APPPHYS383 Thursday 4 February 2010

Summary points

- Reservoir correlation timescale, coupling timescale (Brownian analogy)
- Assumptions leading to reservoir "force" with zero mean and fast fluctuations
  - σ_R a fixed stationary state that commutes with H_R
  - Linear coupling assumption
  - Dense reservoir spectrum
- Secular approximation

Discussion points

- Lindblad form of the Master Equation
- Non-selective versus selective evolution; quantum trajectories

We have the Interaction Picture Schrödinger Equation, written for the density matrix,

$$\begin{aligned}\frac{d}{dt}\tilde{\rho}(t) &= \frac{1}{i\hbar}[\tilde{V}(t),\tilde{\rho}(t)],\\ d\,\tilde{\rho}(t) &= \frac{1}{i\hbar}[\tilde{V}(t),\tilde{\rho}(t)]\,dt,\\ \tilde{\rho}(t+\Delta t)-\tilde{\rho}(t) &= \frac{1}{i\hbar}\int_t^{t+\Delta t}dt'\,[\tilde{V}(t'),\tilde{\rho}(t')].\end{aligned}$$

We can 'iterate' this equation by writing

$$\tilde{\rho}(t') = \tilde{\rho}(t)+\frac{1}{i\hbar}\int_t^{t'}dt''\,[\tilde{V}(t''),\tilde{\rho}(t'')],$$

and inserting this back into the expression for $\Delta\tilde{\rho}(t)\equiv\tilde{\rho}(t+\Delta t)-\tilde{\rho}(t)$,

$$\begin{aligned}\Delta\tilde{\rho}(t) &= \frac{1}{i\hbar}\int_t^{t+\Delta t}dt'\left[\tilde{V}(t'),\tilde{\rho}(t)+\frac{1}{i\hbar}\int_t^{t'}dt''\,[\tilde{V}(t''),\tilde{\rho}(t'')]\right]\\ &= \frac{1}{i\hbar}\int_t^{t+\Delta t}dt'\,[\tilde{V}(t'),\tilde{\rho}(t)]+\left(\frac{1}{i\hbar}\right)^2\int_t^{t+\Delta t}dt'\int_t^{t'}dt''\,[\tilde{V}(t'),[\tilde{V}(t''),\tilde{\rho}(t'')]].\end{aligned}$$

At this point the expression is still exact. We next want to trace over the reservoir degrees of freedom, to obtain

$$\begin{aligned}\Delta\tilde{\sigma}(t) = {}&\frac{1}{i\hbar}\int_t^{t+\Delta t}dt'\,\mathrm{Tr}_R[\tilde{V}(t')\tilde{\rho}(t)-\tilde{\rho}(t)\tilde{V}(t')]\\ &+\left(\frac{1}{i\hbar}\right)^2\int_t^{t+\Delta t}dt'\int_t^{t'}dt''\,\mathrm{Tr}_R[\tilde{V}(t')\tilde{V}(t'')\tilde{\rho}(t'')-\tilde{V}(t')\tilde{\rho}(t'')\tilde{V}(t'')]\\ &-\left(\frac{1}{i\hbar}\right)^2\int_t^{t+\Delta t}dt'\int_t^{t'}dt''\,\mathrm{Tr}_R[\tilde{V}(t'')\tilde{\rho}(t'')\tilde{V}(t')-\tilde{\rho}(t'')\tilde{V}(t'')\tilde{V}(t')].\end{aligned}$$

Under the 'thermodynamic' approximation that the coarse-grained evolution maintains $\tilde{\rho}(t)$ in a factorizable form $\tilde{\rho}(t)=\tilde{\sigma}(t)\otimes\sigma_R$, with σ_R a constant incoherent combination of reservoir energy eigenstates, and recalling Eqs. (B.16-B.18),

$$\tilde{V}(t) = -e^{iH_At/\hbar}Ae^{-iH_At/\hbar}e^{iH_Rt/\hbar}R\,e^{-iH_Rt/\hbar},$$

we have (using cyclic property of the trace)

$$\begin{aligned}\mathrm{Tr}_R[\tilde{V}(t')\tilde{\rho}(t)] &= -\mathrm{Tr}_R\left[e^{iH_At'/\hbar}Ae^{-iH_At'/\hbar}e^{iH_Rt'/\hbar}R\,e^{-iH_Rt'/\hbar}\tilde{\sigma}(t)\otimes\sigma_R\right]\\ &= -\mathrm{Tr}_R\left[e^{iH_At'/\hbar}Ae^{-iH_At'/\hbar}R\,e^{-iH_Rt'/\hbar}\tilde{\sigma}(t)\otimes\sigma_Re^{iH_Rt'/\hbar}\right]\\ &= -\mathrm{Tr}_R\left[e^{iH_At'/\hbar}Ae^{-iH_At'/\hbar}\tilde{\sigma}(t)\otimes\sigma_RR\right]\\ &= -e^{iH_At'/\hbar}Ae^{-iH_At'/\hbar}\tilde{\sigma}(t)\otimes\mathrm{Tr}[\sigma_RR]\\ &= 0,\end{aligned}$$

