Econometrica, Vol. 78, No. 5 (September, 2010), 1749–1772

MONOTONICITY AND IMPLEMENTABILITY

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Consider an environment with a finite number of alternatives, and agents with private values and quasilinear utility functions. A domain of valuation functions for an agent is a monotonicity domain if every finite-valued monotone randomized allocation rule defined on it is implementable in dominant strategies. We fully characterize the set of all monotonicity domains.

KEYWORDS: Monotone, cyclic monotonicity, implementable, dominant strategies.

1. INTRODUCTION

WE CONSIDER AN ENVIRONMENT with a finite set of alternatives A, and agents with private values and quasilinear preferences. We focus on direct revelation mechanisms, which consist of an allocation rule and a payment function. The allocation rule maps each profile of valuations to a probability vector over the set of alternatives.[2] Our interest is in allocation rules that are implementable in dominant strategies. For brevity, such rules will be called just implementable.

Monotonicity is a necessary but not sufficient condition for an allocation rule to be implementable.[3] Rochet (1987) (see also Rockafellar (1970)) showed that a condition called cyclic monotonicity, is both necessary and sufficient for any allocation rule to be implementable. Cyclic monotonicity, however, is a considerably more difficult condition to work with than monotonicity; roughly, monotonicity is a condition on every pair of values, whereas cyclic monotonicity is a condition on every finite sequence of values.[4,5] Therefore, studying in which domains monotonicity is also sufficient for implementing an arbitrary allocation rule is a desired task. Myerson (1981) showed that in a single dimensional domain, monotonicity is sufficient for any allocation rule to

[1]We thank André Berger, Shahar Dobzinski, Ron Lavi, Rudolf Müller, Seyyed Hossein Naeemi, and Maria Polukarov for very helpful discussions. We also thank Jacob Leshno, Scott Kominers as well as the editor and two anonymous referees for useful comments. Dov Monderer thanks the Israeli Science Foundation (ISF) and the Fund for the Promotion of Research at the Technion. Itai Ashlagi and Avinatan Hassidim acknowledge support from Microsoft Research, where they were consultant researchers during this project.

[2]What is called here an allocation rule is often called a randomized allocation rule.

[3]An example of a monotone allocation rule which is not implementable was given by Saks and Yu (2005).

[4]For a good background on monotonicity and cyclic monotonicity, see Bikhchandani et al. (2006) and Vohra (2007). See also Jehiel and Moldovanu (2001), Jehiel, Moldovanu, and Stacchetti (1996), and Krishna and Maenner (2001) for characterizations of Bayesian incentive compatible mechanisms.

[5]See Lavi and Swamy (2009), who presented a mechanism in a scheduling setting and showed that the cyclic monotonicity holds so as to prove that their mechanism is implementable.

 DOI: 10.3982/ECTA8882

be implementable.[6] Bikhchandani et al. (2006) proved that in many convex domains, most notably R_+^A, every monotone deterministic allocation rule[7] is implementable. Gui, Müller, and Vohra (2004) noticed that by a theorem by Roberts (1979), this result holds for the unrestricted domain $D = R^A$, and they proved, in addition, that it holds for every cube. Finally, Saks and Yu (2005) extended this result for any convex domain.[8]

In this paper, we characterize domains for which every finite-valued (finite range) monotone allocation rule is implementable. Such domains are called *monotonicity domains*. We begin with characterization of *proper monotonicity domains*, which are defined similarly to monotonicity domains, except allocation rules can also output subprobability vectors (rather than just probability vectors).[9] Using the characterization of proper monotonicity domains, we are able to characterize monotonicity domains. Our results do not rule out the possibility that a particular monotone allocation rule can be implementable in a nonmonotonicity domain.

It can be shown that every domain with a convex closure is a proper monotonicity domain. One way to see this is to extend the result of Saks and Yu (2005) to finite-valued allocation rules, which also output subprobability vectors, and to domains with a convex closure (instead of just convex domains). Deriving these extensions requires some effort, and the Supplemental Material (Ashlagi, Braveman, Hassidim, and Monderer (2010)) gives an alternative simpler proof. Our main result is the other direction: if the closure of a domain with dimension at least 2 is not convex, there exists a finite-valued monotone allocation rule, which possibly outputs subprobability vectors, that is not implementable. The usefulness of this result is that it helps to identify some domains as not being (proper) monotonicity domains. One such domain is the class of gross substitutes preferences.

Both the finite-valued and randomized properties of the allocation rules are necessary; indeed, Archer and Kleinberg (2008) and Bikhchandani et al. (2006) each gave an example for a monotone allocation rule on a convex domain, with infinitely many outcomes, which is not cyclically monotone. Moreover, both Vohra (2010) and Mu'alem and Schapira (2008) exhibited examples of multidimensional nonconvex domains in which every deterministic monotone allocation rule is implementable.

Knowing that a domain D is a monotonicity domain can serve as a useful tool in other mechanism design problems. For example, one can use this to find a revenue-optimal dominant strategy incentive compatible mechanism on D.

[6]Following Myerson (1981), other authors used the monotonicity condition to prove that their suggested allocation rules are implementable (in single dimensional domains). See, for example, Goldberg et al. (2006) and Lehmann, O'Callaghan, and Shoham (2002).

[7]A deterministic allocation rule always assign probability 1 to some alternative.

[8]Berger, Müller, and Naeemi (2009) extended Saks and Yu's result to convex valuation functions.

[9]Note that every proper monotonicity domain is also a monotonicity domain.

Indeed, proving in the Bayesian setup that every monotone allocation rule is implementable was a key result in finding an optimal single-item auction in Myerson (1981). In monotonicity domains, it is also easier to deal with the important task of finding concrete characterizations of implementable allocation rules, as the one appearing in Roberts (1979), where it was proved that every implementable deterministic allocation rule is an affine maximizer. Roberts proved the theorem using a condition called positive association of differences, which is very similar in spirit to monotonicity (see Bikhchandani et al. (2006) and Lavi, Mu'alem, and Nisan (2003) for details).

Recently, researchers in the area of algorithmic mechanism design have been discussing efficiency bounds: Let g be a desired social choice function and let f be an allocation rule. They are interested in how bad (upper bounds) and how good (lower bounds) the difference between g and f can be (can be measured in various ways) when one insists that f is implementable. Such problems have been extensively analyzed in the computer science literature (see, e.g., Nisan and Ronen (2001), Archer and Tardos (2007), and Lavi and Swamy (2007)). Again, knowing that the domain is a monotonicity domain can be useful in analyzing these bounds.

In the next section, we present the model and our results. In Section 3, we show that if a domain does not have a convex closure, then there exists a finite-valued monotone allocation rule which is not implementable. In other words, we complete the characterization of proper monotonicity domains. In Section 4, we characterize monotonicity domains using the characterization of proper monotonicity domains.

2. MODEL AND RESULTS

We restrict our attention to a model with a single agent. This is without loss of generality, as all relevant definitions can be interpreted by holding all other agents' types fixed. Let A be a finite set of alternatives. Let R^A be the set of all possible valuation functions on A, that is, the set of all real-valued functions defined on A. The value of a for an agent with valuation v is thus v_a. It is convenient to represent each alternative a by its associated unit vector $e^a \in R^A$, where $e^a_a = 1$ and $e^a_b = 0$ for every $b \neq a$. Let $Z(A)$ be the set of all probability vectors $z \in R^A$:

$$Z(A) = \left\{ z \in R^A \;\middle|\; z_a \geq 0 \;\forall a, \sum_{a \in A} z_a = 1 \right\}.$$

Let $D \subseteq R^A$ and let $f: D \to Z(A)$. We think of D as the set of all possible valuations of a given agent with a quasilinear utility function, and f is interpreted as an *allocation rule*. If $f(v) \in \{e^a \mid a \in A\}$ for every $v \in V$, then f is called a deterministic allocation rule. If an agent with valuation v declares w, alternative a is chosen with probability $f_a(w)$ and, therefore, she evaluates

$f(w)$ by the inner product $\langle v, f(w)\rangle = \sum_{a\in A} v_a f_a(w)$. An allocation rule f is *finite-valued* if its range $\{f(v) \mid v \in D\}$ is a finite set.

