

Exercise: explain the comment, "Of the two operators $\tilde{V}(t')$ and $\tilde{V}(t'')$ of (4), one must contain a_i , the other containing a_i^\dagger , so that the trace over R of their product multiplied by σ_R gives a nonzero result in (4)."

$$\frac{\Delta \tilde{\sigma}}{\Delta t} = -\frac{1}{\hbar^2} \frac{1}{\Delta t} \int_0^\infty dt' \text{Tr}_R[\tilde{V}(t'), [\tilde{V}(t''), \tilde{\sigma}(t) \otimes \sigma_R]].$$

Here

$$V = \sum_i g_i b^\dagger a + g_i^* b a_i^\dagger \rightarrow -[b^\dagger R^{(+)} + b R^{(-)}],$$

$$R^{(+)} = -\sum_i g_i a_i, \quad R^{(-)} = -\sum_i g_i^* a_i^\dagger,$$

which in the Interaction Picture merely pick up some oscillating (c-number) exponential time-dependences.

Detailed derivation of Eq. (14)

To begin with we note that in the interaction picture with

$$H_A = \hbar\omega_0 \left(b^\dagger b + \frac{1}{2} \right), \quad H_R = \sum_i \hbar\omega_i \left(a_i^\dagger a_i + \frac{1}{2} \right),$$

we have the simple correspondences

$$\begin{aligned} \tilde{b}(t) &= \exp(i\omega_0 b^\dagger b t) b \exp(-i\omega_0 b^\dagger b t) \\ &= \exp(i\omega_0 b^\dagger b t) b \sum_{n=0}^\infty \frac{(-i\omega_0 t)^n}{n!} (b^\dagger b)^n \\ &= \exp(i\omega_0 b^\dagger b t) \sum_{n=0}^\infty \frac{(-i\omega_0 t)^n}{n!} b b^\dagger (b b^\dagger)^{n-1} b \\ &= \exp(i\omega_0 b^\dagger b t) \sum_{n=0}^\infty \frac{(-i\omega_0 t)^n}{n!} (b^\dagger b + 1) (b^\dagger b + 1)^{n-1} b \\ &= \exp(i\omega_0 b^\dagger b t) \exp(-i\omega_0 b^\dagger b t) \exp(-i\omega_0 t) b \\ &= \exp(-i\omega_0 t) b, \\ \tilde{b}^\dagger(t) &= [\tilde{b}(t)]^\dagger = \exp(i\omega_0 t) b^\dagger, \end{aligned}$$

and similarly

$$\tilde{a}_i = \exp(-i\omega_i t) a_i, \quad \tilde{a}_i^\dagger = \exp(i\omega_i t) a_i^\dagger.$$

We then proceed from the equation above,

$$\begin{aligned} \frac{\Delta \tilde{\sigma}}{\Delta t} &= -\frac{1}{\hbar^2} \frac{1}{\Delta t} \int_0^\infty dt' \text{Tr}_R[\tilde{V}(t'), [\tilde{V}(t''), \tilde{\sigma}(t) \otimes \sigma_R]] \\ &= -\frac{1}{\hbar^2} \frac{1}{\Delta t} \int_0^\infty dt' \text{Tr}_R[\tilde{b}^\dagger(t') \tilde{R}^{(+)}(t') \tilde{b}(t'') \tilde{R}^{(-)}(t''), [\tilde{b}^\dagger(t'') \tilde{R}^{(+)}(t'') + \tilde{b}(t'') \tilde{R}^{(-)}(t''), \tilde{\sigma}(t) \otimes \sigma_R]], \end{aligned}$$

where

$$V = \sum_i g_i b^\dagger a_i + g_i^* b a_i^\dagger \rightarrow -[b^\dagger R^{(+)} + b R^{(-)}],$$

$$R^{(+)} = -\sum_i g_i a_i, \quad R^{(-)} = -\sum_i g_i^* a_i^\dagger,$$

and use the noted fact that the partial trace will only not vanish for certain terms within the double-commutator:

