \delta : s \oplus \delta \geq s' \text{ and } s' \oplus \delta \geq s \right\}.$$

It is straightforward to verify that $(\mathcal{S}^M, \rho)$ is a complete metric space. To show that B is a contraction, we must show that:

$$\rho(Bs, B(s \oplus \delta)) \leq \beta \rho(s, s \oplus \delta)).$$

For this, it suffices to show that for some $\beta < 1$, and all t,$s \in \mathcal{S}^M$, $B_t(s \oplus \delta) \leq (B_t s) \oplus \beta\delta$. Observe that for any t, h_t, z_t this relationship holds with equality if $x_t < \underline{x}$.

Observe also that Assumptions A2 (Action Complementarity), A3 (Uncertainty) and A6 (Parameter Complementarity) imply the following Lipshitz condition:

$$\begin{array}{l} \mathbb{E}\left[\Delta(a_t', a_t, h_t, \vec{a}_t, x_t, x_{-t}) \mid h_t, z_t, x_t', s_{-t}\right] \\ -\mathbb{E}\left[\Delta(a_t', a_t, h_t, \vec{a}_t, x_t, x_{-t}) \mid h_t, z_t, x_t, s_{-t}\right] \end{array} \leq M_\varepsilon(M_a + M_{-x}) \left|x_t' - x_t\right| \left|a_t' - a_t\right|.$$

Let $M_0 = M_\varepsilon(M_a + M_{-x})$.

Now, fixing $t, h_t, z_t, x_t,$ as well as $a_t' > a_t$ and $s \in \mathcal{S}^M$, pick $1 > \beta \geq M_0/(M_x + M_0)$,

$$\begin{aligned} &\mathbb{E}\left[\Delta\left(a_t', a_t, h_t, \vec{a}_t, x_t, x_{-t}\right) \mid h_t, z_t, x_t, s \oplus \delta\right] \\ &\quad \leq \mathbb{E}\left[\Delta\left(a_t', a_t, h_t, \vec{a}_t, x_t, x_{-t}\right) \mid h_t, z_t, x_t + \delta, s\right] \\ &\quad \leq \mathbb{E}\left[\Delta\left(a_t', a_t, h_t, \vec{a}_t, x_t, x_{-t}\right) \mid h_t, z_t, x_t + \beta\delta, s\right] + M_0\delta(1-\beta)\left|a_t' - a_t\right| \\ &\quad \leq \mathbb{E}\left[\Delta\left(a_t', a_t, h_t, \vec{a}_t, x_t + \beta\delta, x_{-t}\right) \mid h_t, z_t, x_t + \beta\delta, s\right] - \left(M_x\beta - M_0(1-\beta)\right)\delta\left|a_t' - a_t\right| \\ &\quad \leq \mathbb{E}\left[\Delta\left(a_t', a_t, h_t, \vec{a}_t, x_t + \beta\delta, x_{-t}\right) \mid h_t, z_t, x_t + \beta\delta, s\right]. \end{aligned}$$

[4]The idea of using a contraction argument for global games is due to Stephen Morris. In unpublished notes, he has used a contraction approach to prove FMP's (2000) result.

The first inequality follows from Positive Shifting and the definition of $s \oplus \delta$, the second from the Lipshitz assumption, and the third from Parameter Complementarity. The final inequality follows from the choice of β. Topkis' Monotonicity Theorem then implies that $B_t(s \oplus \delta)(h_t, z_t, x_t) \leq B_t s(h_t, z_t, x_t + \beta\delta) = (B_t s \oplus \beta\delta)(h_t, z_t, x_t)$. Hence B is a contraction, and by the contraction mapping theorem has a unique fixed point s^* in $\mathcal{S}^M$. This strategy profile is the unique rationalizable and Nash Equilibrium profile. *Q.E.D.*

3.1 Discussion

The logic underlying Proposition 1 is the following: given the structure of the game, if agents $\tau \neq t$ all increase their (monotone) strategies by some fixed amount, then agent t will also want to increase her strategy, *but by a smaller amount*. Thus, the best-response operator is a contraction mapping. The relies on the combination of strategic complementarities and the presence of uncertainty. For instance, if payoffs are known with certainty, only strong assumptions on the shape of the payoff function (for instance, that some players have dominant strategies) will guarantee a unique equilibrium, because an increase in opponent strategies can cause agent t's to increase her action by an arbitrarily large amount.

