APPPHYS383 Thursday 21 January 2010

Summary points

- Restriction to subspace of coupled continuum states; discretization
- Assumptions regarding flat coupling allow explicit computation of coupled eigenvalues, eigenstates
- Lorentzian distribution of discrete state in coupled continuum implies exponential decay
- Spontaneous emission, indirect decay, Fano profiles

The initial setup of the problem involves a total Hilbert space corresponding to the span of $|\varphi\rangle$ and a continuum $|E,\beta\rangle$. The discrete state $|\varphi\rangle$ and each of the continuum states $|E,\beta\rangle$ are assumed to be eigenstates of an unperturbed Hamiltonian H_0, but the total Hamiltonian we consider is the sum of H_0 and a perturbation V, where

$$\langle\varphi|V|\varphi\rangle = \langle E',\beta'|V|E,\beta\rangle = 0,$$

but V does couple $|\varphi\rangle$ to the continuum. We can imagine, for example, that $|\varphi\rangle$ represents something like $|b\rangle\otimes|0\rangle$ where $|b\rangle$ is an atomic excited state and $|0\rangle$ is the global vacuum state of the EM field, while the continuum states are of the form

$$|E,\beta\rangle = |a\rangle\otimes a^\dagger_{\boldsymbol{\varepsilon}}(\boldsymbol{k})|0\rangle.$$

Here $|a\rangle$ is an atomic ground state, while $E\leftrightarrow|\boldsymbol{k}|$ and $\beta\leftrightarrow\left(\hat{\boldsymbol{k}},\boldsymbol{\varepsilon}\right)$. The first important manipulation is to define $|E\rangle$ as the specific state within $\mathrm{span}_\beta\{|E,\beta\rangle\}$ that is coupled to $|\varphi\rangle$ by the interaction Hamiltonian, V. This simplifies subsequent steps in the calculation.

Consider the Hilbert space vector

$$|V\varphi\rangle \equiv V|\varphi\rangle,$$

which we note will not in general have unit norm. We can project $|V\varphi\rangle$ into a subspace with fixed E to obtain the state $|E\rangle$ utilized in the book,

$$|E\rangle \equiv \int d\beta\langle E,\beta|V|\varphi\rangle|E,\beta\rangle.$$

Since $\langle\varphi|V|\varphi\rangle = 0$ we can in fact write

$$V|\varphi\rangle = \int dE|E\rangle,$$

and it follows that if we consider any state $|E_\perp\rangle$ contained within the orthogonal complement of $|E\rangle$ in $\mathrm{span}_\beta\{|E,\beta\rangle\}$,

$$\langle E_\perp|V|\varphi\rangle = \langle\varphi|V|E_\perp\rangle = 0,$$

and we are free simply to remove all such states (that is, the orthogonal complement subspaces) from the problem.

After discretizing the continuum the total Hilbert space for our problem is thus taken to be $\mathrm{span}_k\{|\varphi\rangle,|k\rangle\}$. With the simplifying assumptions

$$\langle k|V|\varphi\rangle = v,$$
$$\langle k|H_0|k\rangle = k\delta,$$
$$\langle\varphi|H_0|\varphi\rangle = 0,$$

(flat coupling, uniform density of states, and E_φ as the origin of the energy scale) we find that it is possible to solve explicitly for the eigenvalues E_μ and eigenvectors $|\psi_\mu\rangle$ of the total Hamiltonian. We start by writing the eigenvalue relation

$$H|\psi_\mu\rangle = E_\mu|\psi_\mu\rangle,$$

and project it first onto the state $|k\rangle$:

$$\langle k|H|\psi_\mu\rangle = E_\mu\langle k|\psi_\mu\rangle,$$
$$\langle k|H_0|\psi_\mu\rangle + \langle k|V|\psi_\mu\rangle = E_\mu\langle k|\psi_\mu\rangle,$$
$$E_k\langle k|\psi_\mu\rangle + \langle k|V|\psi_\mu\rangle = E_\mu\langle k|\psi_\mu\rangle.$$

Note that since we have

$$\langle k|V|\varphi\rangle = v,$$
$$\langle k|V|k'\rangle = 0,$$

and $\mathrm{span}_k\{|\varphi\rangle,|k\rangle\}$ is the entire relevant Hilbert space, we have

$$\langle k|V = v\langle\varphi|,$$

and thus the projection we have been considering simplifies to

$$k\delta\langle k|\psi_\mu\rangle + v\langle\varphi|\psi_\mu\rangle = E_\mu\langle k|\psi_\mu\rangle.$$

We can likewise calculate the projection onto $|\varphi\rangle$,

$$\begin{aligned} \langle\varphi|H|\psi_\mu\rangle &= E_\mu\langle\varphi|\psi_\mu\rangle, \\ \langle\varphi|H_0|\psi_\mu\rangle + \langle\varphi|V|\psi_\mu\rangle &= E_\mu\langle\varphi|\psi_\mu\rangle, \\ \langle\varphi|V|\psi_\mu\rangle &= E_\mu\langle\varphi|\psi_\mu\rangle. \end{aligned}$$

Since

$$\langle\varphi|V|\varphi\rangle = 0,$$

$$\langle k|V|\varphi\rangle = v,$$

we have

$$\langle\varphi|V = \sum_k v\langle k|,$$

and thus

$$\sum_k v\langle k|\psi_\mu\rangle = E_\mu\langle k|\psi_\mu\rangle.$$

As described in the text, it is now straightforward to solve for E_μ, $\langle\varphi|\psi_\mu\rangle$ and $\langle k|\psi_\mu\rangle$, which together tell us everything about the diagonalization of H.

Discussion points

- $|E\rangle$ states for electric dipole decay
- Principle parts
- Quenching and quantum beats in hydrogen [J. Phys. B: Atom. Molec. Phys. **9**, 2017 (1976)]