APPPHYS217 Tuesday 25 May 2010

Our aim today is to take a brief tour of some topics in nonlinear dynamics. Some good references include:

[Perko] Lawrence Perko, *Differential Equations and Dynamical Systems* (Springer-Verlag, New York, 2001)

[Wiggins] Stephen Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos* (Springer-Verlag, New York 1990)

The Stable Manifold Theorem

Consider a linear dynamical system,

$$\dot{x} = Ax,$$

where $x \in R^n$. The origin is an equilibrium point of such a system, and we have seen that the eigenvalues of A determine its stability. In fact, one can write (Perko §1.9 Theorem 1)

$$R^n = E^s \oplus E^u \oplus E^c,$$

where where E^s, E^u and E^c are the stable, unstable and center subspaces associated with eigenvalues of A having negative, positive and zero real parts. The subspaces E^s, E^u and E^c are invariant with respect to the flow $x(t) = \exp(At)x(0)$. As a simple example we can consider

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix},$$

which has eigenvalues and eigenvectors

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \leftrightarrow -3, \quad \begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix} \leftrightarrow -i, \quad \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix} \leftrightarrow +i.$$

Hence the stable subspace E^s is the x_3 axis while the center subspace E^c is the $x_1 - x_2$ plane. Note than an arbitrary initial condition with x_1, x_2 and x_3 all nonzero 'decays into' the center subspace as $t \to +\infty$.

We have seen that it is straightforward to compute the linearization of a nonlinear dynamical system in the neighborhood of an equilibrium point. For example if we consider (Perko §2.7 Problem 7.4)

$$\begin{aligned} \dot{x}_1 &= -x_1, \\ \dot{x}_2 &= -x_2 + x_1^2, \\ \dot{x}_3 &= x_3 + x_2^2, \end{aligned}$$

we note by inspection that the origin is an equilibrium point and assuming the vector notation $\dot{x} = f(x)$ we can derive

$$Df(x) = \begin{bmatrix} -1 & 0 & 0 \\ 2x_1 & -1 & 0 \\ 0 & 2x_2 & 1 \end{bmatrix}, \quad A = Df(0) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It thus appears that the local linearization at the origin has a two-dimensional stable subspace (the $x_1 - x_2$ plane) and a one-dimensional unstable subspace (the x_3 axis). Thinking only about the linearization, any initial condition $x(0)$ that lies in the $x_1 - x_2$ plane should flow to the origin as $t \to +\infty$, and it seems tempting to infer that in the full nonlinear system this should hold true as long as $x(0)$ is also within a small neighborhood of the origin. But is that really true? The Stable Manifold Theorem (Perko §2.7) and the Local Center Manifold Theorem (Perko §2.12) can be combined to yield the following (Perko §2.7):

> Let $f \in C^r(E)$ where E is an open subset of R^n containing the origin and $r \geq 1$. Suppose that $f(0) = 0$ and that $Df(0)$ has k eigenvalues with negative real part, j eigenvalues with positive real part, and $m = n - k - j$ eigenvalues with zero real part. Then there exists an m-dimensional center manifold $W^c(0)$ of class C^r tangent to the center subspace E^c of $Df(0)$, there exists a k-dimensional stable manifold $W^s(0)$ of class C^r tangent to the stable subspace of $Df(0)$ and there exists a j-dimensional unstable manifold $W^u(0)$ of class C^r tangent to the unstable subspace E^u of $Df(0)$; furthermore, $W^c(0)$, $W^s(0)$ and $W^u(0)$ are invariant under the flow of $\dot{x} = f(x)$.

To get a practical feeling for what this means, we note that the (global) stable and unstable manifolds of the above example are

$$\begin{aligned} S &: x_3 = -\frac{1}{3}x_2^2 - \frac{1}{6}x_1^2 x_2 - \frac{1}{30}x_1^4, \\ U &: x_1 = x_2 = 0. \end{aligned}$$