We say that an allocation rule f is *implementable in dominant strategies* (or just implementable) if there exists a payment function $c : D \to R$ such that

$$\text{(1)} \qquad \langle v, f(v)\rangle - c(v) \geq \langle v, f(w)\rangle - c(w) \quad \forall v, w \in D.$$

Inequality (1) implies that given the payment function c, the agent is better off reporting v over w when her value is v. Writing the same inequality while reversing the order of v and w, and summing with (1), one obtains

$$\text{(2)} \qquad \langle f(v) - f(w), v - w\rangle \geq 0 \quad \text{for every } v, w \in D.$$

An allocation rule satisfying (2) is called *monotone*.

It was observed by Rochet (1987) that every implementable allocation rule f satisfies a stronger monotonicity property. f is called *cyclically monotone* if for every $k \geq 2$ and for every k vectors in D (not necessarily distinct), $v_1, v_2, \ldots, v_k$,

$$\text{(3)} \qquad \sum_{i=1}^{k} \langle v_i - v_{i+1}, f(v_i)\rangle \geq 0,$$

where v_{k+1} is defined to be v_1. By taking $k = 2$ in (3), it can be seen that every cyclically monotone allocation rule is monotone. The following characterization of implementability was proved by Rochet (1987).

THEOREM 1—Rochet: *An allocation rule is implementable if and only if it is cyclically monotone*.

We say that a domain of valuation functions is a *monotonicity domain* if every finite-valued monotone allocation rule defined on it is implementable. It is well known that every domain of dimension at most 1 is a monotonicity domain (Myerson (1981)).

To characterize monotonicity domains, we need to consider an equivalent definition which relaxes the allocation rule to output also subprobability vectors. Formally, let $\bar{Z}(A)$ be the set of all subprobability vectors $z \in R^A$:

$$\bar{Z}(A) = \left\{ z \in R^A \;\middle|\; z_a \geq 0 \; \forall a, \sum_{a\in A} z_a \leq 1 \right\}.$$

We say that a domain of valuation functions is a *proper monotonicity domain* if every finite-valued monotone function $f : D \to \bar{Z}(A)$ is implementable.[10] To

[10]Defining implementability, monotonicity, and cyclic monotonicity is similar for functions of the form $f : D \to \bar{Z}$. Moreover, Rochet's theorem holds for such functions.

avoid confusion, only functions that always output probability vectors will be called allocation rules.

An important step in characterizing monotonicity domains is due to Saks and Yu (2005).

THEOREM 2—Saks and Yu: *Every deterministic allocation rule on a convex domain is implementable*.

In the Supplemental Material, we give an alternative simpler proof. Our proof holds under weaker assumptions, only requiring the domain to have a convex closure and the allocation rule to be finite-valued. Furthermore, we allow the allocation rule to output subprobability vectors. In other words, it is shown that *every domain with a convex closure is a proper monotonicity domain*. Other proofs and extensions can be found in Archer and Kleinberg (2008) and Vohra (2007).

In our main result, we complete the characterization of proper monotonicity domains (Theorem 3); that is, if a domain with dimension at least 2 does not have a convex closure, then there exists a monotone finite-valued function $f : D \to \bar{Z}(A)$ which is not implementable.

Finally, we use the characterization for proper monotonicity domains to characterize monotonicity domains (Theorem 11); that is, a domain D is a monotonicity domain if and only if its projection to the hyperplane $H^A = \{v \in R^A : \sum_{a\in A} v_a = 0\}$ is a proper monotonicity domain.[11]

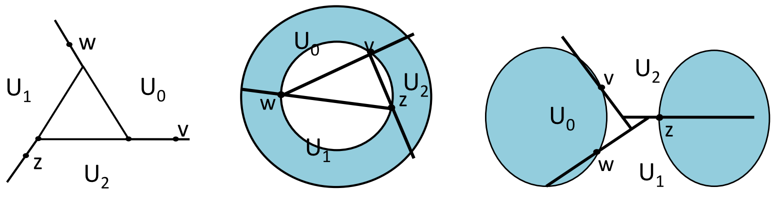
3. DOMAINS WITH A NONCONVEX CLOSURE

For every domain D, let M_D be the linear space generated by all differences $v - w$, where $v, w \in D$. The dimension $d(D)$ of D is defined to be the linear dimension of M_D. It is well known that D is 0 dimensional if and only if D is a singleton. Let $k \geq 1$. It is also well known that $d(D) \geq k$ if and only if there exist $k+1$ distinct valuations in D, $v_0, v_1, \dots, v_k$, such that $v_1 - v_0, v_2 - v_0, \dots, v_k - v_0$ are linearly independent. In this section we prove the following theorem.

THEOREM 3: *If the closure of a domain of dimension at least* 2 *is not convex*, *there exists a monotone finite-valued function* $f : D \to \bar{Z}(A)$ *which is not implementable*. *Alternatively*, *the closure of every proper monotonicity domain of dimension at least* 2 *is convex*.

The proof of Theorem 3 is by construction. We distinguish between domains of dimension 2 and domains of higher dimensions. The proof for $k = 2$ is given

[11]Intuitively, valuations that project onto the same point in H^A reflect identical preferences under probability distributions. Thus there is a natural connection between allocation rules on D and arbitrary functions on the projection of D to H^A.



(a) (b) (c)

FIGURE 1.—(a) The “ideal” case, in which the domain is the entire space except a triangle. (b) and (c) The triangle is embedded such that its interior does not intersect the domain (the shaded region), but the extensions (including the vertices) of the triangle’s edges do.

in Section 3.1 and the proof for $k \geq 3$ is given in Section 3.2. The reason for distinguishing between the dimensions is subtle and will be explained below. We begin with a sketch the proof of Theorem 3 for domains of dimension $k = 2$:

Step 1. First we construct a monotone function f that is not implementable on a domain which is obtained by removing from R^2 the relative interior of a triangle (see Figure 1(a)). The range of f contains exactly three outcomes, each obtained in a different region U_0, U_1, and U_2. Furthermore, f violates the cyclic monotonicity condition with every three valuations that each belong to a different intersection of two of the regions U_0, U_1, and U_2. For example, v, w, and z in Figure 1(a) as well as the three vertices of the triangle violate cyclic monotonicity.

Step 2. Next it is shown that if the domain D has a nonconvex closure, then there exist three valuations such that the relative interior of the convex hull generated by them contains a ball which does not intersect D. Subsequently it is shown that the structure identified in Step 1 can be embedded in D such that the triangle will be located in the ball (see, e.g., Figure 1(b) and (c)).

For further intuition regarding our construction and why we distinguish between domains of dimension 2 and domains of higher dimensions, we use the following theorem which provides a unique way (up to a constant) to assign a utility function (or prices) for a cyclically monotone function on a polygonally connected domain.[12] This property is called *revenue equivalence* (see also Myerson (1981)).

THEOREM 4—Derived From Rockafellar (1970): *Let f be an allocation rule on a domain D. f is cyclically monotone on D if and only if there exists a real-valued function U on D such that*[13]

$$U(v_2) - U(v_1) \geq \langle f(v_1), v_2 - v_1 \rangle \quad \forall v_1, v_2 \in D. \tag{4}$$

[12]A domain D is called polygonally connected if for every two values $v, w \in D$, there is a polygonal path in D from v to w.

[13]$U(v)$ can be interpreted as the utility function of the agent when her valuation is v.

Let $[w, v] \subseteq D$. *Every function* U *satisfying* (4) *satisfies*

$$U(v) - U(w) = \int_0^1 \phi(t)\,dt, \tag{5}$$

where

$$\phi(t) = \langle f(w + t(v - w)), v - w\rangle.^{14} \tag{6}$$

Consequently, *if* D *is polygonally connected*, *any two functions satisfying* (4) *differ by a constant*.

Suppose f is defined on a polygonally connected domain D. By Theorem 4, had f been cyclically monotone, one could define a utility function U by choosing a single valuation v and fixing $U(v)$; then, for any valuation w, $U(w)$ is defined by just taking the integrals of f over some polygonal path from v to w. Hence, if one can provide a pair of valuations v and w and two different polygonal paths from v to w in the domain such that the integrals of f over these paths are not equal, or alternatively the integrals of f over the polygon that is formed by the two paths is not zero, then f is not cyclically monotone. In the proof for domains of dimension $k = 2$, we essentially constructed a monotone function f such that its integral over a polygon (the triangle) in the domain is not zero. We next explain why such an approach may fail in higher dimensions.