To gain some intuition for the contraction argument, suppose the action set is finite. A monotone strategy for t is given by a series of "cut-points" specifying the value of X_t at which the agent should jump up to a higher action. The cut-points depend on h_t and z_t, and on the expected behavior of future agents, and by the Limit Dominance assumption are all in $[\underline{x}, \overline{x}]$. Now, suppose that all agents $\tau \neq t$ simultaneously shift to higher strategies by lowering each of their cut-points by a fixed amount δ. Since the probability distribution over future types is unchanged (given x_t), from agent t's perspective, this change shifts up the probability distribution over future actions. In response, agent t has an incentive to increase her strategy. But by how much? Suppose that x_t^* marked agent t's optimal cutoff between two actions under the original opponent strategies, and she contemplates decreasing this cut-off to $x_t^* - \delta$. Assumption (A4), Positive Shifting, says that under the higher opponent strategies, and given $x_t = x_t^* - \delta$, the probability distribution agent t places on opponent actions will be *no higher than* the probability distribution she placed on opponent actions under the original strategies given that $x_t = x_t^*$. So agent t certainly will not want to shift her strategy by more than δ. But indeed, she cannot even justify a shift this large, because there is also a second effect at work. When agent t's type is $x_t = x_t^* - \delta$, her incremental returns are directly

lower than they were when $x_t = x_t^*$, independent of her beliefs about future behavior. Thus, if agents $\tau \neq t$ increase their strategies by a uniform amount, t's best-response will also increase, but by a strictly smaller amount. The immediate consequence is that there can be at most one equilibrium in monotone strategies, and since the lowest and highest equilibria in games with strategic complementarities are always monotone, no non-monotone equilibria either.

The important assumption made to establish the result is that each agent views the probability distribution over future actions as independent of what she herself chooses to do. One way to view this no influence assumption is simply as a behavioral restriction — agent t simply adopts the view that she cannot influence future play. This clearly involves a departure from rationality in many environments, but in others it is at least approximately consistent with perfectly rational optimizing behavior. I now discuss a few of the latter settings.

(1) Incentive Independence. Suppose that for all t, $a_t', a_t, \vec{a}_t$ that $\Delta_t(a_t', a_t, h_t, \vec{a}_t, x)$ is constant in the behavioral history h_t. That is, each agent's incremental returns are independent of historical behavior. If agents play Markovian strategies (so they condition only on payoff-relevant information), then for all t, s_t will be independent of h_t for all t. Thus, agent t's action indeed will have no influence on the future path of play. Of course, agent t's action may change the payoffs of future players in *levels*. A special case of incentive independence is where agent t's payoff is entirely independent of past actions: i.e. payoffs are given by $u_t(a_t, \vec{a}_t, x)$. Matsui (1999) has proposed an overlapping generations model of bank runs that has exactly this feature. In Matsui's model, payoffs for each agent are given by $u(a_t, a_{t+1}, x_t)$.

(2) Games with Evolving Payoff Aggregates. A no influence presumption can be approximately correct in models where payoffs depend on opponents' actions through some payoff aggregate on which each individual has minimal effect. In this context, no influence is essentially the Chamberlinian assumption employed in many models of monopolistic pricing, macroeconomic investment, search, and technology adoption. For instance, the assumption was motivated above by assuming that $h_t \in \mathbb{R}$ evolves as $h_{t+1} = \lambda h_t + (1-\lambda)a_t$, where $\lambda \approx 1$. This is essentially the set-up in Frankel and Pauzner (2000) and Frankel (2000), where the relevant historical aggregate is a time-weighted fraction of past agents choosing to be involved in the two activities (time is continuous in their model). In principle, the behavioral summary could be more com-

plex. For instance, if $A_t = \{0, 1, ..., K\}$, one might imagine h_t as an "average" of past behavior — a probability distribution on $\{0, 1, ..., K\}$ (and hence an element of the K-dimensional simplex). In this case, some care is needed to incorporate monotonicity — it must be the case that higher values of h_t, a_t lead to values of h_{t+1} that are higher in a particular stochastic order (say, first-order stochastic dominance). Alternatively, h_t need not even involve aggregation: it could be equal to the entire history $(a_\tau)_{\tau<t}$ — in this case, no influence could reasonably approximate a fully rational forecast if incremental returns varied relatively little with either past or future actions.