- (1a) : $\text{Tr}_R[\tilde{b}^\dagger(t') \tilde{R}^{(+)}(t') \tilde{b}(t'') \tilde{R}^{(-)}(t'') \tilde{\sigma}(t) \otimes \sigma_R - \tilde{b}(t'') \tilde{R}^{(-)}(t'') \tilde{\sigma}(t) \otimes \sigma_R \tilde{b}^\dagger(t') \tilde{R}^{(+)}(t')]$,
- (1b) : $\text{Tr}_R[\tilde{\sigma}(t) \otimes \sigma_R \tilde{b}(t'') \tilde{R}^{(-)}(t'') \tilde{b}^\dagger(t') \tilde{R}^{(+)}(t') - \tilde{b}^\dagger(t') \tilde{R}^{(+)}(t') \tilde{\sigma}(t) \otimes \sigma_R \tilde{b}(t'') \tilde{R}^{(-)}(t'')]$,
- (2a) : $\text{Tr}_R[\tilde{b}(t') \tilde{R}^{(-)}(t') \tilde{b}^\dagger(t'') \tilde{R}^{(+)}(t'') \tilde{\sigma}(t) \otimes \sigma_R - \tilde{b}^\dagger(t'') \tilde{R}^{(+)}(t'') \tilde{\sigma}(t) \otimes \sigma_R \tilde{b}(t') \tilde{R}^{(-)}(t')]$,
- (2b) : $\text{Tr}_R[\tilde{\sigma}(t) \otimes \sigma_R \tilde{b}^\dagger(t'') \tilde{R}^{(+)}(t'') \tilde{b}(t') \tilde{R}^{(-)}(t') - \tilde{b}(t') \tilde{R}^{(-)}(t') \tilde{\sigma}(t) \otimes \sigma_R \tilde{b}^\dagger(t'') \tilde{R}^{(+)}(t'')]$.

We note that, since the system and reservoir operators commute and the reduced density matrices are Hermitian, (2a) = (1a)[†] and (2b) = (1b)[†]. Writing out the terms of (1a), using the cyclic property of the trace,

$$\begin{aligned}
(1a) \rightarrow & \text{Tr}_R [\tilde{b}^\dagger(t') \tilde{R}^{(+)}(t') \tilde{b}(t'') \tilde{R}^{(-)}(t'') \tilde{\sigma}(t) \otimes \sigma_R - \tilde{b}(t'') \tilde{R}^{(-)}(t'') \tilde{\sigma}(t) \otimes \sigma_R \tilde{b}^\dagger(t') \tilde{R}^{(+)}(t')] \\
& = \tilde{b}^\dagger(t') \tilde{b}(t'') \tilde{\sigma}(t) \text{Tr}[\tilde{R}^{(+)}(t') \tilde{R}^{(-)}(t'') \sigma_R] - \tilde{b}(t'') \tilde{\sigma}(t) \tilde{b}^\dagger(t') \text{Tr}[\tilde{R}^{(-)}(t'') \sigma_R \tilde{R}^{(+)}(t')] \\
& = (\tilde{b}^\dagger(t') \tilde{b}(t'') \tilde{\sigma}(t) - \tilde{b}(t'') \tilde{\sigma}(t) \tilde{b}^\dagger(t')) \langle \tilde{R}^{(+)}(t') \tilde{R}^{(-)}(t'') \rangle.
\end{aligned}$$