The stable manifold is two-dimensional and is specified in the form of a *graph* over the $x_1 - x_2$ plane (the stable subspace of $Df(0)$); the unstable manifold is one-dimensional and coincides with the x_3 axis (the unstable subspace of $Df(0)$). We can easily confirm that for an initial condition $x(0) \in U$ we have $x(t) \in U$ and in fact $x(t) \to 0$ as $t \to -\infty$. For any initial condition $x(0) \in S$ we have

$$\begin{aligned} \frac{d}{dt}\left\{-\frac{1}{3}x_2^2 - \frac{1}{6}x_1^2 x_2 - \frac{1}{30}x_1^4\right\} &= -\frac{2}{3}x_2\dot{x}_2 - \frac{1}{6}(2x_1\dot{x}_1 x_2 + x_1^2\dot{x}_2) - \frac{4}{30}x_1^3\dot{x}_1 \\ &= -\frac{2}{3}x_2(-x_2 + x_1^2) - \frac{1}{6}(-2x_1^2 x_2 + x_1^2(-x_2 + x_1^2)) + \frac{4}{30}x_1^4 \\ &= \frac{2}{3}x_2^2 - \frac{2}{3}x_1^2 x_2 + \frac{1}{3}x_1^2 x_2 + \frac{1}{6}x_1^2 x_2 - \frac{1}{6}x_1^4 + \frac{4}{30}x_1^4 \\ &= \frac{2}{3}x_2^2 - \frac{1}{6}x_1^2 x_2 - \frac{1}{30}x_1^4 \\ &= x_3 + x_2^2 \\ &= \dot{x}_3, \end{aligned}$$

which confirms that S is an invariant manifold. Since we can easily solve

$$\begin{aligned}
\dot{x}_1 &= -x_1, \\
x_1(t) &= \exp(-t)x_1(0), \\
\dot{x}_2 &= -x_2 + x_1^2 = -x_2 + \exp(-2t)x_1^2(0), \\
x_2(t) &= \exp(-t)x_2(0) + \exp(-t)\int_0^t ds\,\exp(s)\exp(-2s)x_1^2(0) \\
&= \exp(-t)x_2(0) + [\exp(-t) - \exp(-2t)]x_1^2(0),
\end{aligned}$$

we see that for any point $x(0) \in S$ we have $x(t) \to 0$ as $t \to +\infty$. Finally we confirm that S is tangent to the $x_1 - x_2$ plane. For $x_3 = 0$ we have

$$\begin{aligned}
0 &= -\frac{1}{3}x_2^2 - \frac{1}{6}x_1^2x_2 - \frac{1}{30}x_1^4, \\
x_2 &= -\frac{3}{2}\left\{\frac{1}{6}x_1^2 \pm \sqrt{\frac{1}{36}x_1^4 - \frac{4}{90}x_1^4}\right\} \\
&= -\frac{3}{2}\left\{\frac{1}{6}x_1^2 \pm x_1^2\sqrt{\frac{5}{180} - \frac{8}{180}}\right\},
\end{aligned}$$

hence (since x_2 and x_1 must both be real in this solution) S contacts the $x_1 - x_2$ plane only at the origin. The stable manifold S consists of all points (x_1, x_2, x_3) such that

$$F(x_1,x_2,x_3) = x_3 + \frac{1}{3}x_2^2 + \frac{1}{6}x_1^2x_2 + \frac{1}{30}x_1^4 = 0,$$

hence the normal vector to the surface at an arbitrary point on S is

$$\nabla F = \begin{pmatrix} \frac{2}{15}x_1^3 + \frac{1}{3}x_1x_2 \\ \frac{2}{3}x_2 + \frac{1}{6}x_1^2 \\ 1 \end{pmatrix},$$

which clearly points along the x_3-axis at the origin; hence S is tangent to the $x_1 - x_2$ plane at the origin. Note that the answer to our original question is that in the full nonlinear flow, the only point near the origin in the $x_1 - x_2$ plane that goes to the origin as $t \to +\infty$ is the origin! Should we be bothered by this? How should we understand the relevance of the linearized dynamics to the true nonlinear flow in the neighborhood of an equilibrium point?