To construct a monotone function f which is not implementable, it is useful to first identify polygons for which the above process cannot work, that is, the integral of f over the polygon must be 0. First any polygon for which its convex hull belongs to D can be excluded, since one can consider the domain to be exactly the convex hull of the polygon. Furthermore, any polygon that can be contracted through the domain to a point on the domain can be excluded (and in particular if the domain is simply connected,[15] no polygon which is defined on the domain can be chosen): to see this, suppose, for example, that the polygons $\Gamma_1 = \langle v_1, \dots, v_5, v_1\rangle$ and $\Gamma_2 = \langle w_5, \dots, w_1, w_5\rangle$, and the triangles (including their interiors) $\Delta_1, \dots, \Delta_{10}$ in Figure 2 are contained in D. Since f is monotone on D, it is monotone on each Δ_i and, therefore, it is cyclically monotone on each Δ_i (by convexity). Thus the integral of f on the boundary on each triangle Δ_i is 0. This implies that the integral of f on Γ_1 equals the integral of f on Γ_2 (by adding to the integral of f on Γ_1 the integrals of f over all triangles $\Delta_1, \dots, \Delta_{10}$ in the directions as in Figure 2). If Γ_2 can be further contracted in this way to a point, then we obtain that integral of f over Γ_1 is 0.

The proof for domain D with dimension $k \geq 3$ has a similar idea. First it is shown how to create a monotone function over the domain R^k excluding the

[14]Note that if f is monotone, ϕ is necessarily nondecreasing and, therefore, it is in particular Riemann integrable.

[15]In a simply connected domain, every polygon can be contracted to a point.

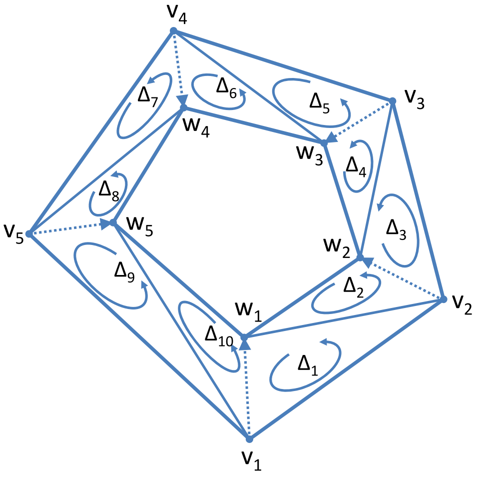


FIGURE 2.—The integral of f over the polygon $\Gamma_1 = \langle v_1, v_2, \ldots, v_5, v_1\rangle$ equals the integral of f over the polygon $\Gamma_2 = \langle w_5, w_1, \ldots, w_5\rangle$.

relative interior of convex polytope[16] (here the polytope will be a k dimensional triangular prism). Then we show how to embed the construction in any domain with a nonconvex closure of dimension k.

However, since the domain we use in the first step for $k \geq 3$ is simply connected, we use a slightly different approach (than for $k = 2$) in constructing the function. First note that implementability of a function implies that for every two valuations v and w such that $f(v) = f(w)$, the price in v and w must be the same. Thus for such v and w, even if the entire segment between (v, w) is not included in the domain, one can still calculate $U(w)$ given the utility $U(v)$. This fact allows us to deal also with polygons where part of them is not defined on the domain. This circumvents the contraction problem, that is, a polygon that has a part not defined on the domain cannot be contracted through the domain to a point.[17]

The following lemma will be useful in our proof; it enables us to embed the constructions in various domains. The first part of the lemma provides that a proper monotonicity domain can be equivalently defined using func-

[16] A convex polytope is a convex hull of a finite set of points.

[17] Another way to see that the direct extension of the first step of $k = 2$ does not work for $k = 3$ is the following example: We wish to construct a monotone function on the faces of a polytope with vertices, v_0, v_1, v_2, and v_4 (i.e., a tetrahedron), which violates cyclic monotonicity on the vertices. Note that f is cyclically monotone on every face of the polytope since each one of the faces is convex. Therefore, for every three vertices v_i, v_j, and v_k, there exists a real-valued function $U_{i,j,k}$ satisfying (4) on the convex hull of v_i, v_j, and v_k. Note that one can shift $U_{0,1,2}$ and $U_{1,2,3}$ so that $U_{0,1,2}(v_1) = U_{1,2,3}(v_1) = U_{0,1,3}(v_1)$. By (5) it must be that $U_{0,1,2}(v_2) = U_{1,2,3}(v_2)$. Note that the function $U : \{v_0, v_1, v_2, v_3\} \to R$ defined by $U(v_i) = U_{0,1,3}(v_i)$ for $i = 0, 1, 3$ and $U(v_2) = U_{1,2,3}(v_2)$ satisfies (4), and, therefore, f is cyclically monotone on the vertices of the polytope D.

tions that do not output necessarily subprobability vectors, that is, by replacing $f:D\to\bar{Z}(A)$ with $f:D\to R^A$. The second part asserts that monotonicity and cyclic monotonicity are invariant under rotations and dilations. Hence when assessing whether a set $D\subseteq R^A$ is a proper monotonicity domain, we can choose the coordinates in any convenient way. The proof is given in the Appendix.

LEMMA 5: (i) *Let $D\subseteq R^A$. If there exists a monotone finite-valued function $f:D\to R^A$ which is not cyclically monotone, then there also exists a function $\tilde{f}:D\to\bar{Z}(A)$ with the same properties.*

(ii) *A domain $D\subseteq R^A$ is a proper monotonicity domain if and only if $L(D)$ is a proper monotonicity domain, where $L(D)$ is a rotation, affine shift, or contraction of D.*

3.1. *Domains of Dimension $k=2$*

In this section, we prove Theorem 3 for domains of dimension $k=2$. We begin by showing that if D is the plane R^2 that excludes an interior of a triangle, one can define a monotone finite-valued function on D which is not cyclically monotone.

3.1.1. *Preparations: The Plane Excluding a Triangle*

A set $L=\{v,w,z\}$ is called affine independent if its dimension is 2. The convex hull of an affine independent set $L=\{v,w,z\}$ is a simplex (triangle) denoted by $\Delta(L)$ and its relative interior is denoted by $\Delta^0(L)$.

Let $\alpha>0$, $\beta>0$ be any nonnegative reals and let

$$S=\left\{(0,0),(1,0),\left(\frac{1}{1+\alpha\beta},\frac{\alpha}{1+\alpha\beta}\right)\right\}. \tag{7}$$

Note that S is affine independent. The complement of $\Delta^0(S)$ is the union of the regions (see Figure 3)

$$U_0=\{v\in R^2:v_1\geq 1-\beta v_2 \text{ and } v_2\geq 0\},$$
$$U_1=\{v\in R^2:v_1\leq 1-\beta v_2 \text{ and } v_2\geq \alpha v_1\},$$

and

$$U_2=\{v\in R^2:v_2\leq \alpha v_1 \text{ and } v_2\leq 0\}.$$

For every $0\leq i,j\leq 2$ let $U_{i,j}=U_i\cap U_j$. In the next proposition, we construct a monotone finite-valued function on $R^2\setminus\Delta^0(S)$. Furthermore, this function will violate the cyclic monotonicity condition for every three points v, w, and z that each one is on an extension of a different edge of the triangle (see Figure 3). Our parametrization will provide such a construction for any triangle, as we will see later.

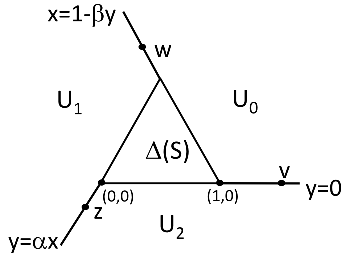


FIGURE 3.—The "ideal" (parameterized) domain for $k = 2$.