by the zero-mean assumption Eq. (B.19) for σ_R. Hence (again with the cyclic property) the first integral vanishes. Looking at the second integral we similarly have $(\tau\equiv t'-t'')$

$$\begin{aligned}\mathrm{Tr}_R[\tilde V(t')\tilde V(t'')\tilde\rho(t'')] &= \mathrm{Tr}_R\left[\tilde A(t')e^{iH_Rt'/\hbar}Re^{-iH_Rt'/\hbar}\tilde A(t'')e^{iH_Rt''/\hbar}Re^{-iH_Rt''/\hbar}\tilde\sigma(t'')\otimes\sigma_R\right]\\ &= \tilde A(t')\tilde A(t'')\tilde\sigma(t'')\mathrm{Tr}\left[e^{iH_Rt'/\hbar}Re^{-iH_R(t'-t'')/\hbar}Re^{-iH_Rt''/\hbar}\sigma_R\right]\\ &= \tilde A(t')\tilde A(t'')\tilde\sigma(t'')\mathrm{Tr}\left[e^{iH_R(t'-t'')/\hbar}Re^{-iH_R(t'-t'')/\hbar}R\sigma_R\right]\\ &= \tilde A(t')\tilde A(t'')\tilde\sigma(t'')\mathrm{Tr}[\tilde R(\tau)\tilde R(0)\sigma_R]\\ &= g(\tau)\tilde A(t')\tilde A(t'')\tilde\sigma(t''),\end{aligned}$$

and thus

$$\begin{aligned}\mathrm{Tr}_R[\tilde V(t')\tilde V(t'')\tilde\rho(t'')-\tilde V(t')\tilde\rho(t'')\tilde V(t'')] &= \mathrm{Tr}_R[\tilde V(t')\tilde V(t'')\tilde\rho(t'')]-\mathrm{Tr}_R[\tilde V(t')\tilde\rho(t'')\tilde V(t'')]\\ &= g(\tau)\tilde A(t')\tilde A(t'')\tilde\sigma(t'')-g(-\tau)\tilde A(t')\tilde\sigma(t'')\tilde A(t'').\end{aligned}$$

Looking at the third integral,

$$\begin{aligned}\mathrm{Tr}_R[\tilde V(t'')\tilde\rho(t'')\tilde V(t')] &= g(\tau)\tilde A(t'')\tilde\sigma(t'')\tilde A(t'),\\ \mathrm{Tr}_R[\tilde\rho(t'')\tilde V(t'')\tilde V(t')] &= g(-\tau)\tilde\sigma(t'')\tilde A(t'')\tilde A(t'),\end{aligned}$$

and thus

$$\mathrm{Tr}_R[\tilde V(t'')\tilde\rho(t'')\tilde V(t')-\tilde\rho(t'')\tilde V(t'')\tilde V(t')] = g(\tau)\tilde A(t'')\tilde\sigma(t'')\tilde A(t')-g(-\tau)\tilde\sigma(t'')\tilde A(t'')\tilde A(t').$$

Putting everything together we are left with

$$\begin{aligned}\Delta\tilde\sigma(t) &= \left(\frac{1}{i\hbar}\right)^2\int_t^{t+\Delta t}dt'\int_t^{t'}dt''\{g(\tau)\tilde A(t')\tilde A(t'')\tilde\sigma(t'')-g(-\tau)\tilde A(t')\tilde\sigma(t'')\tilde A(t'')-g(\tau)\tilde A(t'')\tilde\sigma(t'')\tilde A(t')+g(-\tau)\tilde\sigma(t'')\tilde A(t'')\tilde A(t')\}\\ &= \left(\frac{1}{i\hbar}\right)^2\int_t^{t+\Delta t}dt'\int_t^{t'}dt''\{g(\tau)[\tilde A(t')\tilde A(t'')\tilde\sigma(t'')-\tilde A(t'')\tilde\sigma(t'')\tilde A(t')]-g(-\tau)[\tilde A(t')\tilde\sigma(t'')\tilde A(t'')-\tilde\sigma(t'')\tilde A(t'')\tilde A(t')]\}\\ &= \left(\frac{1}{i\hbar}\right)^2\int_t^{t+\Delta t}dt'\int_t^{t'}dt''\{g(\tau)[\tilde A(t'),\tilde A(t'')\tilde\sigma(t'')]-g(-\tau)[\tilde A(t'),\tilde\sigma(t'')\tilde A(t'')]\},\end{aligned}$$

and it becomes clear that the only reservoir property that survives into the Master Equation, even non-perturbatively, is the correlation function $g(\tau)$. Note that there is almost a nice double-commutator structure in the integrand, which would work out if $g(\tau)$ were actually symmetric about zero.