(3) Lack of Information. Another way to justify no influence is through informational assumptions. I sketch the basic idea. Suppose agent t's payoff depends only on past actions $a_1, ..., a_{t-1}$, the action of her successor a_{t+1} and her type x_t. And suppose agent $t+1$ does not observe a_t or x_t. No influence will be satisfied because agent $t+1$ will base her action on her *beliefs* about a_t and x_t. However, a key condition that may be hard to satisfy is Positive Shifting (A4). In particular, to prove equilibrium uniqueness using the method of Proposition 1, one must show that when agent $t+1$ forms a conditional probability distribution over X_t given $(x_0, ..., x_{t-1})$ and x_{t+1}, the distribution satisfies (or at least approximiately satisfies) Positive Shifting. If X_t evolves as a random walk with independent increments, what this means is that the the increments must become progressively smaller as t increases, since it must be that to form a belief over X_t, agent $t+1$ basically ignores x_{t-1} and conditions on x_{t+1}.[5]

3.2 Examples

Example A. Suppose, as in Matsui (1999), that payoffs for each agent depend only on her own action and type, and the action of her successor, so $u_t = u(a_t, a_{t+1}, x_t)$. Suppose also that u satisfies the assumptions A1, A2, A5 and A6 above, and that each agent has a finite action set $A = \{a_0, a_1, ..., a_M\}$. Consider first the infinite sequential move game where $x_t = x$ is fixed at some constant level over time. Then, for any action a_m with the property that

$$a_m \in \arg\max_{a \in A} u(a, a_m, x)$$

[5]Indeed, this is the same reason why Frankel, Morris and Pauzner (2000) in studying simultaneous games, must assume that either players have no prior information or that signals are nearly perfectly informative: if x_0 is their analogue of "prior" information, they require that $X_\tau | x_t, x_0 \approx X_\tau | x_t$.

there is a corresponding pure-strategy Nash equilibrium in which each agent chooses a_m (thus correctly anticipating and matching her successor). Since $\arg\max_{a\in A} u(a, a_m, x)$ is increasing in a_m by strategic complementarities, it is clear that there may be many such equilibria.

Now suppose instead that X_t evolves as $x_t = x_{t-1} + \eta\varepsilon_t$, where ε_t has a time-independent distribution G with support $\mathbb{R}$ and $\eta > 0$ is a constant. By Proposition 1, there is a unique Nash equilibrium, where each player uses a "cut-off" strategy:

$$s(x_t) = a_m \text{ if } x_m^* \leq x_t \leq x_{m+1}^*$$

where $-\infty = x_0^* \leq x_1^* \leq ... \leq x_M^* \leq x_{M+1}^* = \infty$. For $\eta \approx 0$, each cut-off point x_m^* satisfies:

$$\begin{aligned} 0 &= \mathbb{E}\left[\Delta\left(a_m, a_{m-1}, s(X_{t+1}), x_m^*\right) \mid X_t = x_m^*\right] \\ &\approx \left[1 - G(0)\right]\Delta(a_m, a_{m-1}, a_m, x_m^*) + G(0)\Delta(a_m, a_{m-1}, a_{m-1}, x_m^*), \end{aligned}$$

and if ε_t is symmetric around zero,

$$0 \approx \frac{1}{2}\Delta(a_m, a_{m-1}, a_m, x_m^*) + \frac{1}{2}\Delta(a_m, a_{m-1}, a_{m-1}, x_m^*).$$

That is, an agent switches from a_{m-1} to a_m if a_m yields a higher payoff under the presumption that her successor will choose a_{m-1} and a_m with equal probability. If there are only two actions, this corresponds to an equilibrium where each agent chooses the strategy which is $\frac{1}{2}$-dominant given her type.