PROPOSITION 6: *There exists a monotone finite-valued function* $f : R^2 \setminus \Delta^0(S) \to R^2$ *which is not cyclically monotone. Furthermore, f can be chosen such that its range contains exactly three vectors* $y^0 = (0, 1)$, $y^1 = (-\frac{\alpha}{1+\alpha\beta}, \frac{1}{1+\alpha\beta})$, *and* $y^2 = (0, 0)$, *and the following statements hold*:

(i) *For* $i = 0, 1, 2$, $f(v) = y^i$ *for every* $v \in U_i \setminus U_{i,i+1}$.[18]

(ii) *For every three vectors* v, w, *and* z *such that* $v \in U_{0,2}$, $w \in U_{0,1}$, *and* $z \in U_{1,2}$,

$$\langle v - w, f(v)\rangle + \langle w - z, f(w)\rangle + \langle z - v, f(z)\rangle < 0. \tag{8}$$

PROOF: First we show that f is monotone. We need to show that $\langle v - w, f(v) - f(w)\rangle \geq 0$ for every $v, w \in R^2 \setminus \Delta^0(S)$. Three cases should be considered. Assume that $f(v) = y^0$ and $f(w) = y^1$. Thus $v \in U_0$ and $w \in U_1$. Therefore,

$$\langle v - w, y^0 - y^1\rangle = (v_1 - w_1)\frac{\alpha}{\alpha\beta + 1} + (v_2 - w_2)\frac{\alpha\beta}{\alpha\beta + 1},$$

which is nonnegative if and only if $v_1 - w_1 + \beta(v_2 - w_2) \geq 0$, since α and β are positive. $v \in U_0$ implies that $v_1 + \beta v_2 \geq 1$ and $w \in U_1$ implies that $w_1 + \beta w_2 \leq 1$. Therefore, $v_1 - w_1 + \beta(v_2 - w_2) \geq 0$.

Next assume that $f(v) = y^1$ and $f(w) = y^2$. Thus $v \in U_1$ and $w \in U_2$. Therefore

$$\langle v - w, y^1 - y^2\rangle = (v_1 - w_1)\frac{-\alpha}{\alpha\beta + 1} + (v_2 - w_2)\frac{1}{\alpha\beta + 1},$$

which is nonnegative if and only if $-\alpha(v_1 - w_1) + v_2 - w_2 \geq 0$. This inequality holds since $v \in U_1$ and $w \in U_2$. Finally assume that $f(v) = y^2$ and $f(w) = y^0$. Thus $v \in U_2$ and $w \in U_0$. Therefore,

$$\langle v - w, y^2 - y^0\rangle = w_2 - v_2 \geq 0,$$

[18]As usual $U_{2,3} = U_{2,0}$.

where the last inequality follows since $v \in U_2$ and $w \in U_0$.

To complete the proof, we show that part (ii) holds, which in particular shows that f is not cyclically monotone. Let $v \in U_{0,2}$, $w \in U_{0,1}$, and $z \in U_{1,2}$ (see Figure 3). Then $v = (a, 0)$ for some $a > 0$, $w = (1 - \beta b, b)$ for some $b > 0$, and $z = (-c, -\alpha c)$ for some $c > 0$. Therefore, the left-hand side of (8) equals

$$-b - (1 - \beta b + c)\frac{\alpha}{\alpha\beta + 1} + (b + \alpha c)\frac{1}{\alpha\beta + 1} = -\frac{\alpha}{\alpha\beta + 1} < 0.$$

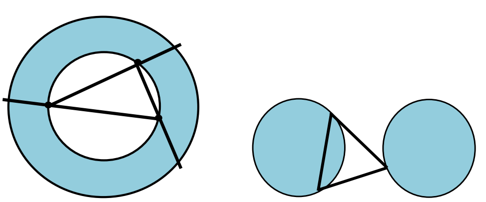
Q.E.D.

3.1.2. *Proof of Theorem 3 for $k = 2$*

Let $D \subseteq R^A$ be a set of dimension $k = 2$ whose closure $\text{cl}(D)$ is nonconvex. Observe that if one can define a monotone function on a set $D' \supseteq D$ and find a finite sequence of valuations $v_1, \ldots, v_k \in D$ which violates cyclic monotonicity, then D is not a proper monotonicity domain (the restriction $f|_D$ is monotone but not cyclically monotone). By Lemma 5 and Proposition 6, D is not a proper monotonicity domain if there exist affine independent valuations v, w, and z in D, such that the relative interior of the simplex generated by them does not intersect D. For example, such valuations can be easily detected for the nonconvex ring in Figure 4(a) by choosing the vertices shown in this figure. However, it is not true that such valuations exist for every nonconvex set, even if it is closed. For example, if D is the union of two disjoint closed disks as shown in Figure 4(b), then, as demonstrated in that figure, for every three valuations in D not on the same line, the relative interior of the triangle generated by them intersects D. Therefore, we need a more delicate procedure, which uses the following claim.

CLAIM 1: *Let D be a set of dimension $k = 2$ whose closure is nonconvex. There exist three affine independent valuations in D such that the relative interior of the simplex Δ generated by them contains a point, say d, for which there exists $\eta > 0$ such that $B(d, \eta) \cap D = \emptyset$, where $B(d, \eta) = \{v \in \Delta^0 \mid \|v - d\| < \eta\}$.*

The proof of Claim 1 is postponed to the end of this proof. Without loss of generality, we can assume that $D \subseteq R^2$. Let v, w, and z be affine independent



(a) (b)

FIGURE 4.—(a) The triangle is embedded "directly" in the domain. (b) The relative interior of every triangle whose vertices belong to the domain intersects the domain.

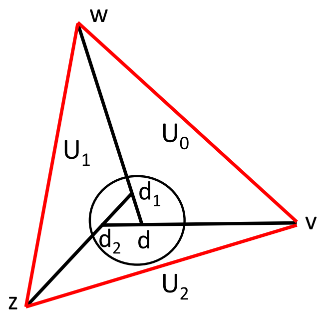


FIGURE 5.—The large triangle is as in Figure 4(b) (v, w, and z belong to the domain). A ball which does not intersect the domain is located in the interior of the large triangle centered at d. A triangle is located in the ball such that v, w, and z are each on an extension of a different edge of the triangle.

valuations in D such that there exist d and η as in Claim 1. We choose η to be small enough such that $B(d, \eta) \subset \Delta^0(\{v, w, z\})$ (see Figure 5). By rotating and shifting the plane, we can assume without loss of generality that $d = (0, 0)$ and $v = (x, 0)$ for some $x > 0$.[19] Consider the line $z + t(d - z - (\varepsilon, 0))$ for $\varepsilon > 0$. Let d_1 and d_2 be the points in which this line intersects the lines $w + t(d - w)$ and the x axis, respectively. There exists a small enough ε such that $d_1 \in B(d, \eta)$ and $d_2 \in B(d, \eta)$, since for $\varepsilon = 0$, all three lines intersect in d.

To finish the proof, note that it is possible to rotate, shift, and scale the plane such that $S = \{d, d_1, d_2\}$ (see (7)), and v, w, and z are vectors as in part (ii) of Proposition 6. We can now apply Proposition 6 to show that D is not a proper monotonicity domain. *Q.E.D.*

To complete the proof of the theorem, it remains to prove Claim 1.

PROOF OF CLAIM 1: The proof is by contradiction. Assume that the claim does not hold. Therefore, for every three affine independent valuations in D, the interior of the simplex Δ generated by them is contained in $\mathrm{cl}(D)$. Therefore, the simplex itself is contained in $\mathrm{cl}(D)$. As the dimension of D is 2, for every $v_0 \neq v_1$ in D, there exists $v_2 \in D$ such that v_0, v_1, and v_2 are affine independent: therefore, the simplex generated by these valuations is contained in $\mathrm{cl}(D)$ and, therefore, in the interval $[v_0, v_1] \subseteq \mathrm{cl}(D)$. Let w_0 and w_1 be in $\mathrm{cl}(D)$. There exist sequences v_0^n and v_1^n in D such that $v_i^n \to w_i$ and $i = 0, 1$. Therefore, every valuation in $[w_0, w_1]$ is a limit of valuations in $\mathrm{cl}(D)$, and hence it belongs to $\mathrm{cl}(D)$. This implies that $\mathrm{cl}(D)$ is convex, contradicting the assumption of the claim. Hence, Claim 1 holds, which completes the proof of the theorem for $k = 2$. *Q.E.D.*

[19] Lemma 5 provides that any shift, rotation, or scaling of D preserves the monotonicity domain property. Hence these operations can be done alternatively on the space itself.

