

# Auctions with Severely Bounded Communication

Liad Blumrosen, Noam Nisan\*

and

Ilya Segal†

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\*Email: {liad,noam}@cs.huji.ac.il. The Hebrew University of Jerusalem, Jerusalem, Israel. Supported by grants from the Israeli Academy of Sciences. School of Engineering and Computer Science. Phone number: 972-2-6585503 (w) fax: 972-2-6585727

†Email: ilya.segal@stanford.edu. Department of Economics, Stanford University, Stanford, CA 94305. Supported by the National Science Foundation.

## Abstract

We study auctions with severe bounds on the communication allowed: each bidder may only transmit  $t$  bits of information to the auctioneer. We consider both welfare-maximizing and profit-maximizing auctions under this communication restriction. For both measures, we determine the optimal auction and show that the loss incurred relative to unconstrained auctions is mild. We prove non-surprising properties of these kinds of auctions, e.g., that in optimal mechanisms bidders simply report the interval in which their valuation lies in, as well as some surprising properties, e.g., that asymmetric auctions are better than symmetric ones and that multi round auctions reduce the communication complexity only by a linear factor.

**Keywords:** Auctions, Communication Complexity, Mechanism Design

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# 1 Introduction

Recent years have seen the emergence of the Internet as a platform of multifaceted economic interaction, from the technical level of computer communication, routing, storage, and computing, to the level of electronic commerce in its many forms. Studying such interactions raises new questions in economics that have to do with the necessity of taking computational considerations into account. This paper deals with one such question: how to design auctions optimally when we are restricted to use a very small amount of communication.

This paper studies the effect of severely restricting the amount of communication allowed in a single-item auction. Each bidder privately knows his real-valued willingness to pay for the item, but is only allowed to send  $k$  possible messages to the auctioneer, who must then allocate the item and determine the price on the basis of the messages received. (For example, a bidder may only be able to send  $t$  bits of information, in which case  $k = 2^t$ ). The simplest case is  $k = 2$ , i.e., each bidder sends a single bit of information. This is in contrast to the usual auction design formulation, in which bidders communicate real numbers.

While communicating a real number may not seem excessively burdensome, there are several motivations for studying auctions with such severe restrictions on the communication. First, if auctions are to be used for allocating low-level computing resources, they should use only a very small amount of computational effort. For example, an auction for routing a single packet on the Internet must require very little communication overhead, certainly not a whole real number. Ideally, one would like to “waste” only a bit or two on the bidding information, perhaps “piggy-backing” on some unused bits in the packet header of existing networking protocols (such as IP or TCP). Second, a restriction on communication may sometimes be viewed as a proxy for other simplicity considerations, such as simple user interface or small number of possible payments to facilitate their electronic handling. We find that single-item auctions may be very close to fully optimal despite the severe communication constraints. This is in contrast to *combinatorial* auctions, in which exact or even approximate efficiency is known to require an exponential amount of communication in the number of goods [15].<sup>1</sup>

We examine the effect of severe communication bounds on both the problem of maximizing social welfare and that of maximizing the seller’s expected profits (the latter under the restrictions of Bayesian incentive-compatibility and interim individual rationality of the bidders, and under a regularity condition on the distribution of bidders’ valuations). In both settings, we show that the

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<sup>1</sup>There have been several other studies considering various computational considerations in auction design: online behavior [8, 1], unbounded supply [3, 5, 1], computational complexity in combinatorial auctions [2, 9, 14, 17], timing uncertainty [16], and more. See also the surveys [13, 6].

optimal 2-bidder auction takes the simple form of a ‘‘priority game,’’ in which the player with the highest bid wins, but ties are broken asymmetrically among the players (i.e., some players have a pre-defined priority over the others when they send the same message). We show how to derive the optimal values for the parameters of the priority game. Furthermore, we show that for any number of players, as the allowed number of messages grows, the loss due to bounded communication is bounded above as  $O(\frac{1}{k^2})$  (i.e., the bound is smaller than  $c \cdot \frac{1}{k^2}$  for some constant  $c > 0$ ). The bound is tight for some distributions of valuations (e.g., for the uniform distribution). In addition, we consider the case in which the number of players grows while each player has exactly 2 possible messages. We show that priority games are optimal for this case as well, and we also characterize the parameters for the optimal mechanisms. We offer an asymptotic bound on the welfare and profit losses due to bounded communication as the number of players grows (it is  $O(\frac{1}{n})$  for the uniform distribution).

Our analysis implies some expected as well as some unexpected results:

- **Low welfare and profit loss:** Even severe bounds on communication result in only a mild loss of efficiency. For example, with two bidders whose valuations are uniformly distributed on  $[0, 1]$ , the optimal 1-bit auction brings expected welfare 0.648, compared to the first-best expected welfare 0.667.
- **Asymmetry helps:** Asymmetric auctions are better than symmetric ones with the same communication bounds. For example, with two bidders whose valuations are uniformly distributed in  $[0, 1]$ , symmetric 1-bit auctions only achieve expected welfare of 0.625, compared to 0.648 for asymmetric ones. We prove that both welfare- and profit-maximizing auctions must be discriminatory in both allocation and payments.
- **Threshold bidding is optimal:** We show that in the optimal auctions with  $k$  messages, bidders simply partition the range of valuations into  $k$  intervals ranges and announce their interval.
- **Dominant-strategy incentive-compatibility is achieved at no additional cost:** The auctions we design have dominant-strategy equilibria and are ex post individually rational, yet are optimal even without any incentive constraints (for welfare maximization), or among all Bayesian-Nash incentive-compatible and interim individually rational auctions (for profit maximization). This generalizes well-known results for the case without any communication constraints.

- **Symmetric mechanisms may also be close to optimal when allowed communication grows:** We show that as the number  $k$  of messages grows while the number of players is fixed, the loss in optimal symmetric mechanisms converges to zero at the same rate as the loss in optimal priority games. However, the loss in optimal priority games is still smaller by a factor. On the other hand, when we fix the number of messages, and let the number  $n$  of players grow, we show that the loss in optimal asymmetric mechanisms converges *asymptotically faster* to zero than in optimal symmetric mechanisms ( $O(\frac{1}{n})$  compared to  $O(\frac{\log n}{n})$ , for the uniform distribution).
- **Sequential mechanisms can do better, but only up to a linear factor:** Allowing players to send messages sequentially rather than simultaneously can achieve a higher payoff than in simultaneous mechanisms. However, the payoff in any such multi-round mechanism among  $n$  players can be achieved by a simultaneous mechanism in which the players send messages which are longer only by a factor of  $n$ . This result is surprising in light of the fact that in general the restriction to simultaneous communication can increase communication complexity exponentially.

The most closely related studies in the economic literature are [10], which considers similar questions in cases of restricting bid levels in oral auctions to discrete levels, and [20, 11], which analyzes the inefficiency caused by discrete priority classes of buyers. In particular, ([20]) shows that as the number  $k$  of priority classes grows, the efficiency loss is asymptotically proportional to  $\frac{1}{k^2}$ . While in [20] the buyers' aggregate demand is known while supply is uncertain, in our model the demand is uncertain as well. Both [20, 10] restrict attention to symmetric mechanisms, while we show that creating endogenous asymmetry among ex ante identical buyers is beneficial.

We start by presenting, in Section 2, a self-contained treatment of the simplest case: 2 bidders with uniformly distributed valuations, each allowed a single bit of communication. We continue with the general case: Section 3 provides the model definition and introduces our notations, Section 4 presents a characterization of the welfare optimal and profit optimal 2-player auctions. Section 5 characterizes optimal mechanisms with arbitrary number of bidders, but 2 possible bids for each player. In Section 6 we give an asymptotic analysis of the minimal welfare and profit losses in the optimal mechanisms and in other, simpler, mechanisms. Finally, Section 7 compares simultaneous and sequential mechanisms with bounded communication.

## 2 2-player Auctions with 2 Possible Bids

We start with a description of the simplest case: auctions among 2 players where every player can send only a single bit to the auctioneer, and the valuations are distributed uniformly. With this simple case, we demonstrate the properties of the optimal solution in the general case, when we allow any number of possible bids or any number of players.

### 2.1 The Simple Model

Consider single item auctions among risk-neutral players, with statistically independent private values, and quasi-linear utilities. In this section, we assume players' valuations are distributed uniformly in  $[0, 1]$  and that the seller's valuation for the item is 0.

Our unique assumption is that players can send only 1 bit messages to the auctioneer, i.e., they have only two possible bids to choose from. Such mechanisms can be described with a  $2 \times 2$  matrix, where the 1st player (Alice) chooses a row, and the 2nd (Bob) chooses a column. Each entry of the matrix specifies the allocation and payments given a bids' combination. The mechanism can toss coins to determine the allocations. Figures 1 and 2 depict examples for *2-player 1-bit* mechanisms.

A *Strategy*  $s_i$  for player  $i$  is a function  $s_i : [0, 1] \rightarrow \{0, 1\}$  which determines the bid of player  $i$  according to his valuation  $v_i$ .

Each selfish bidder wants to maximize her expected utility. As the mechanism's designers, we will try to optimize "social" criteria such as the expected *welfare* and the expected seller *profit*. The *expected welfare* (or efficiency) achieved by a mechanism is the expected valuation of the player that wins the item (if any). The *expected profit* from a mechanism is the expected sum of bidders' payments<sup>2</sup>.

### 2.2 Welfare and Profit in Simple Mechanisms

Let  $g_1$  denote the mechanism described in Figure 1 and  $g_2$  denote the mechanism in Figure 2.

We first note that both  $g_1$  and  $g_2$  have **dominant-strategy equilibria**. Consider the following strategy: "*bid 1 if your valuation is greater than  $\frac{1}{3}$ , else bid 0*". Clearly, this strategy is dominant for player  $A$  in  $g_1$ : when her valuation is smaller than  $\frac{1}{3}$  she will gain a negative utility if she bids "1"; When her valuation is greater than  $\frac{1}{3}$ , bidding "0" gives her a utility of zero, but she can get positive utility by bidding "1". We call this kind of strategies *threshold strategies*. Similarly, a threshold strategy with the threshold  $\frac{2}{3}$  is dominant for player  $B$  in  $g_1$ . The threshold strategies with the values  $\frac{1}{2}, \frac{5}{8}$  are dominant for  $A, B$ , respectively, in  $g_2$ .

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<sup>2</sup>In this section, the seller's valuation for the item is zero.

	$B$		
$A$		0	1
0		$B$ wins and pays 0	$B$ wins and pays 0
1		$A$ wins and pays $\frac{1}{3}$	$B$ wins and pays $\frac{2}{3}$

Figure 1: ( $g_1$ ) A 2-player 1-bit game that achieves maximal expected welfare (efficiency). For example, when Alice (the row player) bids “1” and Bob bids “0”, Alice wins and pays  $\frac{1}{3}$

	$B$		
$A$		0	1
0		No allocation	$B$ wins and pays $\frac{2}{3}$
1		$A$ wins and pays $\frac{1}{2}$	$B$ wins and pays $\frac{2}{3}$

Figure 2: ( $g_2$ ) A 2-player 1-bit game that achieves maximal expected profit

Next, we calculate the expected welfare and profit, in  $g_1$  and  $g_2$ , respectively. The expected welfare from  $g_1$ , when the players play according to their dominant strategies, is  $\frac{35}{54} = 0.648$ :

$$\frac{1}{3} \frac{2}{3} \frac{2}{2} + \frac{1}{3} \left(1 - \frac{2}{3}\right) \frac{\left(1 + \frac{2}{3}\right)}{2} + \left(1 - \frac{1}{3}\right) \frac{2}{3} \frac{\left(1 + \frac{1}{3}\right)}{2} + \left(1 - \frac{1}{3}\right) \left(1 - \frac{2}{3}\right) \frac{\left(1 + \frac{2}{3}\right)}{2} = \frac{35}{54}$$

It turns out that  $g_1$  is efficient, i.e., no other mechanism has an equilibrium that achieves strictly higher welfare than  $\frac{35}{54}$ . Actually, we prove a stronger statement: any mechanism, with any profile of strategies (not necessarily an equilibrium), achieves an expected welfare which is not greater than  $\frac{35}{54}$ . The efficiency of  $g_1$  is shown by 3 steps we sketch below (this result is generalized later in the paper, in Theorem 2):

Step 1: We show that every mechanism  $g$  can achieve its optimal welfare with a pair of *threshold strategies*, i.e., there exists a pair of threshold strategies such that no other strategies achieve strictly greater expected welfare in  $g$ .

Step 2: Consider mechanisms in which the item is allocated to the player with the highest bid, and in case of equal bids, the item is always allocated using a pre-defined order on the players. We call this family of mechanisms *priority games*. (For example,  $g_1$  is a priority game: we always break ties in favor of  $B$ .) Allocating the item to the player with the highest expected valuation shows that priority games achieve optimal welfare (with some strategies).

Step 3: Due to the previous steps, optimal welfare can be achieved in priority games with threshold strategies. Thus, we can express the expected welfare in a priority game as a function of the threshold values  $x, y$  that the players use:

$$w(g, x, y) = xy \frac{x}{2} + x(1 - y) \frac{1 + y}{2} + (1 - x)y \frac{1 + x}{2} + (1 - x)(1 - y) \frac{1 + y}{2}$$

This function achieves a unique maximum  $(x, y \in [0, 1])$  when  $(x, y) = (\frac{1}{3}, \frac{2}{3})$ . Thus,  $g_1$  is efficient.

Recall that with no communication limitations, the optimal welfare is  $\frac{2}{3} = \frac{36}{54}$  (e.g., 2nd-price auction, see [19]). We surprisingly see that despite severely limiting the communication from infinitely many bits to a single bit, the welfare loss is relatively mild (only  $\frac{1}{54}$ ).

The expected profit achieved by the dominant-strategy equilibrium in  $g_2$  is  $\frac{25}{64} = 0.39$ :

$$0 + \frac{1}{2}\left(1 - \frac{5}{8}\right)\frac{5}{8} + \left(1 - \frac{1}{2}\right)\frac{5}{8}\frac{1}{2} + \left(1 - \frac{1}{2}\right)\left(1 - \frac{5}{8}\right)\frac{5}{8} = \frac{25}{64}$$

Similarly, we can show that  $g_2$  achieves optimal profit, i.e., no other mechanism has a Bayesian-Nash equilibrium, with interim individual-rationality, that achieves an expected profit greater than 0.39. The optimal profit in mechanisms with unbounded communication is  $\frac{5}{12}$  (see, e.g., [12]), so the optimal profit loss from limiting the communication to 1 bit is less than 0.03. Note that in the profit-maximizing auction, the seller sometimes keeps the item for himself. This is similar to a reservation price in second price auctions (see, e.g., [12]).

The welfare optimal and the profit optimal mechanisms ( $g_1$  and  $g_2$ ), demonstrate some important properties of the optimal mechanisms in the general case:

- Both the optimal welfare and the optimal profit are achieved when the players use threshold strategies. Thus, the set of strategies for each player reduces to the set of real numbers in  $[0, 1]$ , instead of the set of functions  $f : [0, 1] \rightarrow \{0, 1\}$ .
- The mechanism  $g_1$  achieves the maximal welfare achievable by any 1-bit mechanism and any pair of strategies, without restrictions to any kind of equilibria. Nevertheless, we show a game with an ex-post individually-rational dominant-strategy equilibrium that achieves this welfare.  $g_2$  achieves maximal profit among all the profits achievable in *interim individually-rational Bayesian-Nash* equilibria of 1-bit mechanisms. Yet,  $g_2$  achieves this optimal profit with a *dominant-strategy* equilibrium and *ex-post* IR.
- The welfare-maximizing threshold values  $(x, y) = (\frac{1}{3}, \frac{2}{3})$  are what we call *mutually-centered*, i.e.,  $x$  is the expected valuation of **B** given that  $B$  bids 0 (i.e.,  $x = E(v_B | 0 \leq v_B \leq y) = \frac{0+y}{2}$ ), and  $y$  is the expected valuation of **A** given that  $A$  bids 1 (i.e.,  $y = E(v_A | x \leq v_A \leq 1) = \frac{x+1}{2}$ ).
- The optimal mechanisms are asymmetric, even though the players are identical (priority games are asymmetric by definition). Actually, symmetric mechanisms can achieve a strictly smaller expected welfare. Consider the mechanisms  $g_3$  and  $g_4$  described in Figure 3 and 4 respectively.  $g_3$  achieves the optimal welfare among all the symmetric 1-bit mechanisms<sup>3</sup> and

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<sup>3</sup>To see that, note that a symmetric, efficient mechanism will clearly allocate the item to the player that bids 1

	0	1
0	w.p. $\frac{1}{2}$ $A$ wins, pays 0 w.p. $\frac{1}{2}$ $B$ wins, pays 0	$B$ wins and pays $\frac{1}{4}$
1	$A$ wins and pays $\frac{1}{4}$	w.p. $\frac{1}{2}$ $A$ wins, pays $\frac{1}{2}$ w.p. $\frac{1}{2}$ $B$ wins, pays $\frac{1}{2}$

Figure 3: ( $g_3$ ) 2-player 1-bit symmetric mechanism that achieves optimal welfare

	0	1
0	No allocation	$B$ wins and pays $\frac{1}{\sqrt{3}}$
1	$A$ wins, pays $\frac{1}{\sqrt{3}}$	w.p. $\frac{1}{2}$ $A$ wins, pays $\frac{1}{\sqrt{3}}$ w.p. $\frac{1}{2}$ $B$ wins, pays $\frac{1}{\sqrt{3}}$

Figure 4: ( $g_4$ ) 2-player 1-bit symmetric mechanism that achieves optimal profit

$g_4$  achieves optimal profit among all the symmetric 1-bit, individually-rational mechanisms<sup>4</sup>.  $g_3$  achieves an expected welfare of 0.625, which is smaller than the expected welfare of 0.648 that can be achieved in an asymmetric 1-bit mechanism. Similarly,  $g_4$  achieves an expected profit of 0.385 where asymmetric mechanisms can achieve 0.39.