Example B. Now consider the same model as above, but where each agent's payoff also depends on the average size of the last T actions. Let $h_t = \frac{1}{T}\sum_{\tau=1}^{T} a_{t-\tau}$. And assume that agent t acts under the presumption that h_{t+1} will be equal to h_t — an assumption that is approximately justified if T is large. As above, there may be many equilibria absent uncertainty, but with uncertainty, the unique equilibrium has each player using a cut-off strategy:

$$s(h_t, x_t) = a_m \text{ if } x_m^*(h_t) \leq x_t \leq x_{m+1}^*$$

where $x_m^*(\cdot)$ is decreasing in h_t. Taking η to be small, and ε_t to be symmetric around zero, each cut-off point $x_m^*(h_t)$ must satisfy:

$$0 \approx \frac{1}{2}\Delta\left(a_m, a_{m-1}, h_t, a_m, x_m^*(h_t)\right) + \frac{1}{2}\Delta(a_m, a_{m-1}, h_t, a_{m-1}, x_m^*(h_t)).$$

So an agent switches from a_{m-1} to a_m if, *conditional on history*, a_m yields a higher payoff under the presumption that her successor with choose a_{m-1} and a_m with equal probability.

4 Games with Influence

No influence is violated if agents explicitly account for the effect of their actions on future play. In this case, the argument behind Proposition 1 breaks down. To gain some intuition for this, note that in forming her expected payoff, agent t computes a probability distribution over future actions conditioning on her information x_t, h_t, her action a_t, and opponent strategies s_{-t}. Her best response maximizes:

$$U_t(a_t, h_t, z_t, x_t; s_{-t}) = \mathbb{E}\left[u(a_t, h_t, \vec{a}_t, x) \mid h_t, z_t, x_t, s_{-t}, a_t\right]$$

A profile s is an Nash equilibrium if and only if for all t, h_t, z_t, x_t,

$$s_t(h_t, z_t, x_t) \in \arg\max_{a \in A_t} U_t(a_t, h_t, z_t, x_t; s_{-t}).$$

An immediate difference here is that even if payoffs satisfy the monotonicity conditions assumed above, it need not be the case that the dynamic game has strategic complementarities. In general, if agent $t+1$ raises her strategy from s_{t+1} to s'_{t+1}, this need not raise the incremental returns to agent t of playing a higher action.

Nevertheless, one might still wonder if a global game argument could have some bite, or whether the full power of the argument might extend to particular classes of games. To explore this, I turn attention to an interesting class of binary action games with an element of "pure coordination."[6]

4.1 Binary Coordination Games

Consider the following class of dynamic coordination games. Suppose there are an infinite number of agents who choose from an action set $A = \{\underline{a}, \overline{a}\}$ in sequence. Suppose each agent's payoff depends only on her own action and type, and the actions of her predecessors through the behavioral summary h_t, and the action of her immediate successor. Then agent t's payoff can be expressed as $u_t(a_t, h_t, a_{t+1}, x_t)$. Her incremental returns are $\Delta_t(h_t, a_{t+1}, x_t)$. Assume payoffs have the following element of "pure coordination": conditional on choosing a_t, agent t prefers that her successor match her own action. Define

$$\Lambda_t(a_t, h_t, x_t) = u_t(a_t, h_t, \overline{a}, x_t) - u_t(a_t, h_t, \underline{a}, x_t).$$

[6] Lagunoff and Matsui (1997) prove an equilibrium uniqueness result for alternating move games with no uncertainty where player have *identical* payoffs. Both their model, and their argument, are quite different however.

(BC) For all t, $\Lambda_t(a_t, h_t, x_t) = \Lambda_t(a_t)$ is independent of h_t, x_t, and $\Lambda_t(\overline{a}) \geq 0 \geq \Lambda_t(\underline{a})$.

Proposition 2 *In binary coordination games that satisfy A1–A6 and BC, there is a unique rationalizable and Nash equilibrium profile, given by*

$$s_t(h_t, x_t) = \begin{cases} \overline{a} & \text{if } x_t \geq x_t^*(h_t) \\ \underline{a} & \text{if } x_t < x_t^*(h_t) \end{cases},$$

where for all t, $x_t^*(\cdot)$ is a decreasing function of h_t.