Before answering this, let us briefly note that the stable, unstable and center manifolds are prototypical examples of *invariant manifolds* of a nonlinear dynamical system. It can be useful to know about invariant manifolds as, *e.g.*, no trajectory of the dynamics can cross through one. And while the global stable and unstable manifolds of the above example had no boundaries, this need not always be the case. For example consider the two-dimensional dynamical system

$$\begin{aligned}
\dot{x}_1 &= 1 - x_1^2, \\
\dot{x}_2 &= -x_2.
\end{aligned}$$

We have equilibrium points $(x_1, x_2) = \{(-1,0), (+1,0)\}$ with linearizations

$$Df = \begin{pmatrix} -2x_1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Df(-1,0) = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}, \quad Df(+1,0) = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}.$$

We see that the global stable manifold of the stable equilibrium point $(+1,0)$ corresponds to the half-space $x_1 > -1$. On the interval $x_1 \in [-1,+1]$ the unstable manifold of $(-1,0)$ coincides with the stable manifold of $(+1,0)$.

The Hartman-Grobman Theorem

An equilibrium point of a nonlinear system is called *hyperbolic* if all of the eigenvalues of the linearization there have non-zero real part. The dynamics in the neighborhood of a hyperbolic equilibrium point are guaranteed to be 'simple' in the sense of the following theorem (Perko §2.8):

> (The Hartman-Grobman Theorem) Let E be an open subset of R^n containing the origin, let $f \in C^1(E)$, and let ϕ_t be the flow of the nonlinear system $\dot{x} = f(x)$. Suppose that $f(0) = 0$ and that the matrix $A = Df(0)$ has no eigenvalue with zero real part. Then there exists a homeomorphism H of an open set U containing the origin onto an open set V containing the origin such that for each $x_0 \in U$, there is an open interval $I_0 \subset R$ containing zero such that for all $x_0 \in U$ and $t \in I_0$
>
> $$H \circ \phi_t(x_0) = \exp(At)H(x_0);$$
>
> *i.e.*, H maps trajectories of $\dot{x} = f(x)$ near the origin onto trajectories of $\dot{x} = Ax$ near the origin and preserves the parameterization by time.

Note that the 'flow' $\phi_t(\bullet)$ is defined such that $\phi_t(x(0)) = x(t)$ according to $\dot{x} = f(x)$. In order to get a practical feeling for what the theorem means we consider a simple example (Perko §2.8 Problem 8.1):

$$\begin{aligned} \dot{y}_1 &= -y_1, \\ \dot{y}_2 &= -y_2 + z^2, \\ \dot{z} &= z, \end{aligned}$$

for which the origin is clearly an equilibrium point. The linearization there is

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 2z \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

so the eigenvalues are ± 1 and the equilibrium point at the origin is hyperbolic. We can actually solve the nonlinear system explicitly,

$$\begin{aligned} y_1(t) &= \exp(-t)y_1(0), \\ z(t) &= \exp(t)z(0), \\ y_2(t) &= \exp(-t)y_2(0) + \exp(-t)\int_0^t ds\, \exp(s)\exp(2s)z^2(0) \\ &= \exp(-t)y_2(0) + \frac{1}{3}[\exp(2t) - \exp(-t)]z^2(0). \end{aligned}$$

Using methods discussed in Perko §2.8 (which are straightforward but rather laborious) one can derive

$$H(y_1,y_2,z) = \begin{bmatrix} y_1 \\ y_2 - \frac{1}{3}z^2 \\ z \end{bmatrix}, \quad H^{-1} = \begin{bmatrix} y_1 \\ y_2 + \frac{1}{3}z^2 \\ z \end{bmatrix}.$$

and the theorem reads

$$H \circ \phi_t(x_0) = \exp(At)H(x_0),$$
$$\phi_t(x_0) = H^{-1}\{\exp(At)H(x_0)\},$$

where

$$\exp(At)H(x_0) = \begin{bmatrix} \exp(-t) & 0 & 0 \\ 0 & \exp(-t) & 0 \\ 0 & 0 & \exp(t) \end{bmatrix} \begin{bmatrix} y_1(0) \\ y_2(0) - \frac{1}{3}z^2(0) \\ z(0) \end{bmatrix}$$
$$= \begin{bmatrix} \exp(-t)y_1(0) \\ \exp(-t)y_2(0) - \frac{1}{3}\exp(-t)z^2(0) \\ \exp(t)z(0) \end{bmatrix},$$
$$H^{-1}\{\exp(At)H(x_0)\} = \begin{bmatrix} \exp(-t)y_1(0) \\ \exp(-t)y_2(0) - \frac{1}{3}\exp(-t)z^2(0) + \frac{1}{3}\exp(2t)z^2(0) \\ \exp(t)z(0) \end{bmatrix},$$

which is indeed equal to the nonlinear flow that we obtained by explicit solution of the nonlinear equations.