- When optimizing profit, the (interim) individual-rationality assumption plays an important role. We could alternatively assume (the non-common) ex-ante IR, where the players participate only if their *expected* utility is non-negative. In this case, we could extract the whole surplus from the players (thus, the optimal profit would be  $\frac{35}{54}$ ).
- When the seller has a non-zero valuation for the item (opportunity cost), the welfare-optimal mechanism is not necessarily  $g_1$ . In such cases, the seller may choose to keep the item when her valuation seems to be higher than the bidders' valuations. For example, when the seller's valuation is  $v_0 = 0.4$ , the mechanism  $g_2$  achieves greater expected **welfare** than  $g_1$ .

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when the other player bids 0, and allocate with equal probabilities of  $\frac{1}{2}$  when the bids are equal. With threshold strategies  $(x, y)$  the expected welfare is:

$$w(x, y) = x \cdot y \cdot \left( \frac{1}{2} \cdot \frac{x}{2} + \frac{1}{2} \cdot \frac{y}{2} \right) + x \cdot (1 - y) \cdot \frac{(1+y)}{2} + (1 - x) \cdot y \cdot \frac{(1+x)}{2} + (1 - x) \cdot (1 - y) \cdot \left( \frac{1}{2} \cdot \frac{(1+x)}{2a} + \frac{1}{2} \cdot \frac{(1+y)}{2} \right)$$

Maximum is achieved when  $(x, y) = (\frac{1}{2}, \frac{1}{2})$ . With the given payments, these threshold strategies are dominant with ex-post Individual-Rationality.

<sup>4</sup>Similarly to previous proofs, in the profit-maximizing symmetric mechanism if a player bids 0 and the other bids 1, the latter wins and pays  $a$ . When both players bid 1, they will pay  $\bar{a}$  with equal probabilities. It is easy to see that under the ex-post IR assumption,  $a = \bar{a}$ . The expected profit is thus:  $r(a) = a(1 - a)a + (1 - a)aa + (1 - a)(1 - a)(\frac{1}{2}a + \frac{1}{2}a)$ . Maximum is achieved ( $a \in [0, 1]$ ) when  $a = \frac{1}{\sqrt{3}}$ .

### 3 The General Model

#### 3.1 The Players and the Mechanism

We consider single item, sealed bid auctions among  $n$  risk-neutral players. Player  $i$  has a private valuation for the object  $v_i \in [\underline{a}, \bar{b}]$ . (For simplicity, we use the range  $[0, 1]$  in some parts of the paper.) The valuations are independently drawn from a cumulative probability function  $F_i$ . In some parts of our analysis<sup>5</sup>, we assume the existence of a probability density function  $f_i$ . We will sometime treat the seller as one of the players, numbered 0. The seller has a constant valuation  $v_0$  for the item. We assume a normalized model, i.e., players' valuations for not having the item are  $\underline{a}$ . Players want to maximize their utilities, which are *quasi-linear*. We also assume that the utilities of the players depend only on whether they win the item or not (no externalities).

In our model, each player  $i$  can send a message of  $t_i = \lg(k_i)$  **bits** to the mechanism, i.e., player  $i$  can choose one of possible  $k_i$  **bids** (or messages). Denote the possible set of bids for player  $i$  as  $\beta_i = \{0, 1, 2, \dots, k_i - 1\}$ . In each auction, player  $i$  chooses a bid  $b_i \in \beta_i$ . Let  $b = (b_1, \dots, b_n)$  be a vector of bids. A mechanism should determine the allocation and payments given a vector of bids  $b$ :

**Definition 1.** A *mechanism*  $g$  is composed of a pair  $(a, p)$  where:

- $a : (\beta_1 \times \dots \times \beta_n) \rightarrow [0, 1]^{n+1}$  is the allocation scheme (not necessarily deterministic). We denote the  $i$ 'th coordinate of  $a(b)$  by  $a_i(b)$ , which is player  $i$ 's probability for winning the item when the bidders bid  $b$ . Clearly,  $\forall i \forall b a_i(b) \geq 0$  and  $\forall b \sum_{i=0}^n a_i(b) = 1$ . If  $a_0(b) > 0$ , the seller will keep the item with a positive probability.
- $p : (\beta_1 \times \dots \times \beta_n) \rightarrow \mathfrak{R}^n$  is the payment scheme.  $p_i(b)$  is the payment of the  $i$ th player given a bids' vector  $b$ . (For convenience, we define  $p_0(b) = 1$  for every  $b$ .)<sup>6</sup>

**Definition 2.** In a *mechanism with  $k$ -possible bids*, for every player  $i$ ,  $|\beta_i| = k_i = k$ . We denote the set of all the mechanisms with  $k$ -possible bids among  $n$  players by  $G_{n,k}$ . We denote the set of all the  $n$ -player mechanisms in which  $|\beta_i| = k_i$  for each player  $i$ , by  $G_{n,(k_1, \dots, k_n)}$ .

Next, we define the notion of a strategy for a player, and show how players choose their strategies.

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<sup>5</sup>That is, in the characterization of the optimal mechanisms in Sections 4.2 and 5 and when using the concept of *virtual valuation* in Sections 4.3 and 6.2

<sup>6</sup>Note that we allow non-deterministic allocations, but we ignore non-deterministic payments (since we are interested in expected values, using lottery for the payments has no effect on our results).

**Definition 3.** A Strategy  $s_i$  for player  $i$  in a game  $g \in G_{n,(k_1,\dots,k_n)}$  describes how a player determines his bid according to his valuation, i.e., it is a function  $s_i : [\underline{a}, \bar{b}] \rightarrow \{0, 1, \dots, k_i - 1\}$ .

Denote  $\varphi_{k_i} = \{s \mid s : [\underline{a}, \bar{b}] \rightarrow \{0, 1, \dots, k_i - 1\}\}$  (i.e., the set of all strategies for players with  $k_i$  possible bids).

**Definition 4.** A real vector  $c = (c_0, c_1, \dots, c_k)$  is a *vector of threshold values* if  $c_0 \leq c_1 \leq \dots \leq c_k$ .

**Definition 5.** A strategy  $s_i \in \varphi_{k_i}$  is a *threshold strategy based on a vector of threshold values*  $c = (c_0, c_1, \dots, c_k)$ , if for every bid  $j \in \{0, \dots, k_i - 1\}$  and for every valuation  $v_i \in [c_j, c_{j+1})$ , player  $i$  bids  $j$  when his valuation is  $v_i$ , i.e.,  $s_i(v_i) = j$  (and for every  $v_i, v_i \in [c_0, c_k]$ ). We say that  $s_i$  is a *threshold strategy*, if there exists a vector  $c$  of threshold values such that  $s_i$  is a threshold strategy based on  $c$ .

We use the notations:  $s(v) = (s_1(v_1), \dots, s_n(v_n))$ , when  $s_i$  is a strategy for bidder  $i$ ,  $s = (s_1, \dots, s_n)$  and  $v = (v_1, \dots, v_n)$ . Note that  $b = s(v)$  is a vector of bids. Let  $s_{-i}$  denote the strategies of the players except  $i$ , i.e.,  $s_{-i} = (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$ . We sometimes use the notation  $s = (s_i, s_{-i})$ .

### 3.2 Optimality Criteria

The players in our model choose strategies that maximize their utilities. We are interested in games where such strategies form equilibria.

**Definition 6.** Let  $u_i(g, s)$  be the expected utility of player  $i$  from a game  $g$  when bidders use strategies  $s$ , i.e.,

$$u_i(g, s) = E_{v \in [\underline{a}, \bar{b}]^n} a_i(s(v)) \cdot (v_i - p_i(s(v)))$$

**Definition 7.** A strategy  $s_i$  for player  $i$  is *dominant* in a mechanism  $g \in G_{n,(k_1,\dots,k_n)}$  if regardless of the other players' strategies  $s_{-i}$ ,  $i$  cannot gain a higher utility by changing his strategy, i.e.,

$$\forall \tilde{s}_i \in \varphi_{k_i} \quad \forall s_{-i} \quad u_i(g, (s_i, s_{-i})) \geq u_i(g, (\tilde{s}_i, s_{-i}))$$

We say that a mechanism  $g$  has a *dominant-strategy equilibrium* if for every player  $i$  there exists a strategy  $s_i$  which is a dominant.

**Definition 8.** A profile of strategies  $s = (s_1, \dots, s_n)$  forms a *Bayesian-Nash equilibrium (BNE)* in a mechanism  $g \in G_{n,(k_1,\dots,k_n)}$ , if for every player  $i$ ,  $s_i$  is the best response for the strategies  $s_{-i}$  of the other players, i.e.,

$$\forall i \quad \forall \tilde{s}_i \quad u_i(g, (s_i, s_{-i})) \geq u_i(g, (\tilde{s}_i, s_{-i}))$$

We use standard participation constraints definitions:

We say that a profile of strategies  $s = (s_1, \dots, s_n)$  is *ex-post individually-rational* in a mechanism  $g$ , if every player never pays more than his actual valuation (for any realization of the valuations).

We say that a strategies profile  $s = (s_1, \dots, s_n)$  is *interim Individually Rational* in a mechanism  $g$  if every player  $i$  achieves a non-negative **expected** utility, given any valuation he might have, when the other players play with  $s_{-i}$ .

Our goal is to find optimal, communication-bounded mechanisms. Each selfish bidder wants to maximize her expected utility. As the mechanism designers, we will try to optimize “social” criteria such as *welfare* (efficiency) and the seller’s *profit*.

The *expected welfare* from a mechanism  $g$ , when bidders use the strategies  $s$ , is the expected social surplus. Because the item is indivisible, the social surplus is actually the valuation of the player that receives the item<sup>7</sup>.

**Definition 9.** Let  $w(g, s)$  denote the *expected welfare* (or expected efficiency) in the  $n$ -player game  $g$  when the bidders’ strategies are  $s$ , i.e.,

$$w(g, s) = E_{v \in [\underline{a}, \bar{b}]^n} \left( \sum_{i=0}^n a_i(s(v)) \cdot v_i \right)$$

and let  $w_{n, (k_1, \dots, k_n)}^{opt}$  denote the maximal possible expected welfare from any  $n$ -player game where each player  $i$  has  $k_i$  possible bids, with any vector of strategies allowed, i.e.,

$$w_{n, (k_1, \dots, k_n)}^{opt} = \max_{g \in G_{n, (k_1, \dots, k_n)}, s \in \varphi_{k_1} \times \dots \times \varphi_{k_n}} w(g, s)$$

We denote the optimal welfare achievable in games where all players have  $k$  possible bids by  $w_{n, k}^{opt} = w_{n, (k, \dots, k)}$

**Definition 10.** Let  $r(g, s)$  denote the *expected profit* in the  $n$ -player game  $g$  where the bidders’ strategies are  $s$ , i.e.,

$$r(g, s) = E_{v \in [\underline{a}, \bar{b}]^n} \left( \sum_{i=0}^n a_i(s(v)) \cdot p_i(s(v)) \right)$$

and let  $r_{n, k}^{opt}$  denote the maximal expected profit from an  $n$ -player mechanism with  $k$  possible bids and some vector of interim individually-rational strategies  $s$  that forms a Bayesian-Nash equilibrium

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<sup>7</sup>Note that the expected welfare does not directly depend on the payments in the mechanism.

in  $g$ :

$$r_{n,k}^{opt} = \max_{g \in G_{n,k}} r(g, s)$$

$s \in \times_{i=1}^n \varphi_k$  is interim IR and in BNE in  $g$

When  $v_0 = 0$ , the expected profit is equivalent to the seller's *revenue*.

Note that we define the optimal welfare as the maximal welfare among all mechanisms and strategies, not necessarily in equilibria, and we define the optimal profit as the maximal profit achievable in interim-IR Bayesian-Nash equilibria in any mechanism. Yet, the optimal mechanisms (for both measures) that we present in this paper achieve these optimal values with dominant-strategy equilibria and ex-post IR.

**Definition 11.** We say that a mechanism  $g \in G_{n,k}$  achieves an *expected welfare (resp. profit)* of  $\alpha$ , if  $g$  has an interim-IR Bayesian-Nash equilibrium  $s$  for which the expected welfare (**resp. profit**) is  $\alpha$ , i.e.,  $w(g, s) = \alpha$  ( $\mathbf{r}(g, s) = \alpha$ ).

We say that a mechanism  $g \in G_{n,k}$  incurs a *welfare loss (resp. profit loss)* of  $c$ , if it achieves an expected welfare (**resp. profit**) which is smaller than the optimal welfare (**resp. profit**) with *unbounded communication* by  $c$  (the optimal results with unbounded communications are the best results achievable with interim-IR Bayesian-Nash equilibria).

## 4 Optimal mechanisms for 2 players

In this section we present 2-player mechanisms with bounded communication that achieve optimal welfare and profit. In Section 5 we will present the characterization of the welfare-optimal and profit-optimal  $n$ -player mechanisms with 2 possible bids for each player. The characterization of the optimal mechanisms in the most general case ( $n$  players and  $k$  possible bids) remains an open question. Anyway, our asymptotic analysis of the welfare loss and the profit loss (in Section 6) holds for the general case, and shows *asymptotically* optimal mechanisms.

**Definition 12.** A game is called a *priority game* if it allocates the item to the player  $i$  that bids the highest bid (i.e., when  $b_i > b_j$  for all  $j \neq i$ , the allocation is  $a_i(b) = 1$  and  $a_j(b) = 0$  for  $j \neq i$ ), with ties consistently broken according to a pre-defined order on the players.

**Definition 13.** A game is called a *modified priority game* if it has an allocation as of a priority game, except when all players bid 0, the seller keeps the item.

The term *Priority Games* means that the allocation rule is asymmetric with the players: some players have priority over the others. This is done with an asymmetric, consistent tie breaking rule.

For example, the player with the highest priority will win the item whenever his bid is the highest (even if other players bid similarly). The player with the lowest priority, however, wins the item only when his bid is strictly higher than all other bids.

It is sometimes convenient to look at our model as if it treats the seller as one of the players, with the lowest priority. Then, modified priority games are actually priority games, where the seller always bids his second lowest bid (i.e., “1”). (in “simple” priority games, the seller always bids “0”).

It turns out to be useful, to build the payment scheme of such mechanisms according to a given profile of threshold strategies:

**Definition 14.** An  $n$ -player *priority game based on a profile of threshold values’ vectors*  $\vec{t} = (t^1, \dots, t^n) \in \times_{i=1}^n \mathbb{R}^{k+1}$  (where for every  $i$ ,  $t_0^i \leq t_1^i \leq \dots \leq t_k^i$ ) is a mechanism that its allocation is of a priority game and its payment scheme is as follows: when player  $j$  wins the item for a bids’ vector  $b$  she pays the smallest valuation she might have and still win the item, given that she uses the threshold strategy  $s_j$  based on  $t^j$ , i.e,  $p_j(b) = \min\{v_j | a_j(s_j(v_j), b_{-j}) = 1\}$ . We denote this mechanism as  $PG_k(\vec{t})$ . A modified priority game with a similar payment rule is called a *modified priority game based on a profile of threshold-value vectors*, and is denoted by  $MPG_k(\vec{t})$ .

For 2-player games, we may use the notations  $PG_k(x, y)$ ,  $MPG_k(x, y)$  (where  $x, y$  are some vectors of threshold values). The mechanisms  $PG_k(x, y)$  and  $MPG_k(x, y)$  are presented in Figures 5 and 6, respectively. Note that the only difference in payments between  $PG_k(x, y)$  and  $MPG_k(x, y)$  is when player  $A$  bids “0” and  $B$  wins (i.e., the first line of the game’s matrix).

A well known result in mechanism design states that for any monotone allocation rule there is *some* transfer (i.e., payment) rule that would implement the desired allocation with a dominant-strategy equilibrium. We observe that monotone mechanisms in our model reveal enough information, despite the communication constraints, to find such transfer rule. It follows that the threshold strategies based on some threshold values vector  $\vec{t}$  are dominant in both  $PG_k(\vec{t})$  and  $MPG_k(\vec{t})$ .