With the assumptions made about the reservoir state we can simplify the expression for $g(\tau)$ somewhat:

$$\begin{aligned}g(\tau) &= \mathrm{Tr}[\tilde R(\tau)\tilde R(0)\sigma_R]\\ &= \sum_\mu \mathrm{Tr}[\tilde R(\tau)\tilde R(0)p_\mu|\mu\rangle\langle\mu|]\\ &= \sum_\mu p_\mu\mathrm{Tr}[\tilde R(\tau)\tilde R(0)|\mu\rangle\langle\mu||\mu\rangle\langle\mu|]\\ &= \sum_\mu p_\mu\mathrm{Tr}[|\mu\rangle\langle\mu|\tilde R(\tau)\tilde R(0)|\mu\rangle\langle\mu|]\\ &= \sum_\mu p_\mu\langle\mu|\tilde R(\tau)\tilde R(0)|\mu\rangle\mathrm{Tr}[|\mu\rangle\langle\mu|]\\ &= \sum_\mu p_\mu\langle\mu|\tilde R(\tau)\tilde R(0)|\mu\rangle\\ &= \sum_{\mu,\nu} p_\mu\langle\mu|\tilde R(\tau)|\nu\rangle\langle\nu|\tilde R(0)|\mu\rangle\\ &= \sum_{\mu,\nu} p_\mu\langle\mu|e^{iH_R\tau/\hbar}Re^{-iH_R\tau/\hbar}|\nu\rangle\langle\nu|R|\mu\rangle\\ &= \sum_{\mu,\nu} p_\mu e^{i\omega_{\mu\nu}\tau}|\langle\mu|R|\nu\rangle|^2,\end{aligned}$$

where

$$\omega_{\mu\nu} = \frac{E_\mu-E_\nu}{\hbar}.$$

This being the case we find

$$g(-\tau) = g^*(\tau),$$

as advertised. Under the assumption that the states $|\nu\rangle$ are dense we can argue that $g(\tau)$ decays quickly to zero...

To proceed further we use a perturbative expansion of the integral equation for $\Delta\tilde\rho(t)$, applying Picard iteration to the exact

$$\tilde{\rho}(t+\Delta t) = \tilde{\rho}(t) + \frac{1}{i\hbar}\int_t^{t+\Delta t} dt'\,[\tilde{V}(t'),\tilde{\rho}(t')].$$

We have (with the reasonable assumption that the integrand is C^1 on the relevant interval)

$$\tilde{\rho}(s) = \lim_{k\to\infty}\tilde{\rho}_k(s),$$

with

$$\begin{aligned}
\tilde{\rho}_0(s) &= \tilde{\rho}(t),\\
\tilde{\rho}_1(s) &= \tilde{\rho}(t) + \frac{1}{i\hbar}\int_t^s dt'\,[\tilde{V}(t'),\tilde{\rho}_0(t')]\\
&= \tilde{\rho}(t) + \frac{1}{i\hbar}\int_t^s dt'\,[\tilde{V}(t'),\tilde{\rho}(t)],\\
\tilde{\rho}_2(s) &= \tilde{\rho}(t) + \frac{1}{i\hbar}\int_t^s dt'\,[\tilde{V}(t'),\tilde{\rho}_1(t')]\\
&= \tilde{\rho}(t) + \frac{1}{i\hbar}\int_t^s dt'\,[\tilde{V}(t'),\tilde{\rho}(t)] + \left(\frac{1}{i\hbar}\right)^2\int_t^s dt'\left[\tilde{V}(t'),\int_t^{t'} dt''\,[\tilde{V}(t''),\tilde{\rho}(t)]\right]\\
&= \tilde{\rho}(t) + \frac{1}{i\hbar}\int_t^s dt'\,[\tilde{V}(t'),\tilde{\rho}(t)] + \left(\frac{1}{i\hbar}\right)^2\int_t^s dt'\int_t^{t'} dt''[\tilde{V}(t'),[\tilde{V}(t''),\tilde{\rho}(t)]].
\end{aligned}$$

Recalling that the first integral vanishes because of the zero mean assumption, and taking $s = t+\Delta t,$ we have the second-order approximation

$$\Delta\tilde{\rho}(t) \approx \left(\frac{1}{i\hbar}\right)^2\int_t^{t+\Delta t} dt'\int_t^{t'} dt''[\tilde{V}(t'),[\tilde{V}(t''),\tilde{\rho}(t)]].$$