Similarly we have

$$\begin{aligned}
(1b) \rightarrow & \text{Tr}_R [\tilde{\sigma}(t) \otimes \sigma_R \tilde{b}(t'') \tilde{R}^{(-)}(t'') \tilde{b}^\dagger(t') \tilde{R}^{(+)}(t') - \tilde{b}^\dagger(t') \tilde{R}^{(+)}(t') \tilde{\sigma}(t) \otimes \sigma_R \tilde{b}(t'') \tilde{R}^{(-)}(t'')] \\
& = \tilde{\sigma}(t) \tilde{b}(t'') \tilde{b}^\dagger(t') \text{Tr}[\sigma_R \tilde{R}^{(-)}(t'') \tilde{R}^{(+)}(t')] - \tilde{b}^\dagger(t') \tilde{\sigma}(t) \tilde{b}(t'') \text{Tr}[\tilde{R}^{(+)}(t') \sigma_R \tilde{R}^{(-)}(t'')] \\
& = (\tilde{\sigma}(t) \tilde{b}(t'') \tilde{b}^\dagger(t') - \tilde{b}^\dagger(t') \tilde{\sigma}(t) \tilde{b}(t'')) \langle \tilde{R}^{(-)}(t'') \tilde{R}^{(+)}(t') \rangle.
\end{aligned}$$

Inserting the time-dependences of the Interaction Picture operators,

$$\begin{aligned}
(1a) \rightarrow & \exp(i\omega_0(t' - t'')) (b^\dagger b \tilde{\sigma}(t) - b \tilde{\sigma}(t) b^\dagger) \langle \tilde{R}^{(+)}(t') \tilde{R}^{(-)}(t'') \rangle, \\
(1b) \rightarrow & \exp(i\omega_0(t' - t'')) (\tilde{\sigma}(t) b b^\dagger - b^\dagger \tilde{\sigma}(t) b) \langle \tilde{R}^{(-)}(t'') \tilde{R}^{(+)}(t') \rangle,
\end{aligned}$$

and we compute

$$\begin{aligned}
\langle \tilde{R}^{(+)}(t') \tilde{R}^{(-)}(t'') \rangle &= \sum_{i,j} \text{Tr}[\exp(-i\omega_i t') g_i a_i \exp(i\omega_j t'') g_j^* a_j^\dagger \sigma_R] \\
&= \sum_{i,j} g_i \exp(-i\omega_i t') g_j^* \exp(i\omega_j t'') \sum_{\langle n_i \rangle} p(n_1 \dots n_i \dots) \text{Tr}[a_i a_j^\dagger |n_1 \dots n_i \dots \rangle \langle n_1 \dots n_i \dots|] \\
&= \sum_i |g_i|^2 \exp(i\omega_i(t'' - t')) \sum_{\langle n_i \rangle} p(n_1 \dots n_i \dots) (n_i + 1) \\
&= \sum_i |g_i|^2 (\langle n_i \rangle + 1) \exp(i\omega_i(t'' - t')), \\
\langle \tilde{R}^{(-)}(t'') \tilde{R}^{(+)}(t') \rangle &= \sum_{i,j} g_i \exp(-i\omega_i t') g_j^* \exp(i\omega_j t'') \sum_{\langle n_i \rangle} p(n_1 \dots n_i \dots) \text{Tr}[a_i^\dagger a_j |n_1 \dots n_i \dots \rangle \langle n_1 \dots n_i \dots|] \\
&= \sum_i |g_i|^2 \langle n_i \rangle \exp(i\omega_i(t'' - t')).
\end{aligned}$$