**Proposition 1.** *In any quasilinear environment, if a nonmonetary allocation rule is supportable in dominant strategies with some transfers, then any communication<sup>8</sup> realizing this rule also reveals enough information to construct supporting transfers.*

*Proof.* (sketch) In direct revelation mechanisms (i.e., with unbounded communication), if the allocation rule proves to be monotonic, we can find transfers that support a dominant-strategy equilibrium. The transfers will change according to some allocation-dependent thresholds, e.g.,

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<sup>8</sup>Here we deal with simultaneous communication, i.e., where all players send their messages simultaneously. This proposition is not true for sequential mechanisms (see Section 7).

	0	1	2	..	k-2	k-1
0	$\mathbf{B}, y_0$	$\mathbf{B}, y_0$	$\mathbf{B}, y_0$	...	$\mathbf{B}, y_0$	$\mathbf{B}, y_0$
1	$A, x_1$	$\mathbf{B}, y_1$	$\mathbf{B}, y_1$	...	$\mathbf{B}, y_1$	$\mathbf{B}, y_1$
2	$A, x_1$	$A, x_2$	$\mathbf{B}, y_2$	...	$\mathbf{B}, y_2$	$\mathbf{B}, y_2$
...	...	...	...	...	...	...
k-2	$A, x_1$	$A, x_2$	$A, x_3$	...	$\mathbf{B}, y_{k-2}$	$\mathbf{B}, y_{k-2}$
k-1	$A, x_1$	$A, x_2$	$A, x_3$	...	$A, x_{k-1}$	$\mathbf{B}, y_{k-1}$

Figure 5: A priority game based on the threshold values  $x, y$ . In each entry, the left argument denotes the winning player, and the right argument is the price she pays. When  $x, y$  are mutually-centered and  $v_0 = 0$ , this mechanism achieves optimal welfare, among all the mechanisms and all possible-strategies.

	0	1	2	..	k-2	k-1
0	$\phi$	$\mathbf{B}, y_1$	$\mathbf{B}, y_1$	...	$\mathbf{B}, y_1$	$\mathbf{B}, y_1$
1	$A, x_1$	$\mathbf{B}, y_1$	$\mathbf{B}, y_1$	...	$\mathbf{B}, y_1$	$\mathbf{B}, y_1$
2	$A, x_1$	$A, x_2$	$\mathbf{B}, y_2$	...	$\mathbf{B}, y_2$	$\mathbf{B}, y_2$
...	...	...	...	...	...	...
k-2	$A, x_1$	$A, x_2$	$A, x_3$	...	$\mathbf{B}, y_{k-2}$	$\mathbf{B}, y_{k-2}$
k-1	$A, x_1$	$A, x_2$	$A, x_3$	...	$A, x_{k-1}$	$\mathbf{B}, y_{k-1}$

Figure 6: A modified priority game based on the threshold strategies  $x, y$ . In each entry, the left argument denotes the winning player, and the right argument is the price she pays. For optimally chosen values of  $x, y$ , this mechanism (or a priority game) achieves optimal profit.

for a deterministic allocation rule each player pays the smallest valuation for which she still wins. Any monotonic allocation rule in *bounded communication* mechanisms, can be viewed as a direct revelation mechanism for which we can find these supporting transfers. The supporting transfers depend on the changes in the allocation rule as the valuation of each player increases, so the transfers change as the allocation rule changes. Thus, with the same communication we can reveal the transfers that support a dominant-strategy equilibrium.  $\square$

#### 4.1 The Efficiency of Priority Games

Towards a characterization of the welfare-optimal mechanism, we will first show that the allocation scheme in 2-player priority games is optimal<sup>9</sup>. For the proof, we use a 2-step approach which is standard in Mechanism Design theory: first solve the optimal allocation rule, and then construct transfers that satisfy the desired incentive-compatibility and individual-rationality constraints. Interestingly, the same approach holds for communicationally bounded mechanisms.

**Definition 15.** A mechanism  $g \in G_{n,k}$  is *monotone* if for any bids' vector  $b$  and for any player  $i$ ,

<sup>9</sup>We assume (w.l.o.g) throughout this paper that in 2-player priority games  $B \succ A$ , i.e., the mechanism allocates the item to  $A$  if she bids a higher bid than  $B$ , and otherwise to  $B$

the probability that player  $i$  wins the item cannot decrease, when only his bid is increased, i.e.,

$$\forall b \quad \forall i \quad \forall b'_i > b_i \quad a_i(b_i, b_{-i}) \leq a_i(b'_i, b_{-i})$$

**Theorem 1. (Priority games' efficiency)** For any pair of distribution functions of the players' valuations, and for any  $v_0$ , the optimal welfare (i.e.,  $w_{2,k}^{opt}$ ) is achieved in either a priority game or a modified priority game (with some pair of threshold strategies).

*Proof.* We first prove that when  $v_0 \leq \underline{a}$ , the optimal mechanism is a priority game. We prove this using the following four claims: The first 3 claims hold for the general case of  $n$ -player mechanisms with  $k$  possible bids. We first show that the optimal welfare can always be achieved with threshold strategies. Then we show that this optimal welfare is achieved in deterministic mechanisms (in which the seller never keeps the item), and that these mechanisms are monotone. Finally, we show that for 2-player mechanisms, combinatorial constraints derive that the optimal mechanisms are priority games.

**Claim 1. (Optimality of threshold strategies)** Given any mechanism  $g \in G_{n,(k_1,\dots,k_n)}$ , there exists a vector of threshold strategies  $s \in \times_{i=1}^n \varphi_{k_i}$  that achieve the optimal welfare in  $g$  among all possible strategies, i.e.,  $w(g, s) = \max_{\tilde{s} \in \times_{i=1}^n \varphi_{k_i}} w(g, \tilde{s})$

*Proof.* The proof idea (full proof is given in Appendix A.1): We can modify any welfare-optimizing profile of strategies to be a profile of threshold strategies in a way that the expected welfare will not decrease. We first observe that for every bid that player  $i$  might bid, the expected welfare is a linear function of his valuation  $v_i$  (given the strategies of the other players). The intersection points of these linear functions are the optimal thresholds.  $\square$

**Claim 2. (Optimality of deterministic mechanisms)** For every numbers of possible bids  $k_1, \dots, k_n$  for the players, there exists a mechanism  $g \in G_{n,(k_1,\dots,k_n)}$  that achieves optimal welfare (i.e.,  $w_{n,(k_1,\dots,k_n)}^{opt}$ ) which is deterministic (i.e., the winner is fixed for each combination of bids) and in which the item must be sold to one of the bidders (for every bids' combination).

*Proof.* Let  $g \in G_{n,(k_1,\dots,k_n)}$  be a mechanism that achieves the optimal welfare with a profile of strategies  $s$  (i.e.,  $w(g, s) = w_{n,(k_1,\dots,k_n)}^{opt}$ ). We will modify the allocation scheme of  $g$  to be as required, such that the welfare it achieves with  $s$  does not decrease. For each bids' combination  $b = (b_1, \dots, b_n)$  we will always allocate the item to a player  $i$  with the highest expected valuation, given that the players bid  $b$  (i.e., for some  $i \in \operatorname{argmax}_j (E(v_j | s_j(v_j) = b_j))$  we have  $a_i(b) = 1$ ). This way, we always allocate the item (as long as we assume that  $v_0 \leq \underline{a}$  the seller's valuation is

not greater than any bidder's valuations), and the allocation is deterministic. Since we allocate the item to the player with the highest expected welfare each time, the expected welfare (achieved with  $s$ ) cannot decrease.  $\square$

*Claim 3. (Optimality of monotone mechanisms)* Every mechanism can be modified to be *monotone* and the expected welfare it achieves with a given pair of strategies will not decrease.

*Proof.* Let  $g \in G_{n,(k_1,\dots,k_n)}$  be a mechanism, and  $s$  be a vector of strategies for the players in  $g$ . We can assume, w.l.o.g, that for each bidder  $i$  the bids' names (i.e., "0", "1" etc.) are ordered by the expected welfare (i.e., if  $m > l$  then  $E(v_i|s_i(v_i) = m) \geq E(v_i|s_i(v_i) = l)$ )<sup>10</sup>. Then, if we modify the mechanism as in the proof for Claim 2 above, the mechanism will be not only deterministic in which the item must be sold, but also monotone: given a bids' vector for which player  $i$  has a maximal expected valuation among all players, if he increases his bid, his expected valuation will also increase, and the expected welfare of all the other players will not change, and thus he will still have the maximal expected valuation.  $\square$

*Claim 4.* Consider the matrix representation of a 2-player game with  $k$  possible bids. In a deterministic, monotone mechanism in which the item must be sold, that achieves optimal expected welfare, no two rows (or columns) have an identical allocation scheme.

*Proof.* Consider such an optimal mechanism  $g \in G_{2,k}$  with two identical rows.  $g$ 's monotonicity derives that these rows are adjacent. Thus, there is a mechanism  $\tilde{g} \in G_{2,(k-1,k)}$  that achieves exactly the same expected welfare as  $g$  (when the identical rows are united to one). Thus, this claim is a direct corollary of a claim we prove in Appendix 3. According to this claim, the optimal welfare from a game where both players have  $k$  possible bids cannot be achieved when one of the players has only  $k - 1$  possible bids (i.e.,  $w_{2,k}^{opt} > w_{2,(k-1,k)}^{opt}$ ).  $\square$

Now, due to Claim 2 and Claim 3, there is a deterministic, monotone game in which the item must be sold that achieves  $w_{2,k}^{opt}$ . In such games, the allocation scheme in some row  $i$  looks like  $[A, \dots, A, B \dots B]$ . Due to Claim 4, in the matrix representation of this optimal game, there are no two rows with the same allocation scheme. There are  $k + 1$  possible rows for the game matrix, but our mechanism has only  $k$  rows. Similarly, we have  $k$  different columns (of possible  $k + 1$ ) in the mechanism. Assume that the row  $[B, B, \dots, B]$  is in  $g$ . Then, the column  $[A, A, \dots, A]$  is clearly not in  $g$ . Therefore, our game matrix consists of all the columns except  $[A, A, \dots, A]$ , which compose

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<sup>10</sup>For threshold strategies, for example, it means that the threshold values are ordered from lowest to highest.

the priority game where  $B \succ A$ . If the row  $[B, B, \dots, B]$  is not in  $g$ , then  $g$  is the priority game where  $A \succ B$ .

Next, we complete the proof for any seller's valuation  $v_0$ . Consider a mechanism  $h \in G_{2,k}$  and a pair of threshold strategies based on some threshold-value vectors  $\tilde{x}, \tilde{y}$  that achieve optimal welfare among all mechanisms and strategies (due to Claim 1, such strategies exist). We will modify  $h$ , such that the expected welfare (with  $\tilde{x}, \tilde{y}$ ) will not decrease. Let  $a$  be the smallest index such that  $E(v_A | \tilde{x}_a \leq v_A \leq \tilde{x}_{a+1}) \geq v_0$ . Let  $b$  be the smallest index such that  $E(v_B | \tilde{y}_b \leq v_B \leq \tilde{y}_{b+1}) \geq v_0$ . If  $a = 0$  or  $b = 0$ , the item is never allocated to the seller, since at least one of the players always has an expected valuation greater than  $v_0$ . Therefore, the efficient mechanism in this case is as in the case where  $v_0 \leq \underline{a}$ , i.e., a priority game.

When  $a, b > 0$ , consider some bids' vector  $(i, j)$ . When  $i < a$  and  $j < b$ , the expected valuations of both  $A$  and  $B$  are smaller than  $v_0$ . Thus, the seller should keep the item for optimal welfare. When  $i < a$  and  $j \geq b$ , the expected welfare of player  $B$  is above  $v_0$ , and  $A$ 's expected welfare is below  $v_0$ , thus we can allocate the item to  $B$  and the welfare will not decrease. Similarly, we should allocate the item to  $A$  when  $i \geq a$  and  $j < b$ . When  $i < a$ , the allocation is done regardless to  $i$ , thus we can assume that  $x_a$  is the first threshold (i.e.,  $a = 1$ ), and similarly  $b = 1$ .

Now, we show the optimal allocation for bids' combinations  $(i, j)$  such that  $i \geq a$  and  $j \geq b$ . Here, the item will not be allocated to the seller, so we actually perform an auction with  $k - 1$  possible bids for each player, when the players' valuation are in the range  $[\tilde{x}_1, 1]$ ,  $[\tilde{y}_1, 1]$ . Note that the proof (above) for the case of  $v_0 \leq \underline{a}$  holds for such ranges, so the optimal welfare is achieved in a priority game. Altogether, the optimal mechanism turns out to be a modified priority game.  $\square$

## 4.2 Welfare-optimal 2-player Mechanisms with $k$ Possible Bids

Now, we can finally characterize the efficient mechanisms in our model. It turns out that the optimal threshold values are *mutually-centered*, i.e., each threshold is the expected valuation of *the other* player, given that the valuation of the other player lies between his 2 adjacent thresholds.

**Definition 16.** The threshold values

$$x = (x_0, x_1, \dots, x_{k-1}, x_k), \quad y = (y_0, y_1, \dots, y_{k-1}, y_k)$$

for players  $A, B$  respectively are *mutually-centered*, if the following constraints hold:

$$\forall 1 \leq i \leq k-1 \quad x_i = E(v_B | y_{i-1} \leq v_B \leq y_i) = \frac{\int_{y_{i-1}}^{y_i} f_B(v_B) \cdot v_B dv_B}{F_B(y_i) - F_B(y_{i-1})}$$

$$\forall 1 \leq i \leq k-1 \quad y_i = E(v_A | x_i \leq v_A \leq x_{i+1}) = \frac{\int_{x_i}^{x_{i+1}} f_A(v_A) \cdot v_A dv_A}{F_A(x_{i+1}) - F_A(x_i)}$$

**Lemma 1.** *For any pair of distribution functions of the players' valuations, and for any values of  $x_0$  and  $y_0$ , there exist a unique pair of mutually-centered threshold values  $x, y$  (when  $x_k = y_k$  and w.l.o.g  $y_1 \geq x_1$ ).*

*Proof.* (sketch) Given  $x_0$  and  $y_0$ , and if  $x_1$  is known, we can calculate  $y_1$  (the smallest value that solves  $x_1 = E_{v_B}(v_B | y_0 \leq v_B \leq y_1)$ ), which is unique since the function  $h(z) = E(v | \underline{a} \leq v \leq z)$  is continuous and monotone. Similarly we can calculate  $x_2$ , then  $y_2$ , then  $x_3$  and so on. Finally, we calculate  $y_{k-1}$  such that it solves  $x_{k-1} = E(v_B | y_{k-2} \leq v_B \leq y_{k-1})$ . It is easy to see that all the variables  $x_i$  and  $y_i$  can be viewed as a continuous, monotone functions of  $x_1$ . Now, let  $z$  be the solution for the equation  $y_{k-1} = E(v_A | x_{k-1} \leq v_A \leq z)$ . For satisfying all the  $2(k-1)$  equations,  $z$  must be equal to  $x_k$ . Because  $z$  is also a continuous monotone function of  $x_1$ , there is only a single value of  $x_1$  for which all the equations hold.  $\square$

Let  $x^w = (\underline{a} = x_0^w, x_1^w, \dots, x_{k-1}^w, x_k^w = \bar{b})$  and  $y^w = (\underline{a} = y_0^w, y_1^w, \dots, y_{k-1}^w, y_k^w = \bar{b})$  be mutually-centered threshold values<sup>11</sup>.

Let  $\bar{x} = (\underline{a} = \bar{x}_0, \bar{x}_1, \dots, \bar{x}_{k-1}, \bar{x}_k = \bar{b})$  and  $\bar{y} = (\underline{a} = \bar{y}_0, \bar{y}_1, \dots, \bar{y}_{k-1}, \bar{y}_k = \bar{b})$  be two threshold values vectors for which the following constraints hold:

- $(\bar{x}_1, \dots, \bar{x}_{k-1}, \bar{b})$  and  $(\bar{y}_1, \dots, \bar{y}_{k-1}, \bar{b})$  are mutually centered vectors<sup>12</sup>.
- $\bar{x}_1 = v_0$  and  $\bar{y}_1 = \frac{1}{F_A(\bar{x}_2)} \cdot \left( v_0 F_A(v_0) - \int_{\bar{x}_1}^1 v_A f_A(v_A) dv_A - \sum_{i=3}^k \int_{\bar{x}_{i-1}}^{\bar{x}_i} v_A f_A(v_A) dv_A \right)$

The following theorem says that if the valuation of the seller for the item ( $v_0$ ) is small enough (e.g.,  $\underline{a}$ ), the efficient mechanism is a priority game based on  $x^w$  and  $y^w$  (which are mutually centered). Otherwise, the optimal welfare can be achieved in a modified priority game based on  $\bar{x}$  and  $\bar{y}$  (which are, excluding the first values, mutually centered). The mechanism designer can calculate the expected welfare in both mechanisms to decide which one is *the* efficient mechanism. Note that both  $(x^w, y^w)$  and  $(\bar{x}, \bar{y})$  are determined solely by the distribution functions of the two players.