Tracing over the reservoir and dividing both sides by Δt leads to Eq. (B.30):

$$\begin{aligned}
\frac{\Delta\tilde{\sigma}(t)}{\Delta t} &= \frac{1}{\Delta t}\mathrm{Tr}_R[\Delta\tilde{\sigma}(t)\otimes\sigma_R]\\
&= -\frac{1}{\hbar^2}\frac{1}{\Delta t}\int_t^{t+\Delta t} dt'\int_t^{t'} dt''\mathrm{Tr}_R[\tilde{V}(t'),[\tilde{V}(t''),\tilde{\rho}(t)]]\\
&= -\frac{1}{\hbar^2}\frac{1}{\Delta t}\int_t^{t+\Delta t} dt'\int_t^{t'} dt''\mathrm{Tr}_R[\tilde{V}(t'),[\tilde{V}(t'')\tilde{\rho}(t)-\tilde{\rho}(t)\tilde{V}(t'')]]\\
&= -\frac{1}{\hbar^2}\frac{1}{\Delta t}\int_t^{t+\Delta t} dt'\int_t^{t'} dt''\mathrm{Tr}_R[\tilde{V}(t')\tilde{V}(t'')\tilde{\rho}(t)-\tilde{V}(t')\tilde{\rho}(t)\tilde{V}(t'')-\tilde{V}(t'')\tilde{\rho}(t)\tilde{V}(t')+\tilde{\rho}(t)\tilde{V}(t'')\tilde{V}(t')]\\
&= -\frac{1}{\hbar^2}\frac{1}{\Delta t}\int_t^{t+\Delta t} dt'\int_t^{t'} dt''\{\tilde{A}(t')\tilde{A}(t'')\tilde{\sigma}(t)g(\tau)-\tilde{A}(t')\tilde{\sigma}(t)\tilde{A}(t'')g^*(\tau)-\tilde{A}(t'')\tilde{\sigma}(t)\tilde{A}(t')g(\tau)+\tilde{\sigma}(t)\tilde{A}(t'')\tilde{A}(t')g^*(\tau)\}\\
&= -\frac{1}{\hbar^2}\frac{1}{\Delta t}\int_t^{t+\Delta t} dt'\int_t^{t'} dt''\{g(\tau)[\tilde{A}(t')\tilde{A}(t'')\tilde{\sigma}(t)-\tilde{A}(t'')\tilde{\sigma}(t)\tilde{A}(t')]+g^*(\tau)[\tilde{\sigma}(t)\tilde{A}(t'')\tilde{A}(t')-\tilde{A}(t')\tilde{\sigma}(t)\tilde{A}(t'')]\}\\
&= -\frac{1}{\hbar^2}\frac{1}{\Delta t}\int_t^{t+\Delta t} dt'\int_t^{t'} dt''\{g(\tau)[\tilde{A}(t'),\tilde{A}(t'')\tilde{\sigma}(t)]-g^*(\tau)[\tilde{A}(t'),\tilde{\sigma}(t)\tilde{A}(t'')]\}.
\end{aligned}$$

Since $\tau \equiv t'-t'',$ it seems tempting to try to change variables in the integration so that we are integrating over $d\tau.$ As shown in Fig. 1 of Chapter IV, the domain of integration begins as the triangle $0AB,$ where $0B$ is the line on which $t'' = t'.$ Hence $\tau = 0$ on the line $0B,$ and we see that we could integrate over the same triangle by setting

$$\int_t^{t+\Delta t} dt'\int_t^{t'} dt'' \to \int_t^{t+\Delta t} dt'\int_0^{t'} d\tau.$$

We next note that since $g(\tau)$ decays quickly to zero, we should not make much of an error by extending the integral over τ from 0 to $\infty.$ Hence we can write

$$\frac{\Delta\tilde{\sigma}(t)}{\Delta t} \approx -\frac{1}{\hbar^2}\int_0^\infty d\tau\frac{1}{\Delta t}\int_t^{t+\Delta t} dt'\,\{g(\tau)[\tilde{A}(t'),\tilde{A}(t'-\tau)\tilde{\sigma}(t)]-g^*(\tau)[\tilde{A}(t'),\tilde{\sigma}(t)\tilde{A}(t'-\tau)]\},$$

where we swap the order of integrations, which we can now do, in order to match Eq. (B.33).