Putting this all together,

$$\begin{aligned}
\frac{\Delta \tilde{\sigma}}{\Delta t} \rightarrow & -\frac{1}{\hbar^2} \frac{1}{\Delta t} \int_0^\infty d(t' - t'') \int_t^{t+\Delta t} dt' \exp(i\omega_0(t' - t'')) (b^\dagger b \tilde{\sigma}(t) - b \tilde{\sigma}(t) b^\dagger) \langle \tilde{R}^{(+)}(t') \tilde{R}^{(-)}(t'') \rangle + \text{h.c.} \\
& -\frac{1}{\hbar^2} \frac{1}{\Delta t} \int_0^\infty d(t' - t'') \int_t^{t+\Delta t} dt' \exp(i\omega_0(t' - t'')) (\tilde{\sigma}(t) b b^\dagger - b^\dagger \tilde{\sigma}(t) b) \langle \tilde{R}^{(-)}(t'') \tilde{R}^{(+)}(t') \rangle + \text{h.c.} \\
& = -\frac{1}{\hbar^2} \left(\frac{1}{\Delta t} \int_t^{t+\Delta t} dt' \right) \int_0^\infty d(t' - t'') \exp(i\omega_0(t' - t'')) (b^\dagger b \tilde{\sigma}(t) - b \tilde{\sigma}(t) b^\dagger) \sum_i |g_i|^2 (\langle n_i \rangle + 1) \exp(i\omega_i(t'' - t')) + \text{h.c.} \\
& -\frac{1}{\hbar^2} \left(\frac{1}{\Delta t} \int_t^{t+\Delta t} dt' \right) \int_0^\infty d(t' - t'') \exp(i\omega_0(t' - t'')) (\tilde{\sigma}(t) b b^\dagger - b^\dagger \tilde{\sigma}(t) b) \sum_i |g_i|^2 \langle n_i \rangle \exp(i\omega_i(t'' - t')) + \text{h.c.} \\
& = -\frac{1}{\hbar^2} (b^\dagger b \tilde{\sigma}(t) - b \tilde{\sigma}(t) b^\dagger) \sum_i |g_i|^2 (\langle n_i \rangle + 1) \int_0^\infty d(t' - t'') \exp(i(\omega_0 - \omega_i)(t' - t'')) + \text{h.c.} \\
& -\frac{1}{\hbar^2} (\tilde{\sigma}(t) b b^\dagger - b^\dagger \tilde{\sigma}(t) b) \sum_i |g_i|^2 \langle n_i \rangle \int_0^\infty d(t' - t'') \exp(i(\omega_0 - \omega_i)(t' - t'')) + \text{h.c.}
\end{aligned}$$

Writing $\Delta_i \equiv \omega_0 - \omega_i$, we note that since the Fourier transform of a Heaviside step function $\theta(x)$ is given by

$$\lim_{N \rightarrow \infty} \int_{-N}^{+N} \exp(-2\pi ixs) \theta(x) dx = \frac{1}{2} \left\{ \delta(s) - \frac{i}{\pi} P \frac{1}{s} \right\},$$

we have

$$\begin{aligned}
\frac{1}{2\pi} \lim_{N \rightarrow \infty} \int_{-N}^{+N} \exp(-i\Delta t) \theta(t/2\pi) dt &= \frac{1}{2} \left\{ \delta(\Delta) - \frac{i}{\pi} P \frac{1}{\Delta} \right\}, \\
\int_0^{+\infty} \exp(-i\Delta t) dt &= \pi \delta(\Delta) - iP \frac{1}{\Delta}, \\
\int_0^{+\infty} \exp(i\Delta t) dt &= \pi \delta(\Delta) + iP \frac{1}{\Delta},
\end{aligned}$$

so finally

$$\begin{aligned}\frac{\Delta\tilde{\sigma}}{\Delta t} &\rightarrow -\frac{1}{\hbar^2}(b^\dagger b\tilde{\sigma}(t) + \tilde{\sigma}(t)b^\dagger b - 2b\tilde{\sigma}(t)b^\dagger) \sum_i |g_i|^2 (\langle n_i \rangle + 1) \left[iP \frac{1}{\Delta_i} + \pi\delta(\Delta_i) \right] \\ &\quad - \frac{1}{\hbar^2}(\tilde{\sigma}(t)bb^\dagger + bb^\dagger\tilde{\sigma}(t) - 2b^\dagger\tilde{\sigma}(t)b) \sum_i |g_i|^2 \langle n_i \rangle \left[iP \frac{1}{\Delta_i} + \pi\delta(\Delta_i) \right].\end{aligned}$$