**Theorem 2.** *For any pair of distribution functions of the players' valuations, and for any seller's valuation  $v_0$  for the item, the mechanism  $PG_k(x^w, y^w)$  or the mechanism  $MPG_k(\bar{x}, \bar{y})$  achieves the optimal welfare (i.e.,  $w_{2,k}^{opt}$ ). The optimal welfare is achieved with dominant-strategy equilibrium and ex-post IR. In particular,  $PG_k(x^w, y^w)$  achieves the optimal welfare when  $v_0 = \underline{a}$ .*

<sup>11</sup>w.l.o.g  $y_1^w \geq x_1^w$ . due to Lemma 1 such a pair of vectors exists and it is unique.

<sup>12</sup>Again, a unique solution exists when, w.l.o.g,  $\bar{y}_2 \geq \bar{x}_2$

*Proof.* First, we prove that  $PG_k(x^w, y^w)$  is optimal when  $v_0 = \underline{a}$  (e.g.,  $v_0 = 0$  when the valuations are in  $[0, 1]$ ). According to Theorem 1 there is a pair of threshold values' vectors  $x = (x_0, x_1, \dots, x_k), y = (y_0, y_1, \dots, y_k)$  such that  $PG_k(x, y)$  achieves optimal welfare. Note that  $x_0 = y_0 = \underline{a}$  and  $x_k = y_k = \bar{b}$ , so we have  $2(k-1)$  variables to optimize. Recall that  $F_i$  is the cumulative distribution function for player  $i$  and note that her average valuation when she bids  $m$  is:

$$E(v_i | s_i(v_i) = m) = E(v_i | y_m \leq v_i \leq y_{m+1}) = \frac{\int_{y_m}^{y_{m+1}} f_i(v_i) v_i dv_i}{F_i(y_{m+1}) - F_i(y_m)}$$

We will calculate the total expected welfare by summing first the expected welfare in the entries of the game matrix where  $B$  wins the item, then summing the entries where  $A$  is the winner. Note that the average valuation of  $B$  is fixed per ‘‘column’’, and for  $A$  is fixed per ‘‘row’’.

$$\begin{aligned} w(g, f) &= \sum_{i=1}^k (F_B(y_i) - F_B(y_{i-1})) \cdot (F_A(x_i) - F_A(x_0)) \cdot \frac{\int_{y_{i-1}}^{y_i} f_B(v_B) v_B dv_B}{F_B(y_i) - F_B(y_{i-1})} \\ &\quad + \sum_{i=2}^k (F_A(x_i) - F_A(x_{i-1})) \cdot (F_B(y_{i-1}) - F_B(y_0)) \cdot \frac{\int_{x_{i-1}}^{x_i} f_A(v_A) v_A dv_A}{F_A(x_i) - F_A(x_{i-1})} \\ &= \sum_{i=1}^k F_A(x_i) \cdot \int_{y_{i-1}}^{y_i} f_B(v_B) v_B dv_B + \sum_{i=2}^k F_B(y_{i-1}) \cdot \int_{x_{i-1}}^{x_i} f_A(v_A) v_A dv_A \end{aligned}$$

We assume here that a probability density function exists for each player. Thus, we can express the partial derivatives for all variables (note that by definition  $\frac{\partial F_i(x)}{\partial x_i} = f_i(x_i)$  and that  $\frac{\partial (\int_c^x f(v) \cdot v \cdot dv)}{\partial x} = f(x)x$ ):

$$(w(g, s))'_{x_i} = \left( \int_{y_{i-1}}^{y_i} f_B(v_B) v_B dv_B \right) \cdot f_A(x_i) + f_A(x_i) \cdot x_i \cdot F_B(y_{i-1}) - f_A(x_i) \cdot x_i \cdot F_B(y_i) = 0$$

$$x_i = \frac{\int_{y_{i-1}}^{y_i} f_B(v_B) v_B dv_B}{F_B(y_i) - F_B(y_{i-1})} = E_{v_B}(v_B | y_{i-1} \leq v_B \leq y_i)$$

$$(w(g, s))'_{y_i} = \left( \int_{x_i}^{x_{i+1}} f_A(v_A) v_A dv_A \right) \cdot f_B(y_i) + f_B(y_i) \cdot y_i \cdot F_A(x_i) - f_B(y_i) \cdot y_i \cdot F_A(x_{i+1}) = 0$$

$$y_i = \frac{\int_{x_i}^{x_{i+1}} f_A(v_A) v_A dv_A}{F_A(x_{i+1}) - F_A(x_i)} = E_{v_A}(v_A | x_{i+1} \leq v_A \leq x_i)$$

Thus,  $x, y$  should be mutually centered for optimal efficiency.

Now, we no longer assume  $v_0 = \underline{a}$ :

According to Theorem 1, if the optimal welfare is not achieved in the priority game above, it

will be achieved in a modified priority game. For some threshold values' vectors  $x, y$ , the expected welfare in  $MPG_k(x, y)$  is given by the formula:

$$\begin{aligned}
& F_A(x_1) \cdot F_B(y_1) \cdot v_0 + F_A(x_1) \int_{y_1}^{\bar{b}} v_B f_B v_B dv_B + F_B(y_1) \int_{x_1}^{\bar{b}} v_A f_A(v_A) dv_A \\
& + \sum_{i=2}^k (F_A(x_i) - F_A(x_1)) \int_{y_{i-1}}^{y_i} v_B f_B(v_B) dv_B \\
& + \sum_{i=3}^k (F_B(y_{i-1}) - F_B(y_1)) \int_{x_{i-1}}^{x_i} v_A f_A(v_A) dv_A
\end{aligned}$$

First order condition similarly derive the constraints on  $x_1$  and  $y_1$  given in the definition of  $\bar{x}, \bar{y}$  above, and that  $(x_1, \dots, x_{k-1}, x_k)$  and  $(y_1, \dots, y_{k-1}, y_k)$  should be mutually-centered<sup>13</sup>.

Due to Proposition 1, both  $PG_k(x^w, y^w)$  and  $MPG_k(\bar{x}, \bar{y})$  have a dominant-strategy equilibrium with ex-post IR.  $\square$

Next, we give an explicit solution for the case of uniformly-distributed valuations in  $[0, 1]$ .

**Corollary 1.** *When players' valuations are distributed uniformly on  $[0, 1]$ , the mechanism  $PG_k(x, y)$  achieves optimal welfare where*

$$\begin{aligned}
x &= \left(0, \frac{1}{2k-1}, \frac{3}{2k-1}, \dots, \frac{2k-3}{2k-1}, 1\right) \\
y &= \left(0, \frac{2}{2k-1}, \frac{4}{2k-1}, \dots, \frac{2k-2}{2k-1}, 1\right)
\end{aligned}$$

*The optimal welfare is achieved with a dominant-strategy equilibrium and ex-post IR.*

*Proof.* According to Theorem 2 optimal welfare is achieved with  $PG_k(x, y)$ , when  $x, y$  are mutually centered. With uniform distributions this derives the following constraints:

$$\begin{aligned}
\forall_{1 \leq i \leq k-1} \quad x_i &= E_{v_B}(v_B | y_{i-1} \leq v_B \leq y_i) = \frac{y_{i-1} + y_i}{2} \\
\forall_{1 \leq i \leq k-1} \quad y_i &= E_{v_A}(v_A | x_i \leq v_A \leq x_{i+1}) = \frac{x_i + x_{i+1}}{2}
\end{aligned}$$

The given vectors  $x, y$  are the unique solution for this set of equations. This efficient mechanism is illustrated in Figure 7.  $\square$

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<sup>13</sup>The results are not surprising, since except for the case when one of the players bids 0, we have a priority game's allocation for which the optimal threshold values must be mutually centered (due to the first part of the proof).

	0	1	..	k-2	k-1
0	$B, 0$	$B, 0$	...	$B, 0$	$B, 0$
1	$A, \frac{1}{2^{k-1}}$	$B, \frac{2}{2^{k-1}}$	...	$B, \frac{2}{2^{k-1}}$	$B, \frac{2}{2^{k-1}}$
2	$A, \frac{1}{2^{k-1}}$	$A, \frac{3}{2^{k-1}}$	...	$B, \frac{4}{2^{k-1}}$	$B, \frac{4}{2^{k-1}}$
...	...	...	...	...	...
k-2	$A, \frac{1}{2^{k-1}}$	$A, \frac{3}{2^{k-1}}$	...	$B, \frac{2^{k-4}}{2^{k-1}}$	$B, \frac{2^{k-4}}{2^{k-1}}$
k-1	$A, \frac{1}{2^{k-1}}$	$A, \frac{3}{2^{k-1}}$	...	$A, \frac{2^{k-3}}{2^{k-1}}$	$B, \frac{2^{k-2}}{2^{k-1}}$

Figure 7: an efficient 2-player  $k$ -possible-bids game with dominant-strategy equilibrium and ex-post IR (when valuations are in  $[0, 1]$ )

### 4.3 Profit-optimal 2-player Mechanisms with $k$ Possible Bids

Now, we present profit-maximizing 2-player mechanisms. Most results in the literature on profit-maximizing auctions, assume that the distribution functions of the players' valuations are *regular* (as defined below). When the valuations of all players are distributed with the same regular distribution function, it is well known that Vickrey's 2nd-price auction, with an appropriately chosen reservation price, is profit-optimal ([19, 12, 4]).

**Definition 17.** ([12]) Let  $f$  be a probability density function, and let  $F$  be its cumulative function. We say that  $f$  is *regular*, if the function

$$\tilde{v}(v) = v - \frac{1 - F(v)}{f(v)}$$

is monotone, strictly increasing function of  $v$ . We call the function  $\tilde{v}(\cdot)$  the *virtual valuation* of the player.

For example, when the players valuations are distributed uniformly on  $[0, 1]$ , a player with a valuation  $v$  has a virtual valuation of  $\tilde{v}(v) = 2v - 1$ .

**Definition 18.** The *virtual surplus* in a game is the virtual valuation of the player (including the seller<sup>14</sup>) that receives the item.

The key observation of Myerson ([12]), which we also use, is that in a Bayesian-Nash equilibrium, the expected profit equals the expected virtual-surplus (in interim individually-rational equilibria where losing players are not getting any surplus). We use this property to reduce the profit-optimization problem to a welfare-optimization problem, for which we have already given a full solution. We first observe that Myerson's observation also holds for auctions with bounded communication. This is easy to see: we can construct a direct revelation mechanism that uses the

<sup>14</sup>The seller's virtual valuation is defined to be his "original" valuation ( $v_0$ ).

equilibrium strategies for the players in  $g$  to simulate the game  $g$ . Clearly, this mechanism will have exactly the same allocation and payments as in  $g$  (given any set of valuations), and for this game Myerson's observation directly applies.

According to Theorem 2, the optimal *welfare* is achieved in either a priority game or a modified priority game. In a model where players consider their virtual valuations as their valuations, let  $MPG(\tilde{x}, \tilde{y})$  or  $PG(\bar{x}, \bar{y})$  be the mechanisms which are the candidates to achieve the optimal **welfare** (where  $\bar{x}, \bar{y}$  are mutually-centered and  $\tilde{x}, \tilde{y}$  are also mutually-centered except the first value, see Theorem 2 for a full characterization). Now, consider the same mechanisms, except each payment  $\tilde{c}$  in them is replaced by the respective "true" valuation  $c = \tilde{v}^{-1}(\tilde{c})$  (i.e.,  $\tilde{c} = \tilde{v}(c)$ ). Denote these mechanisms by  $PG_k(x^R, y^R)$ ,  $MPG_k(x^r, y^r)$ . These mechanisms achieve the optimal **profit** in our (original) model (the mechanism designer should calculate the expected profit in both games to determine which one is the optimal):

**Theorem 3.** *When both players' valuations are distributed with regular distribution functions, the mechanism  $MPG_k(x^r, y^r)$  or the mechanism  $PG_k(x^R, y^R)$  (see definitions above) achieve the optimal expected profit among all profits achievable in an interim-IR Bayesian-Nash equilibrium of a mechanism in  $G_{2,k}$  (i.e.,  $r_{2,k}^{opt}$ ). The optimal profit is achieved with a dominant-strategy equilibrium and ex-post IR.*

*Proof.* Consider the threshold values vectors  $(\tilde{x}, \tilde{y})$  and  $(\bar{x}, \bar{y})$  defined above. The mechanism  $MPG(\tilde{x}, \tilde{y})$  is efficient in the model where the players consider their virtual valuations as their valuation (the same proof holds if  $PG(\bar{x}, \bar{y})$  is the efficient mechanism). The density function  $f$  is regular, and therefore the virtual valuation  $\tilde{v}(\cdot)$  is non-decreasing. Thus,  $MPG_k(x^r, y^r)$  (when players use their original valuations) will have exactly the same allocation for every bids' combination as  $MPG_k(\tilde{x}, \tilde{y})$  (when the players consider their virtual-valuations as their valuations). We conclude that  $MPG_k(x^r, y^r)$  achieves the optimal *expected virtual-surplus*. Due to Myerson's observation (that the expected virtual-surplus equals the expected profit),  $MPG_k(x^r, y^r)$  also achieves optimal profit.  $\square$

As in the case of welfare optimization, we give an explicit solution for the case of uniform distribution functions:

**Corollary 2.** *When players' valuations are distributed uniformly on  $[0, 1]$ , the modified priority game  $MPG_k(x, y)$  achieves optimal expected profit among all the profits achievable in interim-IR Bayesian-Nash equilibria of mechanisms in  $G_{2,k}$ , where*

$$x = (0, \frac{1}{2}, t + \frac{1 \cdot (1-t)}{2k-3}, \dots, t + \frac{(2k-5) \cdot (1-t)}{2k-3}, 1)$$

$$y = (0, t, t + \frac{2 \cdot (1 - t)}{2k - 3}, \dots, t + \frac{(2k - 4) \cdot (1 - t)}{2k - 3}, 1)$$

and  $t = \frac{-2\alpha + \sqrt{1+3\alpha}}{2(1-\alpha)}$  for  $\alpha = \frac{1}{(2k-3)^2}$ . The optimal profit is achieved with a dominant-strategy equilibrium and ex-post IR.

*Proof.* This is a direct corollary of Theorem 3. If we construct the mechanism  $MPG_k(x^r, y^r)$  defined in Theorem 3 for uniform distribution functions, we get the given mechanism. Note that the optimal profit is not achieved in a priority game, since for the uniform distribution the expected virtual valuation of both players is necessarily negative when they bid “0”, so allocating the item to the seller achieves a higher expected virtual surplus (and thus expected profit)<sup>15</sup>.  $\square$

Note that when the players’ valuations are distributed uniformly, the transformation  $\tilde{v}^{-1}$  is linear, and thus the threshold values  $x$  and  $y$  from Theorem 2, without the first zero element, are mutually-centered.

## 5 Optimal Mechanisms for $n$ players with 2 possible bids

In this section we consider games among  $n$  players where each player has 2 possible bids (i.e., they can send only 1 bit to the mechanism). We give the characterization of the optimal mechanisms for general distribution functions. In the previous section we gave the characterization for 2-player mechanisms and  $k$  possible bids. The characterization of the optimal  $n$ -player mechanism with  $k$  possible bids seems to be harder, and it remains an open question.

### 5.1 Optimality of Priority Games

We first show that priority games and modified priority games are efficient (with optimal parameters) also for  $n$ -player games with 2 possible bids. This holds for any distribution function, but only when the players are symmetric, i.e., they share the same distribution function.

**Theorem 4.** *When the players’ valuations are distributed with the same distribution function, there exists a vector of threshold values  $\vec{x} = (x_1, \dots, x_n)$  such that  $PG_2(\vec{x})$  or  $MPG_2(\vec{x})$  achieves the maximal expected welfare among all mechanisms and strategies (i.e.,  $w_{n,2}^{opt}$ ).*

*Proof.* First, assume that  $v_0 = \underline{a}$ . Consider a mechanism  $g \in G_{n,2}$  and a profile of strategies  $s^*$  that achieves with  $g$  optimal welfare ( $w_{n,2}^{opt}$ ). We can assume (by Claim 1 in Theorem 1) that  $s^*$  consists

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<sup>15</sup>Note that a direct proof is also easy: we know that the threshold values vectors are mutually-centered, except for the first values  $x_1, y_1$ . First order conditions on these 2 variables yields the same optimal solution.

of threshold strategies. Player  $i$  has a single threshold value  $x_i$  (w.l.o.g.  $x_1 \leq \dots \leq x_n$ ). Because all the distributions functions are identical, if  $x_i \geq x_j$  then  $E(v_i | v_i \in [x_i, \bar{b}]) \geq E(v_j | v_j \in [x_j, \bar{b}])$  and  $E(v_i | v_i \in [\underline{a}, x_i]) \geq E(v_j | v_j \in [\underline{a}, x_j])$ . Thus, it is easy to see that priority-game allocation ( $n \succ \dots \succ 1$ ) always allocates the item to the player with the highest expected valuation. When we modify  $g$  to be a priority game, the welfare it achieves with  $s^*$  does not decrease. When  $v_0 > \underline{a}$ , the seller might decide to keep the item. Observe that in an efficient mechanism the seller will do so only when all players bid 0<sup>16</sup>. Similar considerations show that the efficient mechanism in this case is a modified priority game.  $\square$

## 5.2 The characterization of the optimal mechanisms

We proved that the optimal welfare is achieved in mechanisms with asymmetric allocation (priority games). Following is a characterization of the efficient  $n$ -player mechanism with 2 possible bids. We can see that this mechanism also fully discriminates between the players using the payments: the player with the highest priority in the priority game, pays the highest payment when she wins, and so forth. The payment (i.e., the threshold) for each player (except the last) is calculated from the payment of the previous player using a simple recursive expression.