3.2. *Domains of Dimension $k \geq 3$*

In this section, we prove Theorem 3 for domains of dimension $k \geq 3$. We need the following definition. A domain D is called *good* if for every $v, w \in D$, the projection of D onto $I = [v, w]$ is dense in $[v, w]$. The proof of Theorem 3 for dimensions $k \geq 3$ distinguishes between two cases, namely whether D is a good domain or not. For domains which are not good we apply the result for domains of dimension 2 by first projecting the domain to a plane, then finding on the projection a monotone finite-valued function which is not cyclically monotone, and finally extending the function to be monotone on the entire domain. For good domains, the proof uses a similar idea as for domains of dimension $k = 2$, but employs a more complicated structure.

Note that if D is not a good domain, then its closure is not convex. First we show the following proposition.

PROPOSITION 7: *If domain D is not good, then it is not a proper monotonicity domain.*

PROOF: Let $D \subseteq R^A$ be of dimension k, where $k \geq 3$. For any closed convex set Q, let $\Pi_Q(D)$ denote the projection of D on Q. Since D is not good there exist $v, w \in D$ and an open interval $(a, b) \subseteq [v, w]$ such that $\Pi_{[v,w]}(D) \cap (a, b) = \emptyset$. Let $z \in D$ be a vector in D such that v, w, and z are affine independent. Rotate and shift the space such that v, w, and z are in the XY plane and $[v, w]$ lies on the X axis. There exists $a < d < b$ and $\varepsilon > 0$ such that $B((d, 0), \varepsilon) \cap \Pi_{XY}(D) = \emptyset$, where Π_{XY} is the projection to the XY plane. Therefore, by our proof for dimension 2, $\Pi_{XY}(D)$ is not a proper monotonicity domain. In particular, there exists a monotone finite-valued function $f : \Pi_{XY}(D) \to R^2$ that is not cyclically monotone. Let $\tilde{f} : D \to R^A$ be the function defined by $\tilde{f}(x_1, \dots, x_A) = (f_1(x_1, x_2), f_2(x_1, x_2), 0, \dots, 0)$ for every $x \in D$. Clearly $\tilde{f}$ is finite-valued, monotone, and not cyclically monotone. *Q.E.D.*

By Proposition 7, it remains to deal only with good domains. These are studied in the next two subsections. One example of a good domain which is not convex is the unit sphere in R^3.

3.2.1. *Preparations for the Proof for Good Domains*

REMARK: Throughout this section we will denote by v_l the lth coordinate of v and indices of vectors will be denoted by superscripts.

Let $k \geq 2$ be some integer. Let S^k be the hyperplane in R^{k+1}:

$$S^k = \left\{ v \in R^{k+1} : \sum_{i=1}^{k} v_i = 0 \right\}. \tag{9}$$

For every $\alpha > 0$, $\varepsilon_1, \varepsilon_2 > 0$, $\delta > 0$, and $i = 1, \dots, k$, we define the following regions in S^k as

$$U_i^k(\varepsilon_1, \alpha, \delta) = \left\{ v \in S^k : v_{k+1} \geq \alpha, v_i = \max_{j=1}^{k} v_j, v_{k+1} \geq \alpha + \frac{1 - \delta - v_i}{\varepsilon_1} \right\},$$

$$M_i^k(\alpha) = \left\{ v \in S^k : -\alpha \leq v_{k+1} \leq \alpha, v_i = \max_{j=1}^{k} v_j \geq 1 \right\},$$

and

$$D_i^k(\varepsilon_2, \alpha) = \left\{ v \in S^k : v_{k+1} \leq -\alpha, v_i = \max_{j=1}^{k} v_j, v_{k+1} \leq -\alpha - \frac{(1 - v_i)}{\varepsilon_2} \right\}.$$

Let

$$P_u^k(\varepsilon_1, \alpha, \delta) = \left\{ v \in S^k : v_{k+1} \geq \alpha, \right. \\ \left. \text{for every } i \leq k \ v_{k+1} \leq \alpha + \frac{(1 - \delta - v_i)}{\varepsilon_1} \right\}$$

and

$$P_d^k(\varepsilon_2, \alpha) = \left\{ v \in S^k : v_{k+1} \leq -\alpha, \right. \\ \left. \text{for every } i \leq k \ v_{k+1} \geq -\alpha - \frac{(1 - v_i)}{\varepsilon_2} \right\}.$$

Note that if $v \in P_u$ or $v \in P_d$, then $v_i \leq 1$ for every $i \leq k$. Finally, let

$$T(\alpha) = \{ v \in S^k : \text{for every } i \leq k \ v_i < 1, \text{ and } -\alpha < v_{k+1} < \alpha \}. \tag{10}$$

Figure 6 illustrates the hyperplane S^2 and the regions defined above for $k = 2$.

The superscript k and the arguments ε_1, ε_2, α, and δ will be dropped whenever they are clear from the context. Let $\Omega = \{U_1, \dots, U_k, M_1, \dots, M_k, D_1, \dots, D_k, P_d, P_u\}$ and let $G = \bigcup_{Q \in \Omega} Q$. Observe that $S^k = G \cup T$. For every set L, we denote by $\mathrm{ri}(L)$ the relative interior of L with respect to S^k. In Proposition 8 we show that G is not a proper monotonicity domain. Let u and w be the peaks of P_u and P_d, respectively, as illustrated in Figure 6. For any i, $1 \leq i \leq k$, we show how to construct a monotone function on G such that for every two points $c^1 \in U_i \cap M_i$ and $c^2 \in M_i \cap D_i$, the sequence of points u, w, c^1, and c^2 (see Figure 6) violates the cyclic monotonicity condition (3). This construction will be a key tool in our main proof.

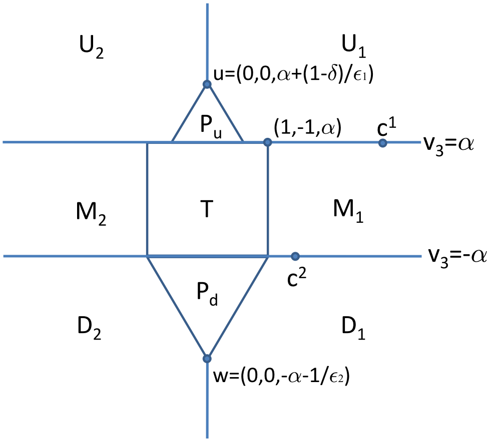


FIGURE 6.—The "ideal" domain for $k = 3$.

PROPOSITION 8: *Let α, ε_1, ε_2, and δ be positive reals such that $2\alpha\varepsilon_1 > \delta$. There exists a monotone finite-valued function $f : G \to R^{k+1}$, which is not cyclically monotone. Moreover, f can be chosen such that its range contains exactly $3k + 1$ distinct vectors $y^{U_1}, \dots, y^{U_k}, y^{M_1}, \dots, y^{M_k}, y^{D_1}, \dots, y^{D_k}, y^P$ and such that for some fixed i, $1 \le i \le k$, the following statements hold*:

(i) *For every set $Q \in \Omega$, $f(v) = y^Q$ for all $v \in \text{ri}(Q)$, where $y^{P_u} = y^{P_d} = y^P$.*[20]

(ii) *$f(v) = y^{U_i}$ for all $v \in U_i \setminus M_i$, $f(v) = y^{M_i}$ for all $v \in M_i \setminus D_i$, $f(v) = y^{D_i}$ for all $v \in D_i \setminus P_d$, and $f(v) = y^P$ for all $v \in P_d$.*

(iii) *For every $v \in G$ other than in* (i) *and* (ii), *let $f(v) = y^Q$ for an arbitrary Q in which $v \in Q$.*

(iv) *For every two distinct vectors c^1 and c^2 in which $c^1 \in U_i \cap M_i$ and $c^2 \in M_i \cap D_i$,*

$$\langle w - u, f(w)\rangle + \langle u - c^1, f(u)\rangle + \langle c^1 - c^2, f(c^1)\rangle + \langle c^2 - w, f(c^2)\rangle < 0, \tag{11}$$

where $w = (0, \dots, 0, -\alpha - 1/\varepsilon_2)$ and $u = (0, \dots, 0, \alpha + (1 - \delta)/\varepsilon_1)$.