Proof. As in the proof of Proposition 1, we first establish strategic complementarities. A monotone strategy for t is $s_t(h_t, x_t)$ with the property that $s_t(h_t, x_t) = \overline{a}$ if and only if $x_t \geq x_t(h_t)$, where $x_t(\cdot)$ is a cut-off function that is decreasing in h_t. Given h_t, x_t and a monotone strategy s_{t+1},

$$\begin{aligned} &D_t(h_t, x_t; s_{t+1}) \\ &\quad \equiv U_t(\overline{a}, h_t, x_t; s_{t+1}) - U_t(\underline{a}, h_t, x_t; s_{t+1}) \\ &\quad = \Pr\left(X_{t+1} \geq x_{t+1}(h_t, \underline{a}) \mid x_t\right) [u_t(\overline{a}, h_t, \overline{a}, x_t) - u_t(\underline{a}, h_t, \overline{a}, x_t)] \\ &\quad + \Pr\left(x_{t+1}(h_t, \overline{a}) \leq X_{t+1} < x_{t+1}(h_{t-1}, \underline{a}) \mid x_t\right) [u_t(\overline{a}, h_t, \overline{a}, x_t) - u_t(\underline{a}, h_t, \underline{a}, x_t)] \\ &\quad + \Pr\left(X_{t+1} < x_{t+1}(h_t, \overline{a}) \mid x_t\right) [u_t(\overline{a}, h_t, \underline{a}, x_t) - u_t(\underline{a}, h_t, \underline{a}, x_t)] \end{aligned}$$

By Limit Dominance $D_t < 0$ for $x_t < \overline{x}$, and $D_t > 0$ for $x_t \geq \overline{x}$. By Assumptions 2,4,6 and BC, $D_t(h_t, x_t; s_{t+1})$ is increasing in h_t and s_{t+1} and strictly increasing in x_t between $\underline{x}$ and $\overline{x}$. So the game has strategic complementarities. Agent t's best response to s_{t+1} is a monotone strategy with cut-off point $x_t(h_t)$, where $D_t(h_t, x_t(h_t); s_{t+1}) = 0$. Note that because D_t is strictly increasing in x_t, these cut-off points are uniquely defined.

To apply a contraction mapping argument as above, observe that $s_t \oplus \delta$ is a cut-off strategy with cut-offs $\max\{x_t(h_t) - \delta, \underline{x}\}$. The idea is to show that for some $\beta < 1$, and all t,h_t, monotone strategies s_{t+1}, and $\delta > 0$, that

$$D_t(h_t, x_t, s_{t+1}) = 0 \Rightarrow D_t(h_t, x_t - \beta\delta; s_{t+1} \oplus \delta) < 0. \tag{2}$$

To verify this, observe that

$$\begin{aligned} &D_t(h_t, x_t - \beta\delta; s_{t+1} \oplus \delta) \\ &\quad = \Pr\left(X_{t+1} + \delta \geq x_{t+1}(h_t, \underline{a}) \mid x_t - \beta\delta\right) \Delta_t(h_t, \overline{a}, x_t - \beta\delta) \\ &\qquad + \left\{ \begin{array}{c} \Pr\left(x_{t+1}(h_t, \overline{a}) \leq X_{t+1} + \delta < x_{t+1}(h_{t-1}, \underline{a}) \mid x_t - \beta\delta\right) \\ \times [u_t(\overline{a}, h_t, \overline{a}, x_t - \beta\delta) - u_t(\underline{a}, h_t, \underline{a}, x_t - \beta\delta)] \end{array} \right\} \end{aligned}$$