We next project the Master Equation onto the energy-state basis of H_A :

$$
\begin{aligned}
\frac{\Delta\tilde{\sigma}_{ab}(t)}{\Delta t} &= \langle a|\frac{\Delta\tilde{\sigma}(t)}{\Delta t}|b\rangle \\
&= -\frac{1}{\hbar^2}\int_0^\infty d\tau \frac{1}{\Delta t}\int_t^{t+\Delta t} dt' \{g(\tau)\langle a|[\tilde{A}(t'),\tilde{A}(t'-\tau)\tilde{\sigma}(t)]|b\rangle - g^*(\tau)\langle a|[\tilde{A}(t'),\tilde{\sigma}(t)\tilde{A}(t'-\tau)]|b\rangle\} \\
&= -\frac{1}{\hbar^2}\int_0^\infty d\tau \frac{1}{\Delta t}\int_t^{t+\Delta t} dt' \sum_{c,d}\{g(\tau)\langle a|[\tilde{A}(t'),\tilde{A}(t'-\tau)|c\rangle\langle c|\tilde{\sigma}(t)|d\rangle\langle d|]|b\rangle - g^*(\tau)\langle a|[\tilde{A}(t'),|c\rangle\langle c|\tilde{\sigma}(t)|d\rangle\langle d|\tilde{A}(t'-\tau)]|b\rangle\} \\
&= -\frac{1}{\hbar^2}\int_0^\infty d\tau \frac{1}{\Delta t}\int_t^{t+\Delta t} dt' \sum_{c,d}\{g(\tau)\langle a|[\tilde{A}(t'),\tilde{A}(t'-\tau)|c\rangle\tilde{\sigma}_{cd}(t)\langle d|]|b\rangle - g^*(\tau)\langle a|[\tilde{A}(t'),|c\rangle\tilde{\sigma}_{cd}(t)\langle d|\tilde{A}(t'-\tau)]|b\rangle\} \\
&= -\frac{1}{\hbar^2}\int_0^\infty d\tau \frac{1}{\Delta t}\int_t^{t+\Delta t} dt' \sum_{c,d}\left\{\begin{array}{c} g(\tau)\tilde{\sigma}_{cd}(t)\langle a|[\tilde{A}(t')\tilde{A}(t'-\tau)|c\rangle\langle d| - \tilde{A}(t'-\tau)|c\rangle\langle d|\tilde{A}(t')]|b\rangle \\ -g^*(\tau)\tilde{\sigma}_{cd}(t)\langle a|[\tilde{A}(t')|c\rangle\langle d|\tilde{A}(t'-\tau) - |c\rangle\langle d|\tilde{A}(t'-\tau)\tilde{A}(t')]|b\rangle \end{array}\right\} \\
&= -\frac{1}{\hbar^2}\int_0^\infty d\tau \frac{1}{\Delta t}\int_t^{t+\Delta t} dt' \sum_{c,d}\left\{\begin{array}{c} g(\tau)\tilde{\sigma}_{cd}(t)[\langle a|\tilde{A}(t')\tilde{A}(t'-\tau)|c\rangle\delta_{bd} - \langle a|\tilde{A}(t'-\tau)|c\rangle\langle d|\tilde{A}(t')|b\rangle] \\ -g^*(\tau)\tilde{\sigma}_{cd}(t)[\langle a|\tilde{A}(t')|c\rangle\langle d|\tilde{A}(t'-\tau)|b\rangle - \delta_{ac}\langle d|\tilde{A}(t'-\tau)\tilde{A}(t')|b\rangle] \end{array}\right\} \\
&= -\frac{1}{\hbar^2}\int_0^\infty d\tau \frac{1}{\Delta t}\int_t^{t+\Delta t} dt' \sum_{c,d}\left\{\begin{array}{c} g(\tau)\tilde{\sigma}_{cd}(t)\left[\delta_{bd}\sum_n\langle a|\tilde{A}(t')|n\rangle\langle n|\tilde{A}(t'-\tau)|c\rangle - e^{i\omega_{ac}(t'-\tau)}e^{i\omega_{db}(t')}A_{ac}A_{db}\right] \\ -g^*(\tau)\tilde{\sigma}_{cd}(t)\left[e^{i\omega_{ac}t'}e^{i\omega_{db}(t'-\tau)}A_{ac}A_{db} - \delta_{ac}\sum_n\langle d|\tilde{A}(t'-\tau)|n\rangle\langle n|\tilde{A}(t')|b\rangle\right] \end{array}\right\} \\
&= -\frac{1}{\hbar^2}\int_0^\infty d\tau \frac{1}{\Delta t}\int_t^{t+\Delta t} dt' \sum_{c,d}\left\{\begin{array}{c} g(\tau)\tilde{\sigma}_{cd}(t)\left[\delta_{bd}\sum_n e^{i\omega_{an}t'}e^{i\omega_{nc}(t'-\tau)}A_{an}A_{nc} - e^{i\omega_{ac}(t'-\tau)}e^{i\omega_{db}(t')}A_{ac}A_{db}\right] \\ +g^*(\tau)\tilde{\sigma}_{cd}(t)\left[\delta_{ac}\sum_n e^{i\omega_{dn}(t'-\tau)}e^{i\omega_{nb}t'}A_{dn}A_{nb} - e^{i\omega_{ac}t'}e^{i\omega_{db}(t'-\tau)}A_{ac}A_{db}\right] \end{array}\right\}.
\end{aligned}
$$