As a consequence of the Hartman-Grobman theorem we can say that the flow in the neighborhood of a hyperbolic equilibrium point is topologically conjugate to that of its linearization. In the neighborhood of a non-hyperbolic equilibrium point (whose linearization has some eigenvalues with zero real part) it is *not* generally possible to find a homeomorphism that transforms the nonlinear flow to that of the linearization; we'll pick up this point again below.

Center Manifold Theory

It follows from the above that if we have an equilibrium point of a nonlinear system whose linearization has eigenvalues with all negative real parts, the equilibrium point is stable. If the linearization has any eigenvalues with positive real part then the equilibrium point is not stable. What if the linearization has no positive eigenvalues but some of its eigenvalues have zero real part? To answer this question we need the Local Center Manifold Theorem (Wiggins §2.1A). Suppose we have a nonlinear dynamical system with an equilibrium point at the origin and with the linearization there having no eigenvalues with positive real part. By a suitable linear transformation we can always rewrite the dynamics in terms of stable coordinates $x \in R^c$ and $y \in R^s$ (where c is the dimension of E^c and s is the dimension of E^s at the origin) in the form

$$\dot{x} = Ax + f(x,y),$$
$$\dot{y} = By + g(x,y),$$

where A is a matrix whose eigenvalues all have zero real part, B is a matrix whose eigenvalues all have negative real part, and

$$f(0,0) = Df(0,0) = 0,$$
$$g(0,0) = Dg(0,0) = 0.$$

The local center manifold of the origin $W^c(0)$ can then be expressed as an invariant manifold

$$W^c(0) = \left\{(x,y) \in R^c \times R^s \mid y = h(x),\ |x| < \delta,\ h(0) = 0,\ Dh(0) = 0\right\},$$

for δ sufficiently small, and the dynamics on the center manifold is, for u sufficiently small, given by

$$\dot{u} = Au + f(u, h(u)), \quad u \in R^c.$$

The non-hyperbolic equilibrium point in question is stable if and only if the dynamics on the center manifold is stable. The form of the all-important function $h(\bullet)$, which specifies $W^c(0)$ as a graph above E^c, can be derived from the equation

$$Dh(x)[Ax + f(x,h(x))] - Bh(x) - g(x,h(x)) = 0.$$

Generally one does this by assuming a low-order polynomial form for $h(x)$ and solving the above equation for the coefficients of powers of x. Note that if we know the local stable manifold $W^c(0)$ exactly, we can extend it to the global stable manifold S by mapping backwards via the flow ϕ_{-t}.

Once again we illustrate the use of this theorem with an example (Wiggins Example 2.1.1):

$$\dot{x} = x^2 y - x^5,$$
$$\dot{y} = -y + x^2.$$

It is clear that the origin is an equilibrium point, and the linearization there is

$$D = \begin{pmatrix} 2xy - 5x^4 & x^2 \\ 2x & -1 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix},$$

which obviously has one zero eigenvalue and one negative eigenvalue; the center subspace is the x-axis and the stable subspace is the y-axis. We can thus write

$$\dot{x} = Ax + f(x,y), \quad \dot{y} = By + g(x,y),$$

with

$$A = 0, \quad f(x,y) = x^2 y - x^5,$$
$$B = -1, \quad g(x,y) = x^2.$$

The equation defining the local center manifold is

$$Dh(x)[Ax + f(x,h(x))] - Bh(x) - g(x,h(x)) = 0,$$
$$Dh(x)[x^2 h(x) - x^5] + h(x) - x^2 = 0,$$

and if we insert a third-order polynomial form for $h(x)$ (note that the coefficients of x^0 and x^1 must vanish because of conditions above),