$$
\begin{aligned}
&\quad + \Pr\left(X_{t+1} + \delta < x_{t+1}(h_t, \overline{a}) \mid x_t - \beta\delta\right) \Delta_t(h_t, \underline{a}, x_t - \beta\delta) \\
&\leq \Pr\left(X_{t+1} + \delta \geq x_{t+1}(h_t, \underline{a}) \mid x_t - \delta\right) \Delta_t(h_t, \overline{a}, x_t) \\
&\quad + \left\{ \begin{array}{c} \Pr\left(x_{t+1}(h_t, \overline{a}) \leq X_{t+1} + \delta < x_{t+1}(h_{t-1}, \underline{a}) \mid x_t - \delta\right) \\ \times \left[u_t(\overline{a}, h_t, \overline{a}, x_t) - u_t(\underline{a}, h_t, \underline{a}, x_t)\right] \end{array} \right\} \\
&\quad + \Pr\left(X_{t+1} + \delta < x_{t+1}(h_t, \overline{a}) \mid x_t - \delta\right) \Delta_t(h_t, \underline{a}, x_t) \\
&\quad + M_\varepsilon M_a(\overline{a} - \underline{a})(1-\beta)\delta - M_x(\overline{a} - \underline{a})\beta\delta \\
&\leq D_t(h_t, x_t, s_{t+1}) - (\overline{a} - \underline{a})\delta\left[M_x\beta - M_\varepsilon M_a(1-\beta)\right].
\end{aligned}
$$

The first inequality follows from the Parameter Complementarity and Lipshitz assumptions, and the second from Positive Shifting. Then (2) follows by taking β sufficiently close to 1. The remainder of the proof follows Proposition 1. *Q.E.D.*

4.2 Examples

The following example applies Proposition 2 to obtain an equilibrium selection result for a particular class of binary coordination games.

Example C. Suppose payoffs for agent t depend on past behavior only through the action of the preceding agent: $h_t = a_{t-1}$, so payoffs are given by $u(a_t, a_{t-1}, a_{t+1}, x_t)$. Suppose that u satisfies Assumptions 1,2,5,6 and BC. Note that if $x_t = x$ is fixed, then, fixing a_0, the overlapping generations game will have multiple Nash equilibria $(\underline{a}, \underline{a}, \underline{a}, ...)$ and $(\overline{a}, \overline{a}, \overline{a}, ...)$ if $\Delta(a_{t-1}, a_{t+1}, x)$ changes sign when a_{t+1} increases from $\underline{a}$ to $\overline{a}$. If this sign-change condition holds regardless of the value of a_{t-1}, both these equilibia will be subgame (and Markov) perfect. Now, suppose that X_t evolves as a random walk with independent increments, so $x_t = x_{t-1} + \eta\varepsilon_t$, where ε_t has a time-independent distribution G with support $\mathbb{R}$ and $\eta > 0$ is a positive constant. The equilibrium for this global game will be a set of cut-off's $x(\overline{a})$, $x(\underline{a})$, such that each agent t optimally plays $a_t = \overline{a}$ if and only if $x_t \geq x(a_{t-1})$. These cut-offs must satisfy:

$$
\begin{aligned}
D(a_{t-1}, x) \quad = \quad & 1 - G(x(\underline{a}) - x)\Delta(a_{t-1}, \overline{a}, x) + G(x(\overline{a}) - x)\Delta(a_{t-1}, \underline{a}, x) \\
& + \left[G(x(\underline{a}) - x) - G(x(\overline{a}) - x)\right]\left[u_t(\overline{a}, a_{t-1}, \overline{a}, x_t) - u_t(\underline{a}, a_{t-1}, \underline{a}, x_t)\right]
\end{aligned}
$$

Solving the implicit equations $D(a_{t-1}, x(a_{t-1})) = 0$ for $x(a_{t-1})$ yields the solution. Observe that if G is symmetric and we take $\eta \to 0$, then the solutions are:

$$
0 = \frac{1}{2}\Delta(a_{t-1}, \overline{a}, x(a_{t-1})) + \frac{1}{2}\Delta(a_{t-1}, \underline{a}, x(a_{t-1})) + \frac{1}{2}\Lambda(a_{t-1}).
$$

The 1/2-dominant strategy for agent t would be to place weights $(1/2, 1/2)$ on agent $t+1$ choosing either $\underline{a}$ or $\overline{a}$. That is, it would be to choose a cut-off $\hat{x}(a_{t-1})$ that satisfied:

$$0 = \frac{1}{2}\Delta(a_{t-1}, \overline{a}, x(a_{t-1})) + \frac{1}{2}\Delta(a_{t-1}, \underline{a}, x(a_{t-1}))$$

With influence, relative to the 1/2-dominant strategy, agents bias their actions toward $\overline{a}$ if the preceding action is $\underline{a}$, and toward $\underline{a}$ if the preceding action was $\overline{a}$. So there is relatively less "history-dependence" when agents take into account their influence on future play.