We next note that

$$
\begin{aligned}
\exp(i\omega_{an}t')\exp(i\omega_{nc}t') &= \exp(i\omega_{ac}t'), \\
\exp(i\omega_{ac}t')\exp(i\omega_{db}t') &= \exp(i(\omega_{ab}-\omega_{cd})t'), \\
\delta_{bd}\exp(i\omega_{ac}t') &= \delta_{bd}\exp(i\omega_{ac}t')\exp(i\omega_{db}t') = \delta_{bd}\exp(i(\omega_{ab}-\omega_{cd})t'), \\
\exp(i\omega_{dn}t')\exp(i\omega_{nb}t') &= \exp(i\omega_{db}t'), \\
\delta_{ac}\exp(i\omega_{db}t') &= \delta_{ac}\exp(i\omega_{ac}t')\exp(i\omega_{db}t') = \delta_{ac}\exp(i(\omega_{ab}-\omega_{cd})t'),
\end{aligned}
$$

hence

$$
\frac{\Delta\tilde{\sigma}_{ab}(t)}{\Delta t} = -\frac{1}{\hbar^2}\int_0^\infty d\tau \frac{1}{\Delta t}\int_t^{t+\Delta t} dt' \exp(i(\omega_{ab}-\omega_{cd})t')\sum_{c,d}\left\{\begin{array}{c} g(\tau)\tilde{\sigma}_{cd}(t)\left[\delta_{bd}\sum_n e^{-i\omega_{nc}\tau}A_{an}A_{nc} - e^{-i\omega_{ac}\tau}A_{ac}A_{db}\right] \\ +g^*(\tau)\tilde{\sigma}_{cd}(t)\left[\delta_{ac}\sum_n e^{-i\omega_{dn}\tau}A_{dn}A_{nb} - e^{-i\omega_{db}\tau}A_{ac}A_{db}\right] \end{array}\right\},
$$

and we follow the book by next applying the integral

$$
\begin{aligned}
\frac{1}{\Delta t}\int_t^{t+\Delta t} dt' \exp(i(\omega_{ab}-\omega_{cd})t') &= \frac{1}{\Delta t}\frac{1}{i(\omega_{ab}-\omega_{cd})}\{\exp(i(\omega_{ab}-\omega_{cd})(t+\Delta t)) - \exp(i(\omega_{ab}-\omega_{cd})t)\} \\
&= \frac{\exp(i(\omega_{ab}-\omega_{cd})t)}{i(\omega_{ab}-\omega_{cd})\Delta t}\{\exp(i(\omega_{ab}-\omega_{cd})\Delta t) - 1\} \\
&= \frac{\exp(i(\omega_{ab}-\omega_{cd})t)}{i(\omega_{ab}-\omega_{cd})\Delta t}\exp(i(\omega_{ab}-\omega_{cd})\Delta t/2)\{\exp(i(\omega_{ab}-\omega_{cd})\Delta t/2) - \exp(-i(\omega_{ab}-\omega_{cd})\Delta t/2)\} \\
&= \frac{\exp(i(\omega_{ab}-\omega_{cd})t)}{i(\omega_{ab}-\omega_{cd})\Delta t}\exp(i(\omega_{ab}-\omega_{cd})\Delta t/2)2i\sin((\omega_{ab}-\omega_{cd})\Delta t/2) \\
&= \exp(i(\omega_{ab}-\omega_{cd})t)\exp(i(\omega_{ab}-\omega_{cd})\Delta t/2)\frac{\sin((\omega_{ab}-\omega_{cd})\Delta t/2)}{(\omega_{ab}-\omega_{cd})\Delta t/2}.
\end{aligned}
$$

The sync function factor indicates that the integral will be very small if $|(\omega_{ab}-\omega_{cd})\Delta t|$ is large, meaning that we can safely ignore any terms in which $|\omega_{ab}-\omega_{cd}| \gg 1/\Delta t$. If $|\omega_{ab}-\omega_{cd}| \ll 1/\Delta t$ then the sync function and the exponential in Δt are approximately equal to one; by a somewhat roundabout argument (see text, last paragraph of IV.B.4) one can therefore motivate the secular approximation in which we neglect all terms in the sum except those for which $|\omega_{ab}-\omega_{cd}| \ll 1/\Delta t$ and approximate the integral over t' accordingly:

$$
\begin{aligned}
\frac{\Delta\tilde{\sigma}_{ab}(t)}{\Delta t} &= -\frac{1}{\hbar^2}\exp(i(\omega_{ab}-\omega_{cd})t)\int_0^\infty d\tau \sum_{c,d}{}^{(\mathrm{sec})}\left\{\begin{array}{c} g(\tau)\tilde{\sigma}_{cd}(t)\left[\delta_{bd}\sum_n e^{-i\omega_{nc}\tau}A_{an}A_{nc} - e^{-i\omega_{ac}\tau}A_{ac}A_{db}\right] \\ +g^*(\tau)\tilde{\sigma}_{cd}(t)\left[\delta_{ac}\sum_n e^{-i\omega_{dn}\tau}A_{dn}A_{nb} - e^{-i\omega_{db}\tau}A_{ac}A_{db}\right] \end{array}\right\} \\
&= \sum_{c,d}{}^{(\mathrm{sec})}\exp(i(\omega_{ab}-\omega_{cd})t)R_{abcd}\tilde{\sigma}_{cd}(t),
\end{aligned}
$$

where

$$R_{abcd} \equiv -\frac{1}{\hbar^2}\int_0^\infty d\tau \left\{ g(\tau)\left[\delta_{bd}\sum_n{}^{(\mathrm{sec})} e^{-i\omega_{nc}\tau} A_{an}A_{nc} - e^{-i\omega_{ac}\tau} A_{ac}A_{db} \right] + g^*(\tau)\left[\delta_{ac}\sum_n{}^{(\mathrm{sec})} e^{-i\omega_{dn}\tau} A_{dn}A_{nb} - e^{-i\omega_{db}\tau} A_{ac}A_{db} \right]\right\}.$$

Using

$$\tilde\sigma_{ab}(t) = \langle a|\exp(iH_A t)\sigma\exp(-iH_A t)|b\rangle = \exp(i\omega_{ab}t)\sigma_{ab},$$
$$\tilde\sigma_{cd}(t) = \langle c|\exp(iH_A t)\sigma\exp(-iH_A t)|d\rangle = \exp(i\omega_{cd}t)\sigma_{cd},$$

we can fully switch back to the Schrödinger Picture, where we should be careful to note that

$$\begin{aligned}\frac{d}{dt}\sigma_{ab}(t) &= \frac{d}{dt}\{\exp(-i\omega_{ab}t)\tilde\sigma_{ab}(t)\}\\ &= -i\omega_{ab}\exp(-i\omega_{ab}t)\tilde\sigma_{ab}(t) + \exp(-i\omega_{ab}t)\frac{d}{dt}\tilde\sigma_{ab}(t)\\ &= -i\omega_{ab}\sigma_{ab}(t) + \exp(-i\omega_{ab}t)\frac{d}{dt}\tilde\sigma_{ab}(t).\end{aligned}$$

Hence by treating the coarse-grained timestep we have been using as a differential, we finally obtain

$$\begin{aligned}\frac{d}{dt}\sigma_{ab}(t) &= -i\omega_{ab}\sigma_{ab}(t) + \exp(-i\omega_{ab}t)\sum_{c,d}{}^{(\mathrm{sec})}\exp(i(\omega_{ab}-\omega_{cd})t)R_{abcd}\exp(i\omega_{cd}t)\sigma_{cd}\\ &= -i\omega_{ab}\sigma_{ab}(t) + \sum_{c,d}{}^{(\mathrm{sec})} R_{abcd}\,\sigma_{cd}.\end{aligned}$$

Noting that the R_{abcd} coefficients have all time variables integrated out, we arrive at the fundamental fact that the Master Equation takes the form of a linear differential equation with time-independent coefficients.

Generally speaking, any Master Equation can be written in the so-called Lindblad form,

$$\dot\sigma = -\frac{i}{\hbar}[H_A,\sigma] + \sum_j \gamma_j\{2L_j\sigma L_j^\dagger - L_j^\dagger L_j\sigma - \sigma L_j^\dagger L_j\}.$$

This form points out that for σ viewed as a numerical matrix, the linear differential equation requires both left- and right-multiplication by coefficient matrices L_j (the Lindblad operators). It is straightforward to show that whatever the form of the $\{L_j\}$, one can transform the Master Equation into a vector form

$$\dot v = Mv,$$

where v is a vector containing all the matrix elements of σ (for example, simply stacking the columns of σ on top of one another) and M is a constant matrix (with dimension equal to the square of the dimension of the A Hilbert space). In this form it is possible to apply various efficient methods for integrating the Master Equation, or finding a steady-state solution v_0 such that $Mv_0 = 0$. In principle, we always have the formal solution

$$v(t) = \exp(Mt)v(0),$$

although in practice M can be too complex to work with analytically and of too large a dimension to permit brute force numerical computation of the expontenial. In such cases one can simply resort to numerical integration methods.