Using the definitions from the book

$$\begin{aligned}\Gamma &= \frac{2\pi}{\hbar} \sum_i |g_i|^2 \delta(\hbar\Delta_i), \\ \Gamma' &= \frac{2\pi}{\hbar} \sum_i |g_i|^2 \langle n_i \rangle \delta(\hbar\Delta_i), \\ \hbar\Delta &= P \sum_i \frac{|g_i|^2}{\hbar\Delta_i}, \\ \hbar\Delta' &= P \sum_i \frac{|g_i|^2 \langle n_i \rangle}{\hbar\Delta_i},\end{aligned}$$

we obtain

$$\begin{aligned}\frac{\Delta\tilde{\sigma}}{\Delta t} &\rightarrow -(b^\dagger b\tilde{\sigma}(t) + \tilde{\sigma}(t)b^\dagger b - 2b\tilde{\sigma}(t)b^\dagger) \left(\frac{\Gamma}{2} + \frac{\Gamma'}{2} \right) - (b^\dagger b\tilde{\sigma}(t) - \tilde{\sigma}(t)b^\dagger b)(i\Delta + i\Delta') \\ &\quad - (\tilde{\sigma}(t)bb^\dagger + bb^\dagger\tilde{\sigma}(t) - 2b^\dagger\tilde{\sigma}(t)b) \frac{\Gamma'}{2} - (\tilde{\sigma}(t)bb^\dagger - bb^\dagger\tilde{\sigma}(t))i\Delta' \\ &= -\frac{\Gamma}{2} \{ \tilde{\sigma}(t), b^\dagger b \} + \Gamma b\tilde{\sigma}(t)b^\dagger - i\Delta [b^\dagger b, \tilde{\sigma}(t)] - i\Delta' (b^\dagger b\tilde{\sigma}(t) - \tilde{\sigma}(t)b^\dagger b + \tilde{\sigma}(t)bb^\dagger - bb^\dagger\tilde{\sigma}(t)) \\ &\quad - \frac{\Gamma'}{2} (b^\dagger b\tilde{\sigma}(t) + \tilde{\sigma}(t)b^\dagger b - 2b\tilde{\sigma}(t)b^\dagger + \tilde{\sigma}(t)bb^\dagger + bb^\dagger\tilde{\sigma}(t) - 2b^\dagger\tilde{\sigma}(t)b) \\ &= -\frac{\Gamma}{2} \{ \tilde{\sigma}(t), b^\dagger b \} - \Gamma' \{ \tilde{\sigma}(t), b^\dagger b \} - i\Delta [b^\dagger b, \tilde{\sigma}(t)] + \Gamma b\tilde{\sigma}(t)b^\dagger + \Gamma' (b^\dagger\tilde{\sigma}(t)b + b\tilde{\sigma}(t)b^\dagger).\end{aligned}$$

If we now use

$$\begin{aligned}\frac{d\sigma}{dt} &= \frac{d}{dt} \{ \exp(-iH_A t/\hbar) \tilde{\sigma} \exp(iH_A t/\hbar) \} \\ &= -\frac{i}{\hbar} H_A \exp(-iH_A t/\hbar) \tilde{\sigma} \exp(iH_A t/\hbar) + \exp(-iH_A t/\hbar) \frac{d}{dt} \{ \tilde{\sigma} \exp(iH_A t/\hbar) \} \\ &= -\frac{i}{\hbar} H_A \exp(-iH_A t/\hbar) \tilde{\sigma} \exp(iH_A t/\hbar) + \exp(-iH_A t/\hbar) \frac{d\tilde{\sigma}}{dt} \exp(iH_A t/\hbar) + \exp(-iH_A t/\hbar) \tilde{\sigma} \frac{i}{\hbar} H_A \exp(iH_A t/\hbar) \\ &= -\frac{i}{\hbar} [H_A, \sigma] + \exp(-iH_A t/\hbar) \frac{d\tilde{\sigma}}{dt} \exp(iH_A t/\hbar),\end{aligned}$$