Let  $\vec{x} = (x_1, \dots, x_n)$  be a profile of threshold values for the  $n$  players such that the following constraints hold:

$$x_1 = E(v_n | \underline{a} \leq v_n \leq \bar{b}) \quad (5.1)$$

$$\forall_{1 \leq m \leq n-2} x_{m+1} = (1 - F(x_m)) \cdot E(v_m | v_m \in [x_m, \bar{b}]) + F(x_m) \cdot x_m \quad (5.2)$$

$$x_n = \frac{\sum_{i=1}^{n-1} (\prod_{j=i+1}^{n-1} F(x_j)) (1 - F(x_i)) E(v_i | v_i \in [x_i, \bar{b}])}{1 - \prod_{i=1}^{n-1} F(x_i)} \quad (5.3)$$

And let  $\vec{y} = (y_1, \dots, y_n)$  be a profile of threshold values for the  $n$  players such that the following constraints hold:

$$y_1 = v_0 \quad (5.4)$$

$$\forall_{1 \leq m \leq n-2} y_{m+1} = (1 - F(y_m)) \cdot E(v_m | v_m \in [y_m, \bar{b}]) + F(y_m) \cdot x_m \quad (5.5)$$

$$y_n = \sum_{i=1}^{n-1} \left( \prod_{j=i+1}^{n-1} F(y_j) \right) (1 - F(y_i)) E(v_i | v_i \in [y_i, \bar{b}]) + \prod_{i=1}^{n-1} F(y_i) \cdot v_0 \quad (5.6)$$

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<sup>16</sup>If the seller keeps the item when some player  $i$  bids “1”, then in an efficient mechanism player  $i$  will clearly never win the item. But then, if player  $i$ ’s threshold is  $v_0 + \epsilon$ , then the same mechanism except allocating the item to  $i$  when he bids “1” and the others bid “0” will clearly achieve a higher expected welfare. Contradiction to the mechanism’s efficiency.

Note that the recursion formulas in both  $\vec{x}$  and  $\vec{y}$  are identical. The differences between the constraints above are only in the first and last expressions. Note that  $\vec{y}$  depends on the seller's valuation  $v_0$  and  $\vec{x}$  does not. As in the 2-player mechanisms with  $k$  possible bids, the efficient mechanism is a priority game or a modified priority game (depends on the seller's valuation  $v_0$ ):

**Theorem 5.** *When the players' valuations are distributed with the same distribution function, the mechanism  $PG_2(\vec{x})$  or the mechanism  $PG_2(\vec{y})$  achieves the optimal expected welfare ( $w_{n,2}^{opt}$ ). In particular, when  $v_0 = \underline{a}$ ,  $PG_2(\vec{x})$  is the efficient mechanism. The optimal welfare is achieved with a dominant-strategy equilibrium with ex-post IR.*

*Proof.* Due to Theorem 4 and Claim 1 (in Theorem 1), there exists a priority game that achieves optimal welfare with threshold strategies. Consider a priority game among  $n$  players, indexed by their priorities (i.e.,  $1 \prec 2 \dots \prec n$ ). In priority games, every player wins the item if he bids 1 and all the players *with higher priorities* bid 0. Thus, the probability that player  $i$  wins is  $\left(\prod_{j=i+1}^n F(x_j)\right) \cdot (1 - F(x_i))$ . When all players bid 0, either player  $n$  wins or the seller keeps the item for herself. The expected welfare from this game, where the players use threshold strategies  $x_1, \dots, x_n$  is:

$$w(g, s) = \sum_{i=1}^n \left( \prod_{j=i+1}^n F(x_j) \right) (1 - F(x_i)) \frac{\int_{x_i}^{\bar{b}} f(v_i) v_i dv_i}{(1 - F(x_i))} + \left( \prod_{i=1}^n F(x_i) \right) E_0$$

Where  $E_0 = E(v_n | v_n \in [\underline{a}, x_n])$  in the priority game and  $E_o = v_0$  in a modified priority game (the second term relates to the case when all the players bid 0). For maximum, the partial derivatives by  $x_1, \dots, x_n$  should equal zero. By rearranging these first order conditions we get a characterization of the optimal solution.

For players  $1 \leq m \leq n-1$  we get (both in the priority game and in the modified priority game):

$$x_m = \sum_{i=1}^{m-1} \left( \prod_{j=i+1}^{m-1} F(x_j) \right) (1 - F(x_i)) E(v_i | x_i \leq v_i \leq \bar{b}) + \left( \prod_{i=1, i \neq m}^{m-1} F(x_i) \right) E(v_n | \underline{a} \leq v_n \leq x_n)$$

Now we can easily reach a recursive formula by calculating  $x_{m+1} - x_m$  (from which Equations 5.2

and 5.5 follows):

$$x_{m+1} - x_m = (1 - F(x_m)) \cdot (E(v_m | x_m \leq v_m \leq \bar{b}) - x_m)$$

For player  $n$  in the priority game the first order conditions yield the constraint in Equation 5.3, and for player  $n$  in the modified priority game we get Equation 5.6.

When  $m = 1$ , we have  $x_1 = E(v_n | \underline{a} \leq v_n \leq x_n)$  (in the priority game) and  $x_1 = v_0$  (in the modified priority game).

□

As in Section 4, we characterize the profit optimal mechanism by a reduction to the welfare optimizing problem. Again, the reduction can be performed only for regular distributions. Consider the model where players take their virtual valuations as their valuations. Let  $PG_2(\tilde{u})$  or  $MPG_2(\tilde{z})$  be the mechanism that achieves the optimal *welfare* in this model (see Theorem 5 above for the exact characterization). Let  $PG_2(u)$  and  $MPG_2(z)$  be similar mechanisms respectively, except each payment  $\tilde{c}$  is replaced with its respective “original” valuation  $c = \tilde{v}^{-1}(\tilde{c})$ .

**Theorem 6.** *When the players’ valuations are distributed with the same regular distribution function, the mechanism  $PG_2(u)$  or the mechanism  $MPG_2(z)$  achieves the optimal expected profit among all the profits achievable with a Bayesian-Nash equilibrium and interim IR ( $r_{n,2}^{opt}$ ). The optimal profit is achieved in a dominant-strategy equilibrium with ex-post IR.*

*Proof.* This is a corollary of Theorem 5. The reduction is done as in Theorem 3, and it is possible due to the regularity of the distribution function. □

Now, we give explicit solutions for uniform distribution functions on the range  $[0, 1]$ . Let  $(x_1, \dots, x_n) \in [0, 1]^n$  be threshold values for players  $1, \dots, n$  respectively. Consider the following recursive constraints:

$$x_1 = \frac{x_n}{2} \tag{5.7}$$

$$\forall m \in \{1, \dots, n-2\} \quad x_{m+1} = \frac{1}{2} + \frac{x_m^2}{2} \tag{5.8}$$

$$x_n = \frac{\sum_{i=1}^{n-1} \left( \prod_{j=i+1}^{n-1} x_j \right) (1 - x_i^2)}{2 \left( 1 - \prod_{i=1}^{n-1} x_i \right)} \tag{5.9}$$

**Corollary 3.** Consider the threshold values  $\vec{x} = (x_1, \dots, x_n)$  for which Equations 5.7, 5.8, 5.9 hold. When the players' valuations are distributed uniformly in  $[0, 1]$  and  $v_0 = 0$ ,  $PG_2(\vec{x})$  achieves optimal welfare (with dominant-strategy equilibrium and ex-post IR).

*Proof.* The given constraints are the constraints given in Theorem 5, for uniform distributions.  $\square$

Let  $\tilde{y} = (\tilde{y}_1, \dots, \tilde{y}_n) \in [0, 1]^n$  be threshold values defined by:

$$\tilde{y}_1 = 0 \tag{5.10}$$

$$\forall m \in \{1, \dots, n-2\} \quad \tilde{y}_{m+1} = \left( \frac{1 + \tilde{y}_m}{2} \right)^2 \tag{5.11}$$

$$\tilde{y}_n = \sum_{i=1}^{n-1} \left( \prod_{j=i}^{n-1} \frac{1 + \tilde{y}_j}{2} \right) \frac{1 - \tilde{y}_i}{2} \tag{5.12}$$

Now, let  $\vec{y} = (y_1, \dots, y_n)$  be a vector of threshold values such that for every  $i$ ,  $\tilde{y}_i = 2y_i - 1$  (i.e.,  $\tilde{y}_i = \tilde{v}_i(y_i)$ ).

**Corollary 4.** Consider the threshold values  $\vec{y} = (y_1, \dots, y_n)$  defined above. When the players' valuations are distributed uniformly in  $[0, 1]$  and  $v_0 = 0$ ,  $MPG_2(\vec{y})$  achieves optimal profit (with dominant-strategy equilibrium and ex-post IR).

*Proof.* The given constraints are the constraints given in Theorem 6, for uniform distributions.  $\square$

## 6 Asymptotic Analysis of the Welfare and Profit Losses

In Section 4 and Section 5 we presented welfare-maximizing and profit-maximizing mechanisms. Given any set of distribution functions for the players, we showed how to construct such mechanisms, and we proved they have some desired properties (dominant-strategy equilibrium, ex-post IR). In this section, we measure the performance of these mechanisms. For simplicity, we assume that the valuations' range is  $[0, 1]$  (all the results apply for a general range  $[\underline{a}, \bar{b}]$  which only changes the constants in our analysis).

In Section 6.1 we show an asymptotically tight upper bound for the welfare loss incurred by the optimal mechanisms: for any number of players, we present mechanisms that incur a welfare loss smaller than  $c \cdot \frac{1}{k^2}$  (for some positive constant  $c$ ). We also show that, under reasonable assumptions, very simple, symmetric mechanisms can also achieve good asymptotic results. Section 6.2 presents similar bounds for the profit loss (which are derived from the bounds for the welfare loss). In Section 6.3 we show that if we generalize the model to deal with general joint distribution functions, we

cannot do asymptotically better than a trivial mechanism. Finally, in Section 6.4, we fix the number of possible bids to 2, and give asymptotic analysis of the welfare and profit losses as a function of the number of players (for the uniform distribution). All the result are asymptotic with respect to the amount of the communication, except in Section 6.4 where it is with respect to the number of players.

We will sometimes use the following (standard and very commonly used in Computer Science) notations, describing upper/lower asymptotic bounds for functions.

**Definition 19.** We say that a function  $f(k)$  is an *asymptotic upper bound* of a function  $g(k)$ , if there exist positive constants  $c$  and  $k_0$  such that  $g(k) \leq c \cdot f(k)$  for all  $k \geq k_0$ . We write this relation by  $\mathbf{g}(\mathbf{k}) = \mathbf{O}(\mathbf{f}(\mathbf{k}))$ .

We say that a function  $f(k)$  is an *asymptotic lower bound* of a function  $g(k)$ , if there exist positive constants  $c$  and  $k_0$  such that  $g(k) \geq c \cdot f(k)$  for all  $k \geq k_0$ . We write this relation by  $\mathbf{g}(\mathbf{k}) = \mathbf{\Omega}(\mathbf{f}(\mathbf{k}))$ .

We say that a function  $f(k)$  is *asymptotically proportional* to a function  $g(k)$ , if both  $\mathbf{g}(\mathbf{k}) = \mathbf{O}(\mathbf{f}(\mathbf{k}))$  and  $\mathbf{g}(\mathbf{k}) = \mathbf{\Omega}(\mathbf{f}(\mathbf{k}))$ . We write this relation by  $\mathbf{g}(\mathbf{k}) = \mathbf{\Theta}(\mathbf{f}(\mathbf{k}))$ .

For example<sup>17</sup>,  $n = O(n^2)$ ,  $\frac{1}{n^2} = \Omega\left(\frac{10000}{n^4}\right)$ ,  $2\log(n) = \Theta(\log(n))$ .

## 6.1 Asymptotic Bounds of the Welfare Loss

The next theorem shows that no matter how the players' valuations are distributed, we can always construct mechanisms such that the welfare loss they incur diminishes quadratically in  $k$ . This is true for any number of players we fix (when  $k > 2n$ ).

**Theorem 7.** *For any (fixed) number of players  $n$ , and for any set of distribution functions of the players' valuations, there exist a set of mechanisms  $g_k \in G_{n,k}$  ( $k = 2n + 1, 2n + 2, \dots$ ), that incur a welfare loss  $\leq c \cdot \frac{1}{k^2}$ , where  $c$  is some positive constant (i.e.,  $O(\frac{1}{k^2})$ ). These mechanisms have dominant-strategy equilibria with ex-post individual-rationality.*

*Proof.* The proof's idea: we construct a priority game in which all players have the same dominant threshold strategy, such that the probability for a player to bid each bid is smaller than  $\frac{n}{k}$ . This is done by dividing the density functions of all the players to  $\frac{k}{n}$  intervals with equal mass, then combining these thresholds to a vector of  $k$  threshold values. Because the players use the same threshold strategy, a welfare loss is possible only when more than one bidder bids the highest bid. This observation leads to the upper bound.

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<sup>17</sup>See [18] for an introduction to asymptotic analysis of functions with more examples.

Let  $\alpha_1, \dots, \alpha_n$  be integers such that  $\sum_{i=1}^n \alpha_i = k - 2$ , and for every  $i$ ,  $\alpha_i \geq \lfloor \frac{k}{n} \rfloor - 1$  (clearly such numbers exist). For every player  $i$ , let  $Y^i = (y_1^i, \dots, y_{\alpha_i}^i)$  be a set of threshold values that divides her distribution function  $f_i$  to  $\alpha_i + 1$  segments with the same mass (when  $y_0^i = 0, y_{\alpha_i+1}^i = 1$ ), i.e., for every bid  $j$ ,  $F_i(y_{j+1}) - F_i(y_j) = \frac{1}{\alpha_i+1}$ .

Let  $X = \{\bigcup_{i=1}^n Y^i\} \cup \{v_0\}$ ,  $|X| = k - 1$ , be the union of all the threshold values (we might add a number of arbitrary threshold values to make the size of  $X$  exactly  $k+1$ ). Let  $x = (0, x_1, \dots, x_{k-1}, 1)$  be a threshold-value vector created by ordering the threshold values in  $X$  from smallest to largest. Now, consider the  $n$ -player mechanism  $MPG_k(\tilde{t})$  where  $\tilde{t} = (x, \dots, x)$ . Due to Proposition 1, the threshold strategy based on  $x$  is dominant for all the players, with ex-post IR. By the construction of the sets  $Y^1, \dots, Y^n$ , every player will bid any particular bid<sup>18</sup> w.p.  $\leq \frac{2n}{k}$ .

Next, we will bound the welfare loss. We divide the possible cases according to the number of players that bid the highest bid. Since all the players use the same threshold strategy, if only one player bids the highest bid, no welfare loss is incurred (he will definitely have the highest valuation). If more than 1 player bid the highest bid  $i$ , the expected welfare loss will not exceed  $x_{i+1} - x_i$ . For a set of players  $T \subseteq N$ , denote the probability that all the players in  $T$  bid  $i$  by  $Pr(T = i)$ , and the probability that all the players not in  $T$  have bids smaller than  $i$  by  $Pr(N \setminus T < i)$ . Thus, the expected welfare loss is smaller than (when  $2n < k$ ):

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<sup>18</sup>For every player  $i$ , and every bid  $j$ ,  $F_i(x_{j+1}) - F_i(x_j) \leq \frac{1}{\lfloor \frac{k}{n} \rfloor} \leq \frac{2n}{k}$

$$\begin{aligned}
& \sum_{\substack{T \subseteq N \\ |T|=2}} \sum_{i=1}^k Pr(T=i) Pr(N \setminus T < i) (x_{i+1} - x_i) + \dots \\
& + \sum_{\substack{T \subseteq N \\ |T|=n}} \sum_{i=1}^k Pr(T=i) Pr(N \setminus T < i) (x_{i+1} - x_i) \\
& \leq \sum_{\substack{T \subseteq N \\ |T|=2}} \sum_{i=1}^k Pr(T=i) (x_{i+1} - x_i) + \dots + \sum_{\substack{T \subseteq N \\ |T|=n}} \sum_{i=1}^k Pr(T=i) (x_{i+1} - x_i) \\
& \leq \sum_1^{\binom{n}{2}} \sum_{i=1}^k \left(\frac{2n}{k}\right)^2 (x_{i+1} - x_i) + \dots + \sum_1^{\binom{n}{n}} \sum_{i=1}^k \left(\frac{2n}{k}\right)^n (x_{i+1} - x_i) \\
& = \binom{n}{2} \left(\frac{2n}{k}\right)^2 + \dots + \binom{n}{n} \left(\frac{2n}{k}\right)^n < 2^n \cdot 4n^2 \cdot \frac{1}{k^2}
\end{aligned}$$

When the valuation of all the players is smaller than  $v_0$ , there is no welfare loss (the seller, with the highest valuation, keeps the item). Note that despite the coefficient of  $\frac{1}{k^2}$  is exponential in  $n$ , we consider it as a constant because  $n$  is fixed. For Example, when  $n = 2$  a similar proof shows a welfare loss smaller than  $\frac{8}{k^2}$  (when  $k > 3$ ).  $\square$

In particular, the efficient mechanism presented in Theorem 2 incurs a welfare loss of  $O(\frac{1}{k^2})$ . Asymptotic quadratic bounds were also given by Wilson in [20], which studied similar settings regarding the effect of discrete priority classes of customers. In [20] the uncertainty was about the supply, while in this paper the demand is uncertain as well. Both results are illustrations for the idea that the deadweight loss is second order in the price distortion. (The price distortion in our model is the maximum difference between the prices that different players are facing for the item given the others' bids, and it can be bounded above by  $\frac{1}{k}$ .) Indeed, a small price distortion ensures both that the probability of an inefficient allocation is small and that the inefficiency is small when it does occur.