The proof of Proposition 8 is straightforward and is given in the Appendix. By Lemma 5 and Proposition 8 we obtain the following corollary.

COROLLARY 9: *For any α, ε_1, ε_2, and δ such that $2\alpha\varepsilon_1 > \delta$, G is not a proper monotonicity domain.*

[20]Since all the sets in Ω are defined with equalities, we first define the function on the interior of every set in Ω and then break ties on the boundaries.

3.2.2. *Proof of Theorem 3 for Good Domains With $k \geq 3$*

PROPOSITION 10: *Let D be a good domain with* $\dim(D) = k \geq 3$ *with a non-convex closure. Then D is not a proper monotonicity domain.*

The technique of this proof resembles the technique of the proof for $k = 2$, as we embed the structure in the previous section in a way that allows us to apply Proposition 8. In particular we show that for any good domain D, there exist parameters α, ε_1, ε_2, and δ as in Proposition 8 such that the set T and the sets in Ω (see Section 3.2.1) can be embedded in the space so that T is in the relative interior of $\text{ConvexHull}(D) \setminus D$ (in the proof of $k = 2$, we located a triangle to be in the relative interior). The peaks of the simplexes $w \in P_d$ and $u \in P_u$ can both be located in D, and there exist c^1 and c^2 as in Proposition 8 that also belong to D. See Figure 7 for an illustration of this construction when D is a sphere (the shaded regions in Figure 7 represent P_u and P_d).

PROOF OF PROPOSITION 10: Since $\text{cl}(D)$ is not convex, there exist $w, u \in D$, $z \in I := [w, u]$, and $r > 0$ such that $B(z, r) \cap \text{cl}(D) = \emptyset$, where $B(z, r) = \{v \in R^k : \|v - z\| < r\}$. We can assume without loss of generality that D is embedded in R^{k+1}. Rotate the space so that the positive x_{k+1} axis is from w to u. All other coordinates are parametrized by $x_1, x_2, \dots, x_k$ such that $\sum_{i=1}^{n} x_i = 0$.

Let $I_z = I_z(r_1)$ be the interval of length $r_1 > 0$ centered in z on I. There exists $r_1 > 0$ such that for every $a = (a_1, \dots, a_k, a_{k+1})$ with $a_{k+1} \in I_z(r_1)$ and $a_i \leq r_1$ for every $i \leq k$, $a \notin \text{cl}(D)$. We scale D by $1/r_1$. Thus, if $a_{k+1} \in I_z$ and $a_i \leq 1$ for every $i \leq k$, then $a \notin \text{cl}(D)$.

Let $a^1, a^2, \dots, a^{k+1}$ be $k + 1$ equally spaced vectors in I_z, that is, there exists $d > 0$ such that for every i, $1 \leq i \leq k$, $a^{i+1}_{k+1} - a^i_{k+1} = d$.

Let Π_{wu} be the projection operator on the interval $I = [w, u]$. Since D is a good set, there exist $k + 1$ distinct vectors $b^1, \dots, b^k, b^{k+1} \in D$ such that $\Pi_{wu}(b^i) \in (a^{i-1}, a^i)$. For any vector $b = (b_1, \dots, b_k)$, we define $\text{indmax}(b)$ to

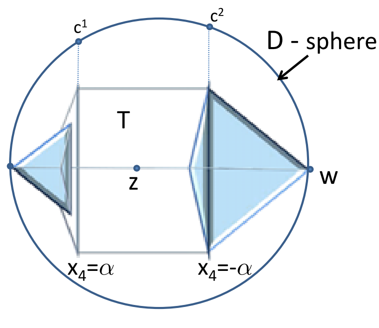


FIGURE 7.—Embedding the "ideal" domain in a sphere.

be some arbitrary index in $\arg\max_{i=1}^{k} b_i$. That is $b_{\mathrm{indmax(b)}} = \max_{i=1}^{k} b_i$. Note that there exist $i, j \leq k+1$ such that $i < j$ and $\mathrm{indmax}(b^i) = \mathrm{indmax}(b^j)$. Define $t := \mathrm{indmax}(b^i)$.

Let $c = (\Pi_{wu}(b^j)_{k+1} + \Pi_{wu}(b^i)_{k+1})/2$. We now shift the set so that c moves to $(0, 0, \dots, 0)$. Therefore, $\Pi_{wu}(b^j) = -\Pi_{wu}(b^i)$. In particular, there exists $\alpha > 0$ such that $\Pi_{wu}(b^j) = (0, \dots, 0, \alpha)$ and $\Pi_{wu}(b^i) = (0, \dots, 0, -\alpha)$. We obtain that

$$w = (0, \dots, 0, -w_{k+1}), \quad u = (0, \dots, 0, u_{k+1}),$$
$$b^i = (\cdot, \dots, -\alpha), \quad b^j = (\cdot, \dots, \alpha),$$

where $w_{k+1}, u_{k+1} > \alpha$.

Define ε_1, ε_2, and δ as

$$\varepsilon_2 = \frac{1}{w_{k+1} - \alpha}, \quad \delta < \min\left(\frac{\alpha}{u_{k+1} - \alpha}, \frac{1}{2}\right), \quad \varepsilon_1 = \frac{1 - \delta}{u_{k+1} - \alpha}.$$

Therefore,

$$\varepsilon_1 > \frac{1}{2(u_{k+1} - \alpha)}, \quad 2\alpha\varepsilon_1 > \frac{\alpha}{u_{k+1} - \alpha} > \delta.$$

Recall that $T(\alpha)$ is the k dimensional prism (see (10)). Therefore, $T(\alpha) \cap D = \emptyset$. We can now apply Proposition 8 with $i = t$, where w and u are as in part (iv) of the proposition and $c^1 = b^j$, $c^2 = b^i$. This implies that D is not a proper monotonicity domain. *Q.E.D.*

4. MONOTONICITY DOMAINS—CHARACTERIZATION

In this section we complete our characterization of monotonicity domains. Recall that a domain D is a *monotonicity domain* if every monotone finite-valued allocation rule is also cyclically monotone. Note that every proper monotonicity domain is a monotonicity domain. We characterize monotonicity domains via proper monotonicity domains.

Let $H^A = \{v \in R^A : \sum_{a \in A} v_a = 0\}$ be the hyperplane which is orthogonal to the vector $(1, \dots, 1) \in R^A$. Denote by $\Pi : R^A \to H^A$ the projection onto the hyperplane H^A.

THEOREM 11: *Let $D \subseteq R^A$. D is a monotonicity domain if and only if $\Pi(D)$ is a proper monotonicity domain.*

PROOF: We first prove that if D is a monotonicity domain, then $\Pi(D)$ is a proper monotonicity domain. Assume for contradiction that $\Pi(D)$ is not a proper monotonicity domain, that is, there exists a function $f^0 : \Pi(D) \to \bar{Z}(A)$ which is monotone but not cyclically monotone. Let f^1 be an allocation rule obtained from f^0 by adding an appropriate multiple of $(1,\dots,1)$ to each value of f^0:

$$f^1(v) := f^0(v) + \frac{1 - \sum_{a\in A} f^0_a(v)}{|A|}(1,\dots,1).$$

Let f^2 be the natural extension of f^1 to D. That is $f^2(v) = f^1(\Pi(v))$ for every $v \in D$. Thus f^2 is also a finite-valued allocation rule. We claim that f^2 is monotone but not cyclically monotone. To see this, it is enough to show that for any $v, w, z \in D$, $\langle v, f^2(w) - f^2(z)\rangle = \langle \Pi(v), f^0(\Pi(w)) - f^0(\Pi(z))\rangle$. Let $v, w, z \in D$. Then

$$\begin{aligned}\langle v, f^2(w) - f^2(z)\rangle &= \big\langle \Pi(v), f^1(\Pi(w)) - f^1(\Pi(z))\big\rangle \\ &\quad + \big\langle v - \Pi(v), f^1(\Pi(w)) - f^1(\Pi(z))\big\rangle.\end{aligned}$$

Since $f^1(\Pi(w)) - f^1(\Pi(z)) \in H^A$, we have that $\langle v - \Pi(v), f^1(\Pi(w)) - f^1(\Pi(z))\rangle = 0$. Therefore,

$$\begin{aligned}\langle v, f^2(w) - f^2(z)\rangle &= \big\langle \Pi(v), f^0(\Pi(w)) - f^0(\Pi(z))\big\rangle \\ &\quad + \langle \Pi(v), c(1,\dots,1)\rangle,\end{aligned}$$

where c is some real number. Since $\langle \Pi(v), c(1,\dots,1)\rangle = 0$, we are done.