and pass to the infinitesimal limit for Δt , then with

$$\begin{aligned}\exp(-iH_A t/\hbar) b\tilde{\sigma}(t)b^\dagger \exp(iH_A t/\hbar) &= \exp(-iH_A t/\hbar) b \exp(iH_A t/\hbar) \sigma \exp(-iH_A t/\hbar) b^\dagger \exp(iH_A t/\hbar) \\ &= \exp(i\omega_0 t) b\sigma \exp(-i\omega_0 t) b^\dagger \\ &= b\sigma b^\dagger, \\ \exp(-iH_A t/\hbar) b^\dagger\tilde{\sigma}(t)b \exp(iH_A t/\hbar) &= b^\dagger\sigma b, \\ [b^\dagger b, H_A] &= 0,\end{aligned}$$

we obtain

$$\begin{aligned}\frac{d\sigma}{dt} &= -i\omega_0 [b^\dagger b, \sigma] - \frac{\Gamma}{2} \{ \tilde{\sigma}(t), b^\dagger b \} - \Gamma' \{ \tilde{\sigma}(t), b^\dagger b \} - \Gamma' \tilde{\sigma}(t) - i\Delta [b^\dagger b, \tilde{\sigma}(t)] + \Gamma b\tilde{\sigma}(t)b^\dagger + \Gamma' (b^\dagger\tilde{\sigma}(t)b + b\tilde{\sigma}(t)b^\dagger) \\ &= -\frac{\Gamma}{2} \{ \sigma, b^\dagger b \} - \Gamma' \{ \sigma, b^\dagger b \} - \Gamma' \sigma - i(\omega_0 + \Delta) [b^\dagger b, \sigma] + \Gamma b\sigma b^\dagger + \Gamma' (b^\dagger\sigma b + b\sigma b^\dagger),\end{aligned}$$

which matches Eq. (14) in the book. Pushing this just a bit further to obtain the standard Lindblad form,

$$\begin{aligned}\frac{d\sigma}{dt} &= -\frac{i}{\hbar} [H'_A, \sigma] + \frac{\Gamma}{2} (2b\sigma b^\dagger - b^\dagger b\sigma - \sigma b^\dagger b) + \Gamma' (b^\dagger\sigma b + b\sigma b^\dagger - b^\dagger b\sigma - \sigma b^\dagger b - \sigma) \\ &= -\frac{i}{\hbar} [H'_A, \sigma] + \frac{\Gamma}{2} (2b\sigma b^\dagger - b^\dagger b\sigma - \sigma b^\dagger b) + \frac{\Gamma'}{2} (2b^\dagger\sigma b - b^\dagger b\sigma - \sigma b^\dagger b - 2\sigma + 2b\sigma b^\dagger - b^\dagger b\sigma - \sigma b^\dagger b) \\ &= -\frac{i}{\hbar} [H'_A, \sigma] + \frac{\Gamma}{2} (2b\sigma b^\dagger - b^\dagger b\sigma - \sigma b^\dagger b) + \frac{\Gamma'}{2} (2b^\dagger\sigma b - b\sigma b^\dagger - \sigma b b^\dagger + 2b\sigma b^\dagger - b^\dagger b\sigma - \sigma b^\dagger b) \\ &= -\frac{i}{\hbar} [H'_A, \sigma] + \frac{\Gamma + \Gamma'}{2} (2b\sigma b^\dagger - b^\dagger b\sigma - \sigma b^\dagger b) + \frac{\Gamma'}{2} (2b^\dagger\sigma b - b\sigma b^\dagger - \sigma b b^\dagger),\end{aligned}$$

where

$$H'_A = \hbar(\omega_0 + \Delta) \left(b^\dagger b + \frac{1}{2} \right),$$

although we see that the $1/2$ has no influence on the equation of motion for σ . Comparing this with the

corresponding form of the master equation for a two-level atom in vacuum,

$$\dot{\sigma} = -\frac{i}{\hbar} [H'_A, \sigma] + \frac{\Gamma}{2} \{2L\sigma L^\dagger - L^\dagger L\sigma - \sigma L^\dagger L\},$$

where $L = |a\rangle\langle b|$ is the atomic lowering operator, how can we interpret our result for the harmonic oscillator?