$$h(x) = ax^2 + bx^3 + O(x^4),$$

we have

$$\{2ax + 3bx^2\}\{ax^4 + bx^5 - x^5\} + ax^2 + bx^3 - x^2 = 0,$$

and we can equate coefficients of the lowest surviving orders in x,

$$ax^2 - x^2 = 0, \quad a = 1,$$
$$bx^3 = 0, \quad b = 0,$$

which implies

$$h(x) = x^2 + O(x^4).$$

According to the theorem, the dynamics on the local center manifold is given by

$$\dot{u} = Au + f(u, h(u)) \rightarrow x^4 + O(x^5),$$

which is not stable ($u(t)$ does not go to zero as $t \rightarrow +\infty$ for all $u(0)$ in any small neighborhood of the origin, because of the points $u(0) > 0$). A diagram of the flow in the neighborhood of the origin, indicating E^c, E^s, $W^c(0)$ and $W^s(0)$ illustrates the 'intrinsically nonlinear' flow.

It should furthermore be evident from this example that in general it is not possible to find a homeomorphism that transforms the nonlinear flow in the neighborhood of a non-hyperbolic equlibrium point to that of its linearization. In general the best that can be done is to transform the dynamics on the stable and unstable manifolds to linear form, and to reduce the dynamics on the center manifold to a 'simplest possible' form called a normal form (Perko §2.13; Wiggins 2.2). Surprisingly, the structure of the normal form is determined entirely by the *linear* part of the dynamics. Normal forms can be quite useful in the classification of local bifurcations at nonhyperbolic equilibrium points. Finding a homeomorphism that transforms the nonlinear flow in the neighborhood of an equilibrium point to linear dynamics in the stable and unstable coordinates and a normal form in the center coordinates is thus conceptually similar to finding an invertible linear transformation that takes a linear dynamical system to its Jordan form.

Structural stability and local bifurcations

One often has cause to consider dynamical systems with parameters,

$$\dot{x} = f(x, \mu), \quad x \in R^n, \quad \mu \in R^p,$$

where the x are the dynamical variables and the μ are parameters that are considered to have fixed values for the purposes of integrating the dynamics, but where we are interested in studying how the nature of the dynamics depend μ. It can be particularly interesting to ask whether there are any critical values μ_0 such that small perturbation $\mu \rightarrow \mu_0 + \varepsilon$ leads to a qualitative change in the nature of the trajectories of $x(t)$. One can get quite technical about what it really means to have a 'qualitative change' but let us simply look at some canonical examples.

(Wiggins §3.1A, Example 3.1.1 and section *iii*) With $n = p = 1$ we can consider

$$\dot{x} = f(x, \mu) = \mu - x^2.$$

The equilibrium points are the solutions of

$$\mu = x^2,$$

hence there are no equilibrium points for $\mu < 0$, a unique equilibrium point at the origin for $\mu = 0$ and two equilibrium points at $x = \pm\sqrt{\mu}$ for $\mu > 0$. It is easy to see the linearizations are

$$Df = -2x,$$

from which we see that when $\mu > 0$ there is one stable ($x > 0$) and one unstable ($x < 0$) equilibrium point, while for $\mu = 0$ the equilibrium at the origin is non-hyperbolic but clearly unstable. The point $(x, \mu) = (0, 0)$ is a *bifurcation* point for this system, as varying μ smoothly from below zero to above zero leads to a qualitative change in the phase portrait (the appearance of new equilibrium points); an intuitive 'bifurcation diagram' can be drawn with μ on the horizontal axis and the equilibrium points in x indicated on the vertical axis as a function of μ. This type of bifurcation, called a saddle-node bifurcation, occurs at a point (x, μ) of a dynamical system where x is an equilibrium point where the linearization has a single zero eigenvalue and the normal form on the center manifold is $\mu \pm x^2$.

(Wiggins §3.1A, Example 3.1.2 and section *iv*) Again with $n = p = 1$ we consider

$$\dot{x} = f(x, \mu) = \mu x - x^2,$$

which has equilibrium points

$$x = 0, \quad x = \mu.$$

The linearizations are

$$Df = \mu - 2x,$$

so both equilibrium points change stability as μ passes through zero. For $\mu < 0$ the origin is stable and the $x = \mu$ point is unstable; for $\mu > 0$ the origin is unstable and the $x = \mu$ point is stable. At $\mu = 0$ the origin is unstable. This type of bifurcation, with normal form $\mu x \mp x^2$, is called a transcritical bifurcation.