Theorem 7 is related to proposition 4 in [15]. In [15], Nisan and Segal showed that discretizing an exactly efficient continuous protocol communicating  $d$  real numbers yields a “truly polynomial” approximation scheme that is proportional to  $d$  (i.e., for any  $\epsilon > 0$  we can realize an approximation

factor of  $1 - \epsilon$  using a number of bits which is polynomial in  $\log(\epsilon^{-1})$ . Here, we discretize a continuous efficient auction (e.g., first-price auction), where  $d$  is the number of players. Discretization then achieves an approximation error that is exponential in the (minus) number of bits sent per player, i.e., asymptotically proportional to  $\frac{1}{k}$ . However, here we care about average-case approximation which is even closer, because worst-case approximation within an error of  $\epsilon$  ensures an average case approximation within  $\epsilon^2$  (the probability that an error is made is itself proportional to  $\epsilon$ ).

Corollary 1 gave an explicit characterization of the efficient mechanism for the case of uniformly-distributed valuations in  $[0, 1]$ . We show that the asymptotic upper bound for the efficiency loss (given in Theorem 7) is tight for any number of players, i.e., for some distribution functions (e.g., the uniform distribution) the minimal welfare loss is proportional to  $\frac{1}{k^2}$ , or  $\Theta(\frac{1}{k^2})$  in CS notations.

**Theorem 8.** *Assume that the players' valuations are uniformly distributed and that  $v_0 = 0$ . Then, the efficient 2-player mechanism  $PG_k(x, y)$  incurs a welfare loss of exactly  $\frac{1}{6 \cdot (2k-1)^2}$  where*

$$x = (0, \frac{1}{2k-1}, \frac{3}{2k-1}, \dots, \frac{2k-3}{2k-1}, 1)$$

$$y = (0, \frac{2}{2k-1}, \frac{4}{2k-1}, \dots, \frac{2k-2}{2k-1}, 1)$$

Moreover, for any (fixed) number of players  $n$  and any  $v_0$ , there exists a positive constant  $c$  such that **any** mechanism  $g \in G_{n,k}$  incurs a welfare loss  $\geq c \cdot \frac{1}{k^2}$ .

*Proof.* We first prove the first part of the theorem, regarding 2-player mechanisms. Note that this mechanism can make non-optimal allocation only for bids' combinations that are on the diagonal or on the lower secondary diagonal in the matrix representation of the 2-player game (i.e., when  $b_A = b_B$  or when  $b_A = b_B + 1$ ). For such bids  $(i, j)$ , the overlapping segment of  $[x_i, x_{i+1}]$  and  $[y_j, y_{j+1}]$  is of size  $\frac{1}{2k-1}$ . Given such bids' vector  $(i, j)$ , if one of the valuations is not in this overlapping segment, the allocation is optimal (note that we allocate the item to  $B$  on the main diagonal, and to  $A$  on the secondary diagonal). The probability that both valuation are in this overlapping range is  $\frac{1}{(2k-1)^2}$ . The expected valuation in our priority game (when both valuation are in this overlapping segment) is exactly in the middle of this segment. The expected valuation in the optimal auction (with unbounded communications) will be in the  $\frac{2}{3}$  point of this range. Thus, the welfare loss, given that both players are in this overlapping segment, is  $\frac{1}{6}$  of the segment, i.e.,  $\frac{1}{6} \cdot \frac{1}{2k-1}$ . Thus, for every bids' vector on the main diagonal or on the secondary-diagonal the expected welfare loss is  $\frac{1}{6} \frac{1}{(2k-1)^2}$ . There are  $(2k-1)$  such bids' vector, thus the total welfare loss is exactly  $\frac{1}{6(2k-1)^2}$ .

A similar argument shows that even when the seller's valuation  $v_0$  is non zero, the welfare loss is asymptotically greater than  $\frac{1}{(2k-1)^2}$ : let  $z_1, \dots, z_m$  be the sizes of the overlapping segments (only when the valuations of both players is greater than  $v_0$ ). Clearly,  $m \leq 2k - 1$  and  $\sum_{i=1}^m z_i \leq 1$ . Then, the welfare loss from the game is at least <sup>19</sup>:

$$(1 - v_0)^2 \cdot \sum_{i=1}^m z_i^2 \cdot \frac{z_i}{6} = \frac{(1 - v_0)^2}{6} \cdot \sum_{i=1}^m z_i^3 \geq \frac{(1 - v_0)^2}{6} \frac{2k - 1}{(2k - 1)^3} \geq \frac{(1 - v_0)^2}{6} \frac{1}{(2k - 1)^2}$$

The proof of the second statement is easily derived: Consider only the case where players 1 and 2 have valuations above  $\frac{1}{2}$ , and the rest of the players have valuations below  $\frac{1}{2}$ . This occurs with the constant probability of  $\frac{1}{2^n}$ . The best a mechanism can do is to always allocate the item to one of 1 or 2. But due to the first part of the theorem, in any 2-player mechanism a welfare loss of proportional to  $\frac{1}{k^2}$  will be incurred (the fact that the valuation range is  $[\frac{1}{2}, 1]$  and not  $[0, 1]$  only changes the constant  $c$ ). This will hold for any opportunity cost  $v_0$  of the seller. Thus, any mechanism will incur a welfare loss of  $\Omega(\frac{1}{k^2})$ .  $\square$

Note that the same asymptotic results hold even if we restrict attention to symmetric mechanisms. Actually, we prove the upper bound in Theorem 7 by constructing a symmetric mechanism (we can allocate the item to all the players that bid the highest bid with equal probabilities). However, asymmetric mechanisms do incur a strictly smaller welfare loss than symmetric mechanisms. For example, when the valuations are distributed uniformly, the optimal welfare loss is  $\frac{1}{6(2k-1)^2}$  (by Theorem 8) compared with an optimal welfare loss of  $\frac{1}{6k^2}$  achieved by symmetric mechanisms<sup>20</sup> (i.e., the welfare loss in asymmetric mechanisms is about 4 times better!). This observation is interesting in light of the results of Rothkopf and Harstad ([10]) and Wilson ([20]). [10] studied symmetric English auctions, and analyzed the optimal price-jumps in such auctions. Our results show that non-anonymous prices (i.e., different jumps for each player) can achieve better results than symmetric (or anonymous) jumps. We also characterize the optimal price-jumps for such auctions (mutually-centered threshold values). [20] also studies only symmetric priority classes in his model, and also gives a convergence rate of  $\frac{1}{n^2}$  for the efficiency loss (where  $n$  is the number of priority classes). We show that asymmetric mechanisms can incur smaller efficiency loss, though the asymptotic convergence rate is the same.

Due to Theorem 2, for any set of distribution functions we can construct a mechanisms that

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<sup>19</sup>In the left inequality we use the fact that when  $z = (z_1, \dots, z_m)$  is in the  $m$ 'th dimensional simplex,  $\sum_{i=1}^m z_i^3 \geq \frac{1}{m^2}$ .

<sup>20</sup>It is easy to show that efficient symmetric mechanisms are similar to priority games, except the item is allocated with equal probabilities in cases of ties. The thresholds of the players simply divide the valuations' range to identical segments. Then, it is straightforward to show that the welfare loss is exactly  $\frac{1}{6k^2}$ .

incurs a welfare loss of  $O(\frac{1}{k^2})$ . But can we design a mechanism that regardless of the distribution functions, will always incur a low welfare loss? The following theorem presents a simple, symmetric mechanism with  $k$ -possible bids that incurs a welfare loss of  $O(\frac{1}{k})$  regardless of the players' distribution functions, and we also show that no mechanism can do asymptotically better for all distributions.

**Proposition 2.** *The  $n$ -player mechanism  $PG_k(x, \dots, x)$ ,  $x = (0, \frac{1}{k}, \frac{2}{k}, \dots, \frac{k-1}{k}, 1)$ , incurs an expected welfare loss  $\leq \frac{1}{k}$  for any set of distribution functions of the players' valuations. Moreover, for any mechanism  $g$  there exists a set of distribution functions for which the expected welfare loss in  $g$  is greater than  $\frac{n-1}{n^2} \cdot \frac{1}{k}$  (i.e.,  $\Omega(\frac{1}{k})$ ).*

*Proof.* When all players use the same threshold strategy in priority games, non-optimal allocation is possible only when more than one bidder bid the highest bid. Since the difference between subsequent thresholds is  $\frac{1}{k}$ , the expected welfare loss is clearly not greater than  $\frac{1}{k}$ .

For proving the lower bound, consider a mechanism  $g \in G_{n,k}$  with an equilibrium  $s_1, \dots, s_n$ . We can prove, similarly to the proof of Claim 1 in Theorem 1, that every mechanism with a Bayesian-Nash equilibrium, has an equilibrium of threshold strategies. Thus, we can assume that  $s_1, \dots, s_n$  are threshold strategies based on some threshold-value vectors  $x^1, \dots, x^n$ . Observe that there are no more than  $nk$  bids' combinations  $b = (b_1, \dots, b_n)$  with overlapping valuations for *all* players, i.e., for every pair of players  $i, j$ :  $[x_{b_i}^i, x_{b_{i+1}}^i] \cap [x_{b_j}^j, x_{b_{j+1}}^j] \neq \emptyset$ . (The maximal number of different thresholds for all players is  $(k-1)n$ , and every two subsequent thresholds define such an "overlapping segment".) In addition, the sum of the sizes of these overlapping segments is 1. Thus, there must be a bids' combination  $\tilde{b} = (\tilde{b}_1, \dots, \tilde{b}_n)$  with an overlapping segment with size of at least  $\frac{1}{nk}$ . Denote this segment as  $[\underline{m}, \overline{m}]$  ( $\overline{m} - \underline{m} \geq \frac{1}{nk}$ ). Assume w.l.o.g that for the bids vector  $\tilde{b}$ , player  $n$  wins the item with probability not greater than  $\frac{1}{n}$  (such a player must exist). Now assume that the players' valuations are distributed such that player  $n$  has the constant valuation  $\overline{m}$  and all the other players have the constant valuation  $\underline{m}$ . Then, the allocation will not be optimal (i.e., player  $n$  will not win) with probability of at least  $\frac{n-1}{n}$ , and the welfare loss is at least  $\overline{m} - \underline{m} \geq \frac{1}{nk}$ . The total welfare loss will therefore be at least  $\frac{n-1}{n^2} \cdot \frac{1}{k}$ . Thus, the expected welfare loss is bounded from below by a proportion of  $\frac{1}{k}$ .  $\square$

If we assume that the distribution functions of the players are bounded from above or from below, we can get even stronger results for this simple mechanism:

**Definition 20.** We say that a probability density function  $f$  is *bounded from above* (resp. *below*) if for every  $x$  in its domain,  $f(x) \leq c$  (resp.  $f(x) \geq c$ ) for some constant  $c > 0$ .

**Proposition 3.** For every set of probability density functions of the players' valuations which are bounded from above, the mechanism  $PG_k(x, \dots, x) \in G_{n,k}$ , where  $x = (0, \frac{1}{k}, \frac{2}{k}, \dots, \frac{k-1}{k}, 1)$ , incurs an expected welfare loss  $\leq c_1 \cdot \frac{1}{k^2}$  for some positive constant  $c_1$  (i.e.,  $O(\frac{1}{k^2})$  in the CS notations).

For every set of probability density functions which are bounded from below, every mechanism incurs an expected welfare loss  $\geq c_2 \cdot \frac{1}{k^2}$  for some positive constant  $c_2$  (i.e.,  $\Omega(\frac{1}{k^2})$ ).

*Proof.* For proving the first statement, say that the distribution function is bounded from above by  $\bar{q}$ . When the players use the same threshold strategy, a welfare loss is only possible when more than one player bid the higher bid. Every subset of players can be the set of players that bids the highest bid, and this bid can be any bid in  $1, \dots, k-1$ . (The welfare added when all player bid "0" is negligible.) The maximal welfare loss is  $\frac{1}{k}$ , thus the expected welfare loss is smaller than:

$$\begin{aligned}
& \sum_{T \subseteq N, |T| \geq 2} \sum_{i=2}^k \left( \bar{q} \cdot \frac{1}{k} \right)^{|T|} \cdot \left( \bar{q} \cdot \frac{i-1}{k} \right)^{n-|T|} \frac{1}{k} \\
& \leq \bar{q}^n \sum_{T \subseteq N, |T| \geq 2} \sum_{i=2}^k \frac{1}{k^{n+1}} \cdot (i-1)^{n-|T|} \\
& \leq \bar{q}^n \sum_{T \subseteq N, |T| \geq 2} \sum_{i=2}^k \frac{1}{k^{n+1}} \cdot (k)^{n-|T|} \\
& \leq \bar{q}^n \sum_{T \subseteq N, |T| \geq 2} \frac{1}{k^n} \cdot (k)^{n-|T|} \\
& \leq \bar{q}^n \sum_{T \subseteq N, |T| \geq 2} \frac{1}{k^2} \leq (2\bar{q})^n \frac{1}{k^2}
\end{aligned}$$

As for the second part of the theorem, we first prove it for 2 players ( $n=2$ ). Assume that the distribution functions of the players are bounded from below by  $\underline{q}$ , and that the players  $A, B$  use the threshold strategies based on  $x = (x_0, \dots, x_k)$  and  $y = (y_0, \dots, y_k)$ . We say that the bids  $i, j$  for players  $A, B$  (respectively) are overlapping, if  $[x_i, x_{i+1}] \cap [y_j, y_{j+1}] \neq \emptyset$ . Consider a mechanism  $g \in G_{2,k}$  such that the threshold strategies based on  $x, y$  are in Bayesian-Nash equilibrium. Let  $m$  be the number of overlapping pairs of bids. It is easy to see that  $m \leq 2k-1$ . For each overlapping pair  $i$  ( $i = 1, \dots, m$ ), let  $z_i$  be the size of the overlapping segment  $i$ . Clearly,  $\sum_{i=1}^m z_i = 1$ . Given that the 2 players' valuations are in the  $i$ th segment, the maximal welfare loss is  $z_i$ . Thus, the expected welfare loss is greater than<sup>21</sup>.

$$\sum_{i=1}^m (\underline{q}z_i) \cdot (\underline{q}z_i) \cdot z_i = \underline{q}^2 \sum_{i=1}^m z_i^3 \geq \underline{q}^2 \cdot \left( \frac{1}{m} \right)^2 \geq \underline{q}^2 \cdot \frac{1}{(2k-1)^2}$$

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<sup>21</sup> Again, we used the fact that when  $z = (z_1, \dots, z_m)$  is in the  $m$ 'th dimensional simplex,  $\sum_{i=1}^m z_i^3 \geq \frac{1}{m^2}$

The proof for  $n$  players is straightforward now: consider only the case where players 1 and 2 have valuations above  $\frac{1}{2}$ , and the other players will have valuations below  $\frac{1}{2}$ . This will occur with probability not smaller than the constant  $\frac{q^n}{2^n}$ . We saw that any 2-player mechanism incurs a loss which is bounded from below by a proportion of  $\frac{1}{k^2}$  (only with a different coefficient)<sup>22</sup>.  $\square$

One can interpret Proposition 3 as a contest between “nature” and the mechanism designers: when they choose a mechanism first, and then “nature” chooses the distribution functions, the designers can ensure a welfare loss of no more than a proportion of  $\frac{1}{k^2}$ . When “nature” chooses the distribution function first, and then we choose the mechanism, “nature” ensures that the welfare loss will be at least proportional to  $\frac{1}{k^2}$ .

## 6.2 Asymptotic Bounds of the Profit Loss

As done in Theorem 3, the profit optimization problem can be reduced to a welfare optimization problem, by maximizing the expected virtual surplus.