The next example shows how, for two-action games that do not satisfy the pure coordination assumption (BC), iterated dominance may reduce the equilibrium set, without pinning down a unique equilibrium.

Example D. Suppose payoffs for agent t depend on only her own action and type and the action of the following agent, so $u(a_t, a_{t+1}, x_t)$, and u is increasing in a_{t+1} (so $\Lambda(\overline{a}) \geq \Lambda(\underline{a}) > 0$ and BC is not satisfied). Fixing $x_t = x$, action $\overline{a}$ will be dominant if $x \geq \overline{x}$, that is, for any x such that $\Delta(\overline{a}, x) + \Lambda(\underline{a}) \geq 0$, and $\underline{a}$ will be dominant if $x < \underline{x}$, i.e. if $\Delta(\underline{a}, x) - \Lambda(\underline{a}) < 0$. For $x \in [\underline{x}, \overline{x})$, there are multiple Nash equilibria. There is always a Nash (and subgame perfect) equilibrium where all agents choose $\overline{a}$, even if u is decreasing in a_t. If $\Delta(\underline{a}, x) < 0$, there is a Nash (and subgame perfect equilibrium) where all agents choose $\underline{a}$. The game is then analogous to Samuelson's (1958) model of intergenerational exchange. If $\Delta(\underline{a}, x) \geq 0$, there is a Nash (and subgame perfect) equilibrium where play follows $(\underline{a}, \overline{a}, \underline{a}, ...)$, one where play follows $(\underline{a}, \overline{a}, \overline{a}, \underline{a}, ...)$ and so on.

Now suppose that X_t evolves as a random walk, so $x_t = x_{t-1} + \eta\varepsilon_t$, where again ε_t has a full-support symmetric distribution G, and $\eta > 0$ is a constant. In the limit as $\eta \to 0$, $\overline{a}$ will be iteratively dominant for any x such that $\Delta(\overline{a}, x) + \frac{1}{2}\Lambda(\underline{a}) \geq 0$, and $\underline{a}$ will be iteratively dominant for any x such that $\Delta(\underline{a}, x) - \frac{1}{2}\Lambda(\underline{a}) < 0$. In the non-empty region $[\underline{x}^*, \overline{x}^*)$ where iterated dominance fails to uniquely pin down play, there will be multiple equilibria. So here adding a small amount of uncertainty about payoffs narrows the set of possible equilibria, but fails to uniquely identify an equilibrium.

5 References

Burdzy, Krzysztof, David Frankel and Ady Pauzner, 2001, "Fast Equilibrium Selection

by Rational Players Living in a Changing World," *Econometrica*, 68, 163–190.

Carlsson, Hans and Eric van Damme, 1993, "Global Games and Equilibrium Selection," *Econometrica*, 61, 989–1018.

Frankel, David, 2001, "Payoff Shocks and Equilibrium Selection with Endogenous Flexibility," mimeo.

Frankel, David, Stephen Morris and Ady Pauzner, 2000, "Equilibrium Selection in Global Games with Strategic Complementarities," mimeo.

Frankel, David and Ady Pauzner, 2000, "Resolving Indeterminacy in Dynamic Settings: The Role of Shocks," *Quarterly Journal of Economics*, 115, 283–304.

Lagunoff, R. and A. Matsui, 1997, "Asynchronous Choice in Repeated Coordination Games," *Econometrica*, 65, 1467–1477.

Matsui, A, 1999, "Multiple Investors and Currency Crises" mimeo.

Milgrom, Paul and John Roberts, 1990, "Rationalizability, Learning and Equilibrium in Games with Strategic Complementarities," *Econometrica*, 58, 1255–1277.

Morris, Stephen and Hyun Shin, 2001, "Global Games: Theory and Applications," mimeo.

Samuelson, Paul, 1958, "An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money, *Journal of Political Economy,* 66, 467–482.