We proceed to prove the other direction. Assume $\Pi(D)$ is a proper monotonicity domain and suppose that D is not a monotonicity domain, that is, there exists an allocation rule f^0 on D that is monotone but not cyclically monotone. Let $v_1,\dots,v_k$ be a shortest sequence of valuations which violates the cyclic monotonicity condition:

$$\sum_{i=1}^{k}\langle v_i, f^0(v_i) - f^0(v_{i-1})\rangle < 0. \tag{12}$$

We have that

$$\begin{aligned}(12) &= \sum_{i=1}^{k}\langle \Pi(v_i), f^0(v_i) - f^0(v_{i-1})\rangle \\ &\quad + \sum_{i=1}^{k}\langle v_i - \Pi(v_i), f^0(v_i) - f^0(v_{i-1})\rangle \\ &= \sum_{i=1}^{k}\langle \Pi(v_i) - \Pi(v_{i+1}), f^0(v_i)\rangle,\end{aligned} \tag{13}$$

where the second equality follows since $v_i - \Pi(v_i)$ is orthogonal to $f^0(v_i) - f^0(v_{i-1})$.

CLAIM 2: *For any* $i \neq j$, $\Pi(v_i) \neq \Pi(v_j)$.

PROOF: Suppose $i < j$ and $\Pi(v_i) = \Pi(v_j)$. Taking all indices modulo k we have

$$\begin{aligned}
&\sum_{l=1}^{k} \langle \Pi(v_l) - \Pi(v_{l+1}), f^0(v_l)\rangle \\
&\quad = \langle \Pi(v_i) - \Pi(v_{i+1}), f^0(v_i)\rangle + \cdots + \langle \Pi(v_{j-1}) - \Pi(v_j), f^0(v_{j-1})\rangle \\
&\qquad + \langle \Pi(v_j) - \Pi(v_{j+1}), f^0(v_j)\rangle + \cdots + \langle \Pi(v_{i-1}) - \Pi(v_i), f^0(v_{i-1})\rangle \\
&\quad = \langle \Pi(v_i) - \Pi(v_{i+1}), f^0(v_i)\rangle + \cdots + \langle \Pi(v_{j-1}) - \Pi(v_i), f^0(v_{j-1})\rangle \qquad (14)\\
&\qquad + \langle \Pi(v_j) - \Pi(v_{j+1}), f^0(v_j)\rangle + \cdots + \langle \Pi(v_{i-1}) - \Pi(v_j), f^0(v_{i-1})\rangle \qquad (15)\\
&\quad < 0.
\end{aligned}$$

Clearly at least one of (14) or (15) is negative, contradicting the minimality of k. *Q.E.D.*

Next we say that f^1 on $\Pi(D)$ is a projection of f^0 if for any $v \in \Pi(D)$, $f^1(v) \in f^0(\Pi^{-1}(v))$.

CLAIM 3: *Any projection* f^1 *of* f^0 *is monotone*.

PROOF: For any $v, w \in \Pi(D)$ there is $\tilde{v}, \tilde{w} \in D$ such that $\Pi(\tilde{v}) = v$, $\Pi(\tilde{w}) = w$, $f^0(\tilde{v}) = f^1(v)$, and $f^0(\tilde{w}) = f^1(w)$. We have

$$\begin{aligned}
\langle v - w, f^1(v) - f^1(w)\rangle &= \langle v - w, f^0(\tilde{v}) - f^0(\tilde{w})\rangle \\
&= \langle \tilde{v} - \tilde{w}, f^0(\tilde{v}) - f^0(\tilde{w})\rangle.
\end{aligned}$$

Q.E.D.

Finally, since, by Claim 2, all the $\Pi(v_i)$'s are distinct, we can select a projection f^1 of f^0 such that $f^1(\Pi(v_i)) = f^0(v_i)$ for all $i = 1, \dots, k$. Therefore, f^1 is monotone but not cyclically monotone:

$$\begin{aligned}
&\sum_{i=1}^{k} \big\langle \Pi(v_i), f^1(\Pi(v_i)) - f^1(\Pi(v_{i-1}))\big\rangle \\
&\quad = \sum_{i=1}^{k} \big\langle v_i, f^1(\Pi(v_i)) - f^1(\Pi(v_{i-1}))\big\rangle = \sum_{i=1}^{k} \langle v_i, f^0(v_i) - f^0(v_{i-1})\rangle < 0.
\end{aligned}$$

This contradicts that $\Pi(D)$ is a proper monotonicity domain. *Q.E.D.*

APPENDIX

PROOF OF LEMMA 5: We begin with the first part. Let D be a domain and let $f : D \to R^A$ be a monotone finite-valued function which is not cyclically monotone. Let $y^1, \dots, y^m$ be the distinct values of f. There exist $\alpha > 0$ and $y \in R^A$ such that for every $i = 1, \dots, m$, $\tilde{y}^i = \alpha(y^i + y) \in \bar{Z}(A)$. Let $\tilde{f}$ be the function defined by $\tilde{f}(v) = \tilde{y}^i$ if and only if $f(v) = y^i$. Thus for every $v, w \in D$,

$$\langle v, \tilde{f}(v) - \tilde{f}(w) \rangle = \alpha \langle v, f(v) - f(w) \rangle. \tag{16}$$

Therefore, all inner products in (16) are multiplied by the same positive factor, implying that $\tilde{f}$ is monotone and not cyclically monotone.

To prove the second part, we first notice that by the first part we do not need to restrict ourselves to functions that output only subprobability vectors. Assume that D is not a proper monotonicity domain and let $f : D \to R^A$ be a monotone function which is not cyclically monotone. We show that there exits a monotone function $\tilde{f} : L(D) \to R^A$ which is not cyclically monotone.

Suppose $L(D)$ is a rotation. Thus, there exists a unitary matrix U such that for every $y \in L(D)$, there exists $x \in D$ such that $Ux = y$. For all $x \in L(D)$, let $\tilde{f}(x) = Uf(U^{-1}x)$. For every three points $x, y, z \in D$, we have

$$\langle x - y, f(z) \rangle = \langle Ux - Uy, Uf(z) \rangle = \langle Ux - Uy, \tilde{f}(Uz) \rangle$$

as U is unitary. Since all the monotonicity and cyclic monotonicity constraints are defined via inner products, $\tilde{f}$ is monotone but not cyclic monotone over $L(D)$. Suppose now that $L(D)$ is an affine shift by some fixed vector $\mathbf{t}$. For every $x \in L(D)$, let $\tilde{f}(x) = f(x - \mathbf{t})$. Therefore $\langle x - y, f(z) \rangle = \langle (x - \mathbf{t}) - (y - \mathbf{t}), f(z - \mathbf{t}) \rangle$ which implies the result. Finally, suppose $L(D)$ is a contraction by a constant $c > 0$. For every $x \in L(D)$, let $\tilde{f}(x) = f(cx)$. In this case, all the inner products are multiplied by $c > 0$, and the result follows. *Q.E.D.*

PROOF OF PROPOSITION 8: To define the range of f, we make use of the following notation. Let $e^j(\gamma) \in R^{k+1}$ denote the sum $e^j + (0, \dots, 0, \gamma)$, where both vectors are in R^{k+1}. The range of f is defined as

$$y^Q = \begin{cases} e^j(\varepsilon_1), & Q = U_j, \\ e^j, & Q = M_j, \\ e^j(-\varepsilon_2), & Q = D_j, \\ \bar{0}, & Q = P_d \text{ or } Q = P_u. \end{cases} \tag{17}$$

We first show that f is not cyclically monotone. To see this, it is enough to verify that (11) holds. Let w, v, c^1, and c^2 be as in part (iv) of the proposition.