**Proposition 4.** *Assume that the players’ valuations are distributed with regular distribution functions. Then, for any number of players  $n$ , there exist a set of mechanisms  $g_k \in G_{n,k}$  ( $k = 2n + 1, 2n + 2, \dots$ ) that incur a profit loss  $\leq c \cdot \frac{1}{k^2}$ , where  $c$  is some positive constant. The profit loss is compared with the optimal, individually-rational mechanism that is unconstrained in communication.*

*Proof.* Consider the model where players consider their virtual valuations as their valuations. As the range of the valuations in this model, we take the union of the ranges of all the players’ virtual valuations. Denote this range as  $[\alpha, \beta]$ . Let  $\tilde{g} \in G_{n,k}$  be the mechanism that achieves maximal welfare in this model. Due to Theorem 7,  $\tilde{g}$  incurs a welfare loss smaller than  $c \cdot \frac{1}{k^2}$ , for some positive constant  $c$  (the constant takes into account the size of the virtual valuations’ range  $\beta - \alpha$ )<sup>23</sup>. Let  $g$  be the mechanism with the same allocation as in  $\tilde{g}$ , only each payment  $\tilde{q}_i$  for player  $i$  in  $\tilde{g}$  is replaced with  $q_i = \tilde{v}_i^{-1}(\tilde{q}_i)$  in  $g$ , i.e.,  $\tilde{q}_i = \tilde{v}_i(q_i)$  (where  $\tilde{v}_i$  is the virtual valuation function of player  $i$ ). Since the virtual valuation functions of the players are non-decreasing (the distribution functions are regular), the allocation in  $g$  and  $\tilde{g}$  is identical, for every bids’ combination. Thus,  $g$  achieves the maximal expected virtual surplus, and the loss of expected virtual surplus is smaller than  $c \cdot \frac{1}{k^2}$ . Due to the equality between the expected virtual-surplus and the expected profit, we conclude that the  $g$  achieves optimal profit loss, and therefore the optimal profit loss is smaller than  $c \cdot \frac{1}{k^2}$ .  $\square$

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<sup>22</sup>see Theorem 8 for similar analysis

<sup>23</sup>Theorem 7 holds when  $k > 2n$ .

Now, we show that this upper bound is asymptotically tight. We show that for some distribution functions (specifically, for the uniform distribution), any mechanism incurs a profit loss of  $\Omega(\frac{1}{k^2})$

**Proposition 5.** *Assume that the players' valuations are distributed uniformly. Then, for any (fixed) number of players  $n$ , there exists a positive constant  $c$  such that **any** mechanism  $g \in G_{n,k}$  incurs a profit loss  $\geq c \cdot \frac{1}{k^2}$ .*

*Proof.* Consider some mechanism  $g \in G_{n,k}$ . Let  $\tilde{g}$  be the mechanism similar to  $g$ , except each payment  $p_i$  for player  $i$  in  $g$  is replaced in  $\tilde{g}$  with  $\tilde{v}_i(p_i)$ . Due to Theorem 8,  $\tilde{g}$  incurs a welfare loss greater than  $\frac{c}{k^2}$  for some positive constant  $c$  in the model were players consider their virtual valuations as their valuations. As in Theorem 4, it follows that  $g$  incurs an expected virtual surplus loss of at least  $\frac{c}{k^2}$ , and thus the profit loss in  $g$  holds this bound as well.  $\square$

### 6.3 Bounds for joint distributions

So far, we assumed that the players' valuations are drawn from statistically independent distributions. Next, we relax this assumption and deal with general joint distributions of the valuations. For this case, we show that a very simple mechanism is actually the best we can do (asymptotically). Particularly, it derives an asymptotically tight upper bound of  $\frac{1}{k}$  for the efficiency loss in  $n$ -player games.

**Theorem 9.** *The mechanism  $PG_k(x, \dots, x) \in G_{n,k}$  where  $x = (0, \frac{1}{k}, \frac{2}{k}, \dots, \frac{k-1}{k}, 1)$  incurs an expected welfare loss  $\leq \frac{1}{k}$  for any joint distribution  $\phi$  on the players' valuations.*

*Moreover, for every  $k$  there is a joint distribution function  $\phi_k$  such that any mechanism  $g \in G_{n,k}$  incurs a welfare loss  $\geq c \cdot \frac{1}{k}$  (where  $c$  is a positive constant independent of  $k$ ).*

*Proof.* The straightforward proof of the first statement is identical to the case of independent valuations (see Proposition 2).

We first prove the second statement for  $n = 2$ . For every  $k$ , we construct a joint distribution that incurs, for any mechanism  $g \in G_{2,k}$ , an efficiency loss which is greater than  $\frac{1}{16k}$ . Consider the following joint distribution:  $v_A$  is distributed uniformly in the range  $[\frac{1}{4k}, 1 - \frac{1}{4k}]$ , and  $v_B$  is  $v_A + \frac{1}{4k}$  or  $v_A - \frac{1}{4k}$  with equal probabilities. We say that  $v_A$  is *dominated*, if there is a threshold  $y_j$  of player  $B$ , such that  $|v_A - y_j| \leq \frac{1}{4k}$ .  $B$ 's thresholds 0 and 1 clearly cannot dominate any  $v_A \in [\frac{1}{4k}, 1 - \frac{1}{4k}]$ . Each one of the other  $k - 1$  thresholds of  $B$  dominates a range of size  $\frac{1}{2k}$ , so the total range of dominated values for  $A$  is at most  $\frac{k-1}{2k}$ . The probability that a random  $v_A$  will be dominated is thus at most  $\frac{\frac{k-1}{2k}}{1 - \frac{1}{2k}} < \frac{1}{2}$ . When  $v_A$  is not dominated,  $v_A, v_A + \frac{1}{4k}$  and  $v_A - \frac{1}{4k}$  will lie within the same entry in the matrix representation of  $g$ . This will therefore happen with probability  $> \frac{1}{2}$  (the

probability that  $v_A$  is not dominated).  $v_B$  is determined randomly (and uniformly), thus whatever allocation is made in this entry, welfare loss will be incurred with probability  $\geq \frac{1}{2}$ . The welfare loss (if incurred) will clearly be of at least  $\frac{1}{4k}$ . Thus, the expected efficiency loss will be greater than  $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{4k} = \frac{1}{16k}$ . The generalization for  $n$  players is easy now (see e.g., Proposition 3).  $\square$

## 6.4 Asymptotic bounds for an increasing number of players

So far, we gave asymptotic analysis with respect to the amount of communication in the mechanism. In this subsection we fix the size of communication allowed (to 2 possible bids), and we show asymptotic bounds as a function of the number of players. Unfortunately, we have been able to prove such bounds only for the uniform distribution. First, we analyze mechanisms with *symmetric* allocation and payments. In this case, all the players play according to the same threshold value  $x$ . It turns out that the welfare-maximizing threshold is  $x = n^{-\frac{1}{n}}$ , and the optimal welfare loss is thus proportional to  $\frac{\log n}{n}$  (for the uniform distribution)<sup>24</sup>. Similarly, the optimal profit loss incurred by a symmetric mechanism is also  $\Theta(\frac{\log n}{n})$ .

We now show that optimal *asymmetric* mechanisms incur asymptotically smaller welfare and profit losses of  $O(\frac{1}{n})$ . These mechanisms fully discriminate among the agents (no two players have the same payment or allocation schemes).

**Theorem 10.** *Consider the efficient mechanism  $PG_2(\vec{x})$  described in Corollary 3. When the players' valuations are distributed uniformly,  $PG_2(\vec{x})$  incurs a welfare loss  $\leq \frac{9}{n}$ .*

*Proof.* Let  $x$  be the thresholds' vector from Corollary 4, except that the constraint 5.11 holds for the  $n$ th player as well. We will bound the welfare in  $PG_2(x)$  (by Theorem 3 the given mechanism is efficient, thus it incurs even a smaller welfare loss). We assume w.l.o.g. that in  $g$ , players are indexed according to their priorities (i.e.,  $1 \prec 2 \dots \prec n$ ). When a player wins after bidding "1", the maximal welfare loss is  $1 - x_i$ . When all players bid "0", we use the trivial upper bound of 1 for the welfare loss. Therefore, we can bound the welfare loss with:

$$\sum_{i=1}^n \left( \prod_{j=i+1}^n x_j \right) (1 - x_i) (1 - x_i) + \prod_{i=1}^n x_i \quad (6.1)$$

We first consider the following 2 claims by induction (see proofs in Appendix A.3):

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<sup>24</sup>The expected welfare then is given by:  $x^n \cdot \frac{x}{2} + (1 - x^n) \cdot \frac{1+x}{2}$ . A maximum is achieved (first order conditions) with:  $x = n^{-\frac{1}{n}}$ . The welfare loss is thus:  $\frac{n}{n+1} - \frac{1}{2} \left( 1 - n^{-\frac{1}{n-1}} \left( \frac{1}{n} - 1 \right) \right)$ . It is easy to see that if  $1 - \left( \frac{1}{n} \right)^{\frac{1}{n}}$  converges to  $\frac{\log n}{n}$  then the welfare loss also converges to  $\frac{\log n}{n}$ . And indeed,  $1 - \left( \frac{1}{n} \right)^{\frac{1}{n}} = 1 - e^{-\frac{\log n}{n}} \approx \frac{\log n}{n}$  (since  $1 - e^{-x} \approx x$  for small  $x$ 's).

*Claim 5.*  $\forall_n \quad 1 - x_n \leq \frac{2}{n}$

*Claim 6.*  $\forall_{n \geq 15} \quad x_n \leq \frac{2n-3}{2n}$

Now, we prove by induction on  $n$  that the first summand in Equation 6.1 is  $\leq \frac{8}{n}$ . Denote this first term by  $\overline{wl}_n$ . Note that  $\overline{wl}_{n+1} = (1 - x_{n+1})^2 + x_{n+1}\overline{wl}_n$ . Assuming that  $\overline{wl}_n \leq \frac{8}{n}$ , and using the two claims above, it is easy to prove that  $\overline{wl}_{n+1} \leq \frac{8}{n+1}$  for  $n > 14$ . (the reader can verify that this also holds for  $n \leq 14$ .)

Next, we prove (again by induction on  $n$ ) that the second expression is  $\leq \frac{1}{n}$ . We assume  $\prod_{i=1}^n x_i \leq \frac{1}{n}$  and prove that  $\prod_{i=1}^{n+1} x_i \leq \frac{1}{n+1}$  (using Claim 6) :

$$\prod_{i=1}^{n+1} x_i = x_{n+1} \prod_{i=1}^n x_i \leq x_{n+1} \frac{1}{n} \leq \frac{2n-1}{2n+2} \cdot \frac{1}{n} < \frac{2n-1}{2n+2} \cdot \frac{1}{n} + \frac{1}{2n(n+1)} = \frac{1}{n+1}$$

Thus, the expected welfare loss is  $\leq \frac{8}{n} + \frac{1}{n} = \frac{9}{n}$  □

**Corollary 5.** *Consider the profit-optimal mechanism  $MPG_2(\vec{y})$  described in Corollary 4. When the players' valuations are distributed uniformly,  $MPG_2(\vec{y})$  incurs a profit loss  $\leq \frac{8}{n}$ .*

*Proof.* The  $O(\frac{1}{n})$  upper bound for the profit loss is a corollary of the previous theorem (again we can use the regularity of the uniform distribution to reduce profit optimization to welfare optimization). Nevertheless, a direct proof is straightforward: with the same thresholds  $x$  from Theorem 10 above, the profit loss is bounded from above by<sup>25</sup>:

$$\sum_{i=1}^n \left( \prod_{j=i+1}^n x_j \right) \cdot (1 - x_i) \cdot (1 - x_i)$$

We showed in Theorem 10 above that this expression is  $\leq \frac{8}{n}$ . □

## 7 Multi-round auctions

In previous sections, we analyzed auctions with bounded communication in which players simultaneously send their bids to the mechanism. We presented the optimal mechanisms and gave tight upper bounds for the profit loss and the efficiency loss they incur. But can we get better results with sequential mechanisms? That is, mechanisms in which players split their bids to smaller messages and send them in alternating order? In this section, we show that sequential mechanisms can achieve better results. However, the additional gain (in the amount of communication) is up

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<sup>25</sup>In the priority games based on the thresholds  $\vec{y}$ , if player  $i$  wins the item, he pays  $y_i$ . Thus, the maximal profit loss when player  $i$  wins is  $1 - y_i$ .

	$B$	0	1
$A$			
0		$A, 0$	$B, \frac{1}{4}$
1		$A, \frac{1}{3}$	$B, \frac{3}{4}$

Figure 8: ( $h_1$ ) This sequential game (when  $A$  bids first) achieves a higher expected welfare than any simultaneous mechanism with the same communication requirement (2 bits). The welfare is achieved with Bayesian-Nash equilibrium.

to a linear factor in the number of players. This result is somewhat surprising since we know (see e.g. [7]) that in general, multi-round protocols can reduce the communication communication by an exponential factor.

## 7.1 Sequential Mechanisms Can Do Better

The definitions in this section are similar in spirit to the model described in Section 3. Therefore, we present the model for sequential mechanisms less formally.

**Definition 21.** A *sequential (or multi-round) mechanism* is a mechanism in which each player sends several messages, in some arbitrary order among the players<sup>26</sup>. In each stage, each player knows what messages the other players have sent so far. After all the messages were sent, the mechanism determines the allocation and payments. The allocation scheme and the payment scheme are known, of course, to all players in advance. In addition, the sizes of the messages, their number and the order in which they are sent are also commonly known in advance.

**Definition 22.** The *communication requirement of a mechanism* is the total amount of bits which are sent by the players.

**Definition 23.** A *strategy* for a player in a sequential mechanism is the way she determines the messages she sends to the mechanism, in every stage of the mechanism, given her valuation and given the other players' messages up to each stage.

A strategy for a player in a sequential mechanism is called a *threshold strategy* if in each stage  $i$  of the game the player determines the message she sends by comparing her valuation to some threshold values  $x_1, \dots, x_{\alpha_i}$  (where this player has  $\alpha_i + 1$  possible bids in stage  $i$ ). For example, for 1 bit messages, if her valuation is smaller than  $x_1$  she bids 0, or bids 1 otherwise.

Denote the following sequential mechanism by  $h_1$  ( see Figure 8 ): Alice sends one bit to the mechanism first. Bob, knowing Alice's bid, also sends one bit. When Alice bids 0: Bob wins if he

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<sup>26</sup>Any order of the messages in possible, not necessarily in a round-robin fashion.

bids 1 and pays  $\frac{1}{4}$ ; If he bids zero Alice wins and pays zero. When Alice bids 1: Bob also wins when he bids 1, but now he pays  $\frac{3}{4}$ ; If he bids zero, Alice wins again, but now she pays  $\frac{1}{3}$ .

The communication requirement of this mechanism is 2 (each player sends one bit to the mechanism). When the players' valuations are distributed uniformly, this mechanism achieves an expected welfare which is greater than the optimal welfare from simultaneous mechanisms with the same communication requirement: It is easy to see that  $h_1$  has a Bayesian-Nash equilibrium<sup>27</sup> that achieves an expected welfare of 0.653. We saw that the efficient simultaneous mechanism with a communication requirement of 2 bits is 0.648 (see Section 2 ). Thus, we can gain more efficiency with sequential mechanisms.

## 7.2 The extra gain from sequential mechanisms is limited

How significant is the extra gain from sequential mechanisms over simultaneous mechanisms? The following theorem states that sequential mechanisms can save communication only up to a linear factor in the number of players. That is for every sequential mechanism with a communication requirement of  $m$  there exists a simultaneous mechanism that achieves at least the same welfare with a communication requirement of  $nm$  (where  $n$  is the number of bidders)<sup>28</sup>. We start again by proving that the optimal welfare can be achieved when all the players use threshold strategies.

**Lemma 2.** *Given a sequential mechanism  $h$  and a profile of strategies  $s = (s_1, \dots, s_n)$  of the players, there exists a profile of **threshold** strategies  $\bar{s} = (\bar{s}_1, \dots, \bar{s}_n)$  that achieves at least the same welfare with  $h$  as  $s$ .*

*Proof.* For an arbitrary player  $i$ , assume w.l.o.g that the strategies for the other players  $s_{-i}$  are fixed. We will modify  $s_i$  to be a threshold strategy  $\bar{s}_i$ , such that the welfare of  $(\bar{s}_i, s_{-i})$  does not decrease.

First, we prove by (backward) induction that in each stage, the expected welfare is a linear function of the valuation of player  $i$  ( $v_i$ ), for every message player  $i$  might send. For the last message that player  $i$  sends: if he is the last player, the expected welfare when he sends the message  $x$  will be  $a_i(x, b_{-i})v_i + \sum_{j \neq i} a_j(x, b_{-i})v_j$  which is linear in  $v_i$ . If other players sent messages after player  $i$  (i.e., player  $i$  was not the last player to play), each bids' combination will have a fixed probability ( $s_{-i}$  is fixed) and therefore the expected welfare will be a linear combination of linear functions, which forms a linear function. In earlier stages of the game, the expected welfare for each bit that

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<sup>27</sup>The following strategies are in Bayesian-Nash equilibrium: Alice uses a threshold strategy based on the threshold value  $\frac{1}{2}$ , and Bob uses the threshold  $\frac{1}{4}$  when Alice bids "0" and the threshold  $\frac{3}{4}$  when Alice bids 1.

<sup>28</sup>Note that in sequential mechanisms the players must be informed about the bits the other players sent (we do not take this into account in our analysis), so the total gain in communication can be very mild.

player  $i$  might send is a linear combination of the expected welfare in the future stages (which are linear functions of  $v_i$  according to the induction hypothesis). A linear combination of linear functions is a linear function by itself, and thus the proofs by induction is complete.