In the simple case of a two-level atom coupled to a vacuum field, the textbook arrives at

$$\frac{d}{dt}\sigma_{bb} = -\Gamma\sigma_{bb}, \quad \frac{d}{dt}\sigma_{aa} = \Gamma\sigma_{bb},$$
$$\frac{d}{dt}\sigma_{ba} = -i(\omega_{ba}+\Delta_{ba})\sigma_{ba} - \frac{\Gamma}{2}\sigma_{ba}.$$

Note that we can write

$$-i(\omega_{ba}+\Delta_{ba})\sigma_{ba} = -\frac{i}{\hbar}[H_A',\sigma], \quad H_A' = \hbar(\omega_{ba}+\Delta_{ba})|b\rangle\langle b|,$$

and hence

$$\dot\sigma = -\frac{i}{\hbar}[H_A',\sigma] + \frac{\Gamma}{2}\{2L\sigma L^\dagger - L^\dagger L\sigma - \sigma L^\dagger L\},$$

if we set

$$L = |a\rangle\langle b|, \quad L^\dagger L = |b\rangle\langle b|,$$

where L is the atomic lowering operator. To verify,

$$\sigma = \sigma_{aa}|a\rangle\langle a| + \sigma_{ab}|a\rangle\langle b| + \sigma_{ba}|b\rangle\langle a| + \sigma_{bb}|b\rangle\langle b|,$$

$$\begin{aligned}-\frac{i}{\hbar}[H_A',\sigma] &= -i(\omega_{ba}+\Delta_{ba})\{|b\rangle\langle b|\sigma - \sigma|b\rangle\langle b|\}\\ &= -i(\omega_{ba}+\Delta_{ba})\{\sigma_{ba}|b\rangle\langle a| + \sigma_{bb}|b\rangle\langle b| - \sigma_{ab}|a\rangle\langle b| - \sigma_{bb}|b\rangle\langle b|\}\\ &= -i(\omega_{ba}+\Delta_{ba})\{\sigma_{ba}|b\rangle\langle a| - \sigma_{ab}|a\rangle\langle b|\},\end{aligned}$$

$$\begin{aligned}
L\sigma L^\dagger &= |a\rangle\langle b|\{\sigma_{aa}|a\rangle\langle a| + \sigma_{ab}|a\rangle\langle b| + \sigma_{ba}|b\rangle\langle a| + \sigma_{bb}|b\rangle\langle b|\}|b\rangle\langle a| \\
&= \sigma_{bb}|a\rangle\langle a|, \\
L^\dagger L\sigma &= |b\rangle\langle b|\{\sigma_{aa}|a\rangle\langle a| + \sigma_{ab}|a\rangle\langle b| + \sigma_{ba}|b\rangle\langle a| + \sigma_{bb}|b\rangle\langle b|\} \\
&= \sigma_{ba}|b\rangle\langle a| + \sigma_{bb}|b\rangle\langle b|, \\
\sigma L^\dagger L &= \{\sigma_{aa}|a\rangle\langle a| + \sigma_{ab}|a\rangle\langle b| + \sigma_{ba}|b\rangle\langle a| + \sigma_{bb}|b\rangle\langle b|\}|b\rangle\langle b| \\
&= \sigma_{ab}|a\rangle\langle b| + \sigma_{bb}|b\rangle\langle b|,
\end{aligned}$$

hence

$$\begin{aligned}
-\frac{i}{\hbar}[H_A', \sigma] + \frac{\Gamma}{2}\{L\sigma L^\dagger - L^\dagger L\sigma - \sigma L^\dagger L\} &= -i(\omega_{ba} + \Delta_{ba})\{\sigma_{ba}|b\rangle\langle a| - \sigma_{ab}|a\rangle\langle b|\} \\
&\quad + \Gamma\sigma_{bb}|a\rangle\langle a| - \frac{\Gamma}{2}\sigma_{ba}|b\rangle\langle a| - \frac{\Gamma}{2}\sigma_{ab}|a\rangle\langle b| - \Gamma\sigma_{bb}|b\rangle\langle b|,
\end{aligned}$$

and we can read off

$$\begin{aligned}
\dot{\sigma}_{aa} &= \Gamma\sigma_{bb}, \quad \dot{\sigma}_{bb} = -\Gamma\sigma_{bb}, \\
\dot{\sigma}_{ab} &= i(\omega_{ba} + \Delta_{ba})\sigma_{ab} - \frac{\Gamma}{2}\sigma_{ab}, \\
\dot{\sigma}_{ba} &= -i(\omega_{ba} + \Delta_{ba})\sigma_{ba} - \frac{\Gamma}{2}\sigma_{ba},
\end{aligned}$$

in agreement with expectations. Note that the radiative level shift Δ_{ba} simply merges with the ‘bare’ energy difference ω_{ba} in an effective Hamiltonian for A.