Since $f(w) = \bar{0}$, then $\langle w - u, f(w)\rangle = 0$. Since $c^1 \in U_i \cap M_i$ it has the form $c^1 = (c^1_1, \ldots, c^1_k, \alpha)$. Similarly $c^2 = (c^2_1, \ldots, c^2_k, -\alpha)$. Therefore,

$$\langle u - c^1, f(u)\rangle = -c^1_i + \frac{1-\delta}{\varepsilon_1} \cdot \varepsilon_1 = -c^1_i + 1 - \delta,$$
$$\langle c^1 - c^2, f(c^1)\rangle = c^1_i - c^2_i,$$

and

$$\langle c^2 - w, f(c^2)\rangle = c^2_i + \frac{1}{\varepsilon_2} \cdot (-\varepsilon_2) = c^2_i - 1.$$

Summing up all the terms, we obtain that (11) $= -\delta < 0$.

To complete the proof, we need to show that f is monotone on G. Let $v = (v_1, \ldots, v_{k+1})$ and $w = (w_1, \ldots, w_{k+1})$ be any two vectors in G. Let e^0 denote the zero vector. We distinguish between the following cases (i will be used now as an arbitrary index):

(i) $f(v) = y^{U_i}$ and $f(w) = y^{U_j}$. Thus $v \in U_i$ and $w \in U_j$. Therefore,

$$\begin{aligned}\langle v - w, f(v) - f(w)\rangle &= \langle v - w, e^i - e^j\rangle = (v_i - w_i) + (v_j - w_j)\\ &= (v_i - v_j) + (w_j - w_i) \geq 0,\end{aligned}$$

where the last inequality follows since $v_i \geq v_j$ and $w_j \geq w_i$.

(ii) $f(v) = y^{U_i}$ and $f(w) = y^{M_i}$. Thus $v \in U_i$ and $w \in M_i$, implying that

$$\langle v - w, f(v) - f(w)\rangle = \langle v - w, e^0(\varepsilon_1)\rangle = (v_{k+1} - w_{k+1}) \cdot \varepsilon_1 \geq 0,$$

since $v_{k+1} \geq \alpha$, $w_{k+1} \leq \alpha$, and $\varepsilon_1 > 0$.

(iii) $f(v) = y^{U_i}$ and $f(w) = y^{M_j}$ for $i \neq j$. Thus $v \in U_i$ and $w \in M_j$. Therefore,

$$\begin{aligned}\langle v - w, f(v) - f(w)\rangle &= \langle v - w, e^i(\varepsilon_1) - e^j\rangle\\ &= (v_i - w_i) + (v_j - w_j) + (v_{k+1} - w_{k+1}) \cdot \varepsilon_1 \geq 0,\end{aligned}$$

where the last inequality follows since $v_i \geq w_i$, $v_j \geq w_j$, and $(v_{k+1} - w_{k+1})\varepsilon_1 \geq 0$.

(iv) $f(v) = y^{U_i}$ and $f(w) = y^{D_i}$. Thus $v \in U_i$ and $w \in D_i$. Therefore,

$$\begin{aligned}\langle v - w, f(v) - f(w)\rangle &= \langle v - w, e_0(\varepsilon_1) - e^0(-\varepsilon_2)\rangle\\ &= (v_{k+1} - w_{k+1}) \cdot (\varepsilon_1 + \varepsilon_2) \geq 0,\end{aligned}$$

where the last inequality follows since $v_{k+1} \geq \alpha$, $w_{k+1} \leq -\alpha$, and $\varepsilon_1, \varepsilon_2 > 0$.

(v) $f(v) = y^{U_i}$ and $f(w) = y^{D_j}$; $x \in U_i$ and $w \in D_j$. Then

$$\begin{aligned}&\langle v - w, f(v) - f(w)\rangle\\ &\quad= \langle v - w, e^i(\varepsilon_1) - e^j(-\varepsilon_2)\rangle\\ &\quad= (v_i - w_i) + (v_j - w_j) + (v_{k+1} - w_{k+1}) \cdot (\varepsilon_1 + \varepsilon_2) \geq 0,\end{aligned}$$

where the last inequality follows since $v_i \geq w_i$, $v_j \geq w_j$, and $v_{k+1} \geq w_{k+1}$.

(vi) $f(v) = y^{M_i}$ and $f(w) = y^{M_j}$. Thus $x \in M_i$ and $y \in M_j$. Therefore,

$$\begin{aligned}\langle v - w, f(v) - f(w)\rangle &= \langle v - w, e^i - e^j\rangle = (v_i - w_i) + (v_j - w_j)\\ &= (v_i - v_j) + (w_j - w_i) \geq 0,\end{aligned}$$

where the last inequality follows since $v_i \geq v_j$ and $w_j \geq w_i$.

(vii) $f(v) = y^{U_i}$ and $f(w) = y^P$. Thus $v \in U_i$ and $w \in P_u \cup P_d$. Then

$$\begin{aligned}\langle v - w, f(v) - f(w)\rangle &= \langle v - w, e^i(\varepsilon_1)\rangle\\ &= (v_i - w_i) + \varepsilon_1 \cdot (v_{k+1} - w_{k+1})\\ &\geq v_i - w_i + (\alpha \cdot \varepsilon_1 + 1 - \delta - v_i) - \varepsilon_1 w_{k+1},\end{aligned} \tag{18}$$

where the last inequality follows since $v_{k+1} \geq \alpha + (1 - \delta - v_i)/\varepsilon_1$. If $w \in P_u$, then $w_{k+1} \leq \alpha + (1 - \delta - w_i)/\varepsilon_1$ and, therefore,

$$(18) \geq v_i - w_i + (\alpha \cdot \varepsilon_1 + 1 - \delta - v_i) - (\alpha \cdot \varepsilon_1 + 1 - \delta - w_i) = 0.$$

If $w \in P_d$, then $w_{k+1} \leq -\alpha$ and $w_i \leq 1$. Therefore, since $2\alpha\varepsilon_1 \geq \delta$,

$$\begin{aligned}(18) &\geq v_i - w_i + (\alpha \cdot \varepsilon_1 + 1 - \delta - v_i) - (-\alpha \cdot \varepsilon_1)\\ &= 1 - w_i - \delta + 2\alpha\varepsilon_1 \geq 0.\end{aligned}$$

(viii) $f(v) = y^{M_i}$ and $f(w) = y^P$. Thus $v \in M_i$ and $w \in P_u \cup P_d$. Therefore,

$$\langle v - w, f(v) - f(w)\rangle = \langle v - w, e^i\rangle = v_i - w_i \geq 0,$$

where the last inequality follows since $v_i \geq 1$ and $w_i \leq 1$.

(ix) $f(v) = y^{D_i}$ and $f(w) = y^P$. Thus $v \in D_i$ and $w \in P_u \cup P_d$. Then

$$\begin{aligned}\langle v - w, f(v) - f(w)\rangle &= \langle v - w, e^i(-\varepsilon_2)\rangle\\ &= (v_i - w_i) + \varepsilon_2 \cdot (w_{k+1} - v_{k+1}).\end{aligned} \tag{19}$$

If $w \in P_u$, then $w_{k+1} \geq \alpha > -\alpha - (1 - w_i)/\varepsilon_2$, since $w_i \leq 1$. If $w \in P_d$, then, by definition, $w_{k+1} \geq -\alpha - (1 - w_i)/\varepsilon_2$. In either case, since $v \in D_i$, $v_{k+1} \leq -\alpha - (1 - v_i)/\varepsilon_2$. Therefore,

$$(19) \geq v_i - w_i + (\alpha \cdot \varepsilon_2 + 1 - v_i) - (\alpha \cdot \varepsilon_2 + 1 - w_i) = 0.$$

The other cases in which $f(v) = y^{D_i}$ are very similar to those in which $f(v) = y^{U_i}$. In fact, it is easier for monotonicity to hold in these cases since P_d is a larger set than P_u. *Q.E.D.*

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Manuscript received October, 2009; final revision received March, 2010.