So in each stage where player  $i$  has to send a message of size  $\alpha$ , his decision actually chooses between  $2^\alpha$  linear functions for the expected welfare. (Note that we are not concerned about players' incentives here.) When his thresholds are the crossing points of these linear functions, the expected welfare is maximized.  $\square$

**Theorem 11.** *Let  $h$  be an  $n$ -player multi-round mechanism with communication requirement  $m$ . Then, there exists a **simultaneous** mechanism  $g$  that achieves at least the same expected welfare as  $h$ , with communication requirement smaller than  $nm - \frac{n(n-3)}{2}$ .*

*Proof.* Consider an  $n$ -player mechanism  $h$  with a Bayesian-Nash equilibrium, and with communication requirement  $m$  (for simplicity, assume  $n$  divides  $m$ , i.e., each player sends  $\frac{m}{n}$  bits. A similar proof holds for the other case). Due to Lemma 2, the decision of each player in each stage is a threshold decision: therefore, there exists a profile  $s = (s_1, \dots, s_n)$  of threshold strategies that achieves at least the same expected welfare on  $h$  as the given welfare. First, we give an upper bound for the total number of thresholds each player uses in the game. For a player  $i$ , let  $\alpha_1^i, \dots, \alpha_{k_i}^i$  be the (positive) sizes of the  $k_i$  messages she sends in  $h$ . Let  $\beta_j^i$  ( $1 \leq j \leq k_i$ ) be the number of bits that were sent by all the players (including  $i$ ), *before* player  $i$  sends his  $j$ th message. When choosing a message of size  $\alpha_j^i$ , the player uses up to  $2^{\alpha_j^i} - 1$  thresholds. In each stage, every player can use a different set of thresholds, for every possible history of the game. Thus, for sending his  $j$ th message she can use up to  $2^{\beta_j^i} \cdot (2^{\alpha_j^i} - 1)$  different thresholds. Summing up, player  $i$  uses no more than  $T(i) = \sum_{j=1}^{k_i} 2^{\beta_j^i} \cdot (2^{\alpha_j^i} - 1)$  thresholds. Now, assume w.l.o.g that the players are numbered according to the order they send their *last* messages (i.e.,  $\beta_{k_1}^1 > \beta_{k_2}^2 > \dots > \beta_{k_n}^n$ ). Recall that the total number of bits sent by the players is  $m$ . When sending the last message, player 1 thus uses  $2^{m-\alpha_{k_1}^1} \cdot (2^{\alpha_{k_1}^1} - 1) < 2^m$  different thresholds. Because all the messages have positive sizes, player 2 will have no more than  $2^{m-1-\alpha_{k_2}^2} \cdot (2^{\alpha_{k_2}^2} - 1) < 2^{m-1}$  different thresholds for the last stage. Similarly, every player  $i$  can use at most  $2^{m-i+1}$  thresholds for his last message. But therefore, for her before-last message player  $i$  uses at most  $2^{m-i-1}$  different thresholds (the worst case occurs when one player sends one bit between player  $i$ 's 2 last messages). It follows that the

maximal number of different thresholds for player  $i$  is:

$$T(i) = \sum_{j=1}^{k_i} 2^{\beta_j^i} \cdot (2^{\alpha_j^i} - 1) \quad (7.1)$$

$$< 2^{m-i+1} + 2^{m-i-1} + \sum_{j=1}^{k_i-2} 2^{\beta_j^i} \cdot (2^{\alpha_j^i} - 1) \quad (7.2)$$

$$< 2^{m-i+1} + 2^{m-i-1} + \sum_{j=1}^{m-i-2} 2^j \quad (7.3)$$

$$< 2^{m-i+1} + 2^{m-i-1} + 2^{m-i-1} < 2^{m-i+2} \quad (7.4)$$

Now, let  $g$  be a simultaneous mechanisms in which each player simply “informs” the mechanism between which of the thresholds he uses in  $h$ , his valuation lies. Clearly, for every set of valuations of the players, this allocation in  $g$  and  $h$  is identical. Due to inequality 7.4,  $m - i + 2$  bits suffice for player  $i$  to express this number. We conclude that the number of bits sent by all the players in  $g$  is smaller than:

$$\sum_{i=1}^n (m - i + 2) = nm - \frac{n(n-3)}{2}$$

Finally, we set the allocation scheme and the payment scheme in  $g$  such that the threshold-strategies based on the thresholds in  $s$  will be an equilibrium, and the expected welfare will not decrease. In the first step we change  $g$  to be a deterministic mechanism in which the item must be sold. This will be done by allocating the item to the player with the highest expected valuation, for each bid combination. That is for the bids’ vector  $b$  the winner is  $\operatorname{argmax}_i \{E(v_i | v_i \in [b_i, b_{i+1}])\}$ . As we saw in Section 4, this modification will result in a monotone mechanism. In monotone mechanisms we can set the payment scheme such that  $s$  will form a dominant-strategy equilibrium. Since we gave the item to the player with the highest expected welfare for each bits’ combination, the expected welfare has not decreased.  $\square$

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## A Welfare Optimization in 2-player Mechanisms

### A.1 Optimality of Threshold Strategies

*Claim 7.* Given any mechanism  $g \in G_{n, (k_1, \dots, k_n)}$ , there exists a vector of threshold strategies  $s \in \times_{i=1}^n \varphi_{k_i}$  that achieve the optimal welfare in  $g$  among all possible strategies, i.e.,

$$w(g, s) = \max_{\tilde{s} \in \times_{i=1}^n \varphi_{k_i}} w(g, \tilde{s})$$

*Proof.* Given a vector of strategies  $s^*$  which achieve optimal welfare in  $g$  (i.e.,  $\max_{\tilde{s} \in \times_{i=1}^n \varphi_{k_i}} w(g, \tilde{s})$ ), we will show that for every player  $i$  we can modify  $s_i^*$  to be a threshold strategy, and the welfare will not decrease.

Assume  $s_i^*$  is not a threshold strategy. Therefore, there must be  $\alpha, \beta, \gamma \in [\underline{a}, \bar{b}]$ ,  $\alpha < \beta < \gamma$  such that  $s_i^*(\alpha) = s_i^*(\gamma) = m$  but  $s_i^*(\beta) \neq m$  (where  $m$  is some bid of player  $i$ ). We will show that a strategies' vector  $s$  identical to  $s^*$ , except  $s_i(\beta) = m$ , holds  $w(g, s) \geq w(g, s^*)$ .

Denote the probability that all players except  $i$  bids  $b_{-i}$  as  $Pr(b_{-i})$ . Thus, the expected welfare from a game  $g$  given that bidder  $i$  with valuation  $v_i$  bids  $m$  and that the other players use strategies  $s_{-i}^*$  is:

$$\sum_{b_{-i}} Pr(b_{-i}) \left( a_i(m, b_{-i}) \cdot v_i + \sum_{j \neq i} a_j(m, b_{-i}) \cdot E(v_j | s_j^*(v_j) = b_j) \right)$$

Note that this expected welfare is a linear function of  $v_i$ , and we denote it by  $h(m) \cdot v_i + t(m)$  (the constants  $h(m)$  and  $t(m)$  depend on the bid  $m$ ).

We know that  $s^*$  achieve optimal welfare in  $g$  and that  $s_i^*(\alpha) = m$ . Therefore, there is no other bid  $l$  such that if  $s_i^*(\alpha) = l$ , the expected welfare will increase, i.e.:

$$\forall l \neq m \quad h(m) \cdot \alpha + t(m) \geq h(l) \cdot \alpha + t(l) \quad (\text{A.1})$$

Similarly, because  $s_i^*(\gamma) = m$ :

$$\forall l \neq m \quad h(m) \cdot \gamma + t(m) \geq h(l) \cdot \gamma + t(l) \quad (\text{A.2})$$

Because  $\beta$  is a convex combination of  $\alpha$  and  $\gamma$ , and due to Equations A.1 and A.2:

$$\forall l \neq m \quad h(m) \cdot \beta + t(m) \geq h(l) \cdot \beta + t(l)$$

Thus, the expected welfare for player  $i$ , given  $v_i = \beta$ , is maximal when she bids  $m$ . Therefore, when modifying  $s_i^*$  such that  $s_i^*(\beta) = m$  the total expected welfare will not decrease. We can repeat this process until  $s_i^*$  becomes a threshold strategy.  $\square$

## A.2 More Possible Bids Enables a Higher Welfare

**Lemma 3.**  $w_{2,(k,k)}^{opt} > w_{2,(k-1,k)}^{opt}$  for every  $k > 1$ .

*Proof.* From Claims 2, 3 and 1 we know that there is a monotone, deterministic mechanism  $g \in G_{2,(k-1,k)}$  in which the item must be sold, and threshold strategies based on some threshold-value vectors  $(x, y)$  that achieve with  $g$  the optimal welfare (i.e.,  $w_{2,(k-1,k)}^{opt}$ ). In such mechanism, each row is of the form  $[A, \dots, A, B, \dots, B]$ , and let  $l_i$  be the first index in row  $i$  in which the item is allocated to player  $B$ .  $l_i$  can have  $k + 1$  values (between 0 and  $k$ ), but since we have  $k - 1$  rows, there are 2 row forms which are missing. We will modify  $g$  to  $\tilde{g} \in G_{2,(k,k)}$  by adding some missing row, and change the threshold strategy  $x$  to  $\tilde{x} \in \mathfrak{R}^{k+1}$ , such that  $\tilde{g}$  achieves with the threshold strategies based on  $\tilde{x}$  and  $y$  an expected welfare that is strictly greater than the originals. Note that the monotonicity also tells us that the index where the item starts to be allocated to  $B$  cannot decrease with the row number (i.e.,  $l_i \leq l_{i+1}$ ). We can also assume that the thresholds of each player are unique (i.e.,  $0 < x_1 < \dots < x_{k-1} < 1$ ,  $0 < y_1 < \dots < 1$ ). (If not, we can omit the duplicates, and add them back after inserting the new line.)

We treat two cases separately: When the row  $[B, \dots, B]$  is one of the rows in the game's matrix, or when it is not.

*Case 1.* We can insert the new line as a first line in the game's matrix representation (i.e., when

both players bid “0”,  $A$  wins the item, so we can add the line  $[B, \dots, B]$ .

Let  $x'_1 = \frac{E_{v_B}(v_B|0 \leq v_B \leq y_1)}{2}$ , and let  $\tilde{x} = (0, x'_1, x_1, x_2, \dots, x_{k-2}, 1)$ . We will create a new  $k \times k$  mechanism  $\tilde{g}$  by adding the line  $[B, \dots, B]$  as the first line in the game matrix. It is easy to see that for every two valuations of the players, the allocation in  $g$  and  $\tilde{g}$  is identical, except when the players  $A, B$  bid  $(0, 0)$  or  $(0, 1)$ . Both players will bid 0 when  $v_A \in [0, x'_1]$  and  $v_B \in [0, y_1]$ . This situation will occur with a positive probability (since the distribution functions are always positive). In  $g$ , we allocated the item to  $A$  in these cases, and in  $\tilde{g}$  we allocate the item to  $B$ . Note that  $E(v_A|v_A \in [0, x'_1]) < x'_1 < E_{v_B}(v_B|0 \leq v_B \leq y_1)$ , thus the expected valuation of the players who won the item has increased, so the total expected welfare has increased. If  $g$  allocated the item to  $A$  for the bids vector  $(0, 1)$ , the allocation is now different when  $v_A \in [0, x'_1]$  and  $v_B \in [y_1, y_2]$ . Since  $E_{v_A}(v_A|0 \leq v_A \leq x'_1) < E_{v_B}(v_B|0 \leq v_B \leq y_1)$ , the expected valuation will be higher in  $\tilde{g}$  also in this case.

*Case 2.* The row  $[B, \dots, B]$  does not appear in  $g$ 's game matrix.

Now,  $g$  must have 2 rows  $i$  and  $i + 1$  and 2 columns  $j$  and  $j + 1$  such that we allocate the item to  $B$  when the bids are  $(i, j), (i, j + 1)$  and to  $A$  when the bids are  $(i + 1, j), (i + 1, j + 1)$  (see Figure 9). We will create a new mechanism  $\tilde{g}$  by adding a row  $i'$  identical to row  $i$  except for the index  $j + 1$  where  $B$  gets the item. The way we construct the new threshold strategy  $\tilde{x}$  for  $A$  depends on whether the expected valuation of  $B$  when bidding  $j$  is smaller than  $x_i$  or not:

**When  $E(v_B|y_{j-1} \leq v_B \leq y_j) < x_i$ :**

Let  $x'_i = E(v_B|y_{j-1} \leq v_B \leq y_j)$ , and let  $\tilde{x} = (0, x_1, \dots, x_{i-1}, x'_i, x_i, \dots, 1)$ . As in previous cases, the expected welfare in all entries hasn't changed, except a strictly positive increase in the  $(i', j)$  index (because  $E(v_A|x'_i \leq v_B \leq x_i) > x_i$ ), so the total welfare has increased. See Figure 9 part (a).

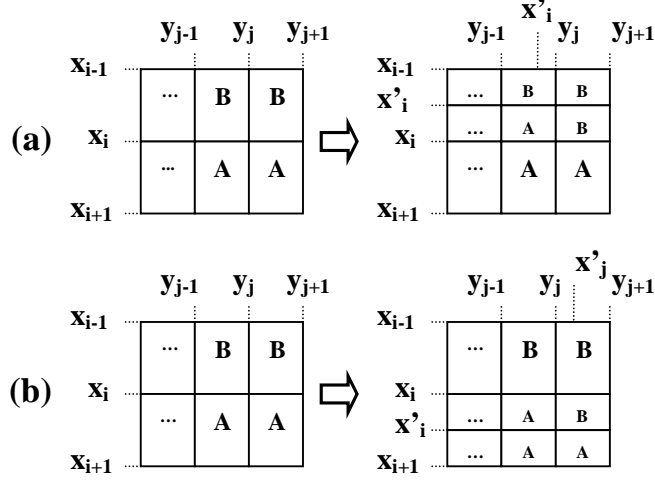
**When  $E(v_B|y_{j-1} \leq v_B \leq y_j) \geq x_i$ :**

Because the probabilities are always positive,  $E(v_B|y_j \leq v_B \leq y_{j+1}) > E(v_B|y_j \leq v_B \leq y_{j+1}) \geq x_i$ . Player  $A$  wins the item in entry  $(i + 1, j + 1)$  thus  $E(v_B|y_j \leq v_B \leq y_{j+1}) \leq E(v_A|x_i \leq v_A \leq x_{i+1}) < x_{i+1}$  so  $x_i < x'_i < x_{i+1}$ . Let  $\tilde{x} = (0, x_1, \dots, x_i, x'_i, x_{i+1}, \dots, 1)$ . Again, the expected welfare has changed only in the entry  $(i', j)$ , where it has increased (because  $x'_i > E(v_A|x_i \leq v_A \leq x'_i)$ ), so the total welfare has increased. See Figure 9 part (b).  $\square$

### A.3 n-player Mechanisms

*Claim 8.*  $\forall_k \quad 1 - x_k \leq \frac{2}{k}$

Figure 9: Adding a middle row to  $(k-1 \times k)$  game with optimal welfare



*Proof.* By induction on  $k$ . For  $k = 1$ , we know  $x_k = \frac{1}{2}$ , and  $1 - x_k = \frac{1}{2} < \frac{2}{k}$ . We assume  $1 - x_k \leq \frac{2}{k}$ , and prove that  $1 - x_{k+1} \leq \frac{2}{k+1}$  (we use the fact that  $\frac{k^2}{(k-1)(k+1)} \geq 1$  for every  $k > 1$ ):

$$\begin{aligned}
 1 - x_{k+1} &= 1 - \left(\frac{1}{2} + \frac{x_k^2}{2}\right) \leq \frac{1}{2} - \frac{\left(1 - \frac{2}{k}\right)^2}{2} = \frac{2(k-1)}{k^2} \\
 &\leq \frac{2(k-1)}{k^2} \cdot \frac{k^2}{(k-1)(k+1)} = \frac{2}{k+1}
 \end{aligned}$$

□

*Claim 9.*  $\forall_{k \geq 15} \quad x_k \leq \frac{2k-3}{2k}$

*Proof.* Again, by induction on  $k$ . For  $k = 15$ ,  $x_{15} = 0.899 \leq 0.9 = \frac{2 \cdot 15 - 3}{2 \cdot 15}$ . We assume  $x_k \leq \frac{2k-3}{2k}$ , and we prove that  $x_{k+1} \leq \frac{2(k+1)-3}{2(k+1)}$ .

$$x_{k+1} = \frac{1}{2} + \frac{x_k^2}{2} \leq \frac{1}{2} + \frac{\left(\frac{2k-3}{2k}\right)^2}{2} = \frac{8k^2 - 12k + 9}{8k^2}$$

It suffices to prove that  $\frac{2(k+1)-3}{2(k+1)} - \frac{8k^2-12k+9}{8k^2} \geq 0$ , and indeed:

$$\frac{2(k+1)-3}{2(k+1)} - \frac{8k^2-12k+9}{8k^2} = \frac{6k-18}{16k^2(k+1)} \geq 0$$

□