(Wiggins §3.1A, Example 3.1.3 and section *v*) Again with $n = p = 1$ we consider

$$\dot{x} = f(x, \mu) = \mu x - x^3,$$

which has equilibrium points

$$x = 0, \quad x = \pm\sqrt{\mu}.$$

Hence there is a unique equilibrium point at the origin for $\mu \le 0$ and three equilibrium points for $\mu > 0$. The linearizations are

$$Df = \mu - 3x^2,$$

so for $\mu < 0$ the origin is stable. For $\mu = 0$ the origin is still stable, but for $\mu > 0$ the origin is unstable while the points at $\pm\sqrt{\mu}$ are stable. This type of bifurcation, with normal form $\mu x \mp x^3$, is called a pitchfork bifurcation.

Bifurcations are responsible for a wide variety of switching and hysteresis behaviors in nonlinear dynamical systems, with important examples in domains ranging from physical precision measurement to biology. A few examples:

- D. Battogtokh and J. Tyson, "Bifurcation analysis of a model of the budding yeast cell cycle," Chaos **14**, 653 (2004).

- R. Vijay, M. H. Devoret and I. Siddiqi, "Invited Review Article: The Josephson Bifurcation Amplifier," Rev. Sci. Instrum. **80**, 111101 (2009).
- J. Lu, H. W. Engl and P. Schuster, "Inverse bifurcation analysis: application to simple gene systems," Algorithms for Molecular Biology **1**:11 (2006).
- C. H. Tseng, D. Enzer, G. Gabrielse, and F. L. Walls, "1-bit memory using one electron: Parametric oscilliations in a Penning trap," Phys. Rev. A **59**, 2094 (1999).

Periodic orbits and limit cycles

Consider the system (Perko §3.2, Example 1)

$$\dot{x} = -y + x(1 - x^2 - y^2),$$
$$\dot{y} = x + y(1 - x^2 - y^2),$$

which is easier to analyze in polar coordinates:

$$\dot{r} = r(1 - r^2),$$
$$\dot{\theta} = 1.$$

This system has an unstable equilibrium at the origin and a *periodic orbit* at $r = 1$. Inspection of the vector field makes it clear that the periodic orbit is actually a *stable limit cycle* for the flow. Such a limit cycle is an example of a non-equilibrium *attractor*; limit cycles can appear and disappear in (Hopf) bifurcations, just like equilibrium points.

Lyapunov/Liapunov stability

For completeness, we conclude with brief mention of a very general technique for proving the stability of an equilibrium point of a nonlinear dynamical system, which can potentially be used in situations where center manifold methods are too cumbersome. The method relies on the following theorem (Perko §2.9):

> Let E be an open subset of R^n containing x_0. Suppose that $f \in C^1(E)$ and that $f(x_0) = 0$. Suppose further that there exists a real valued function $V \in C^1(E)$ satisfying $V(x_0) = 0$ and $V(x) > 0$ if $x \neq x_0$. Then (a) if $\dot{V}(x) \leq 0$ for all $x \in E$, x_0 is stable; (b) if $\dot{V}(x) < 0$ for all $x \in E \backslash \{x_0\}$, x_0 is asymptotically stable; (c) if $\dot{V}(x) > 0$ for all $x \in E \backslash \{x_0\}$, x_0 is unstable.

Recall that an equilibrium point x_0 is stable if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in N_\delta(x_0)$ and $t \geq 0$ we have $\phi_t(x) \in N_\varepsilon(x_0)$, and that x_0 is asymptotically stable if it is stable and there exists a $\delta > 0$ such that for all $x \in N_\delta(x_0)$ we have $\lim_{t\to\infty} \phi_t(x) = x_0$.

The difficult part about applying the Lyapunov (sometimes written Liapunov) stability theorem lies in finding an appropiate Lyapunov function $V(x)$. Unfortunately no generally applicable method exists for deriving one from the form of the dynamics, but sometimes one can intuit a workable form using the idea that $V(x)$ is something like an energy. Once again we provide an illustrative example (Perko §2.9, Example 3):

$$\begin{aligned}\dot{x}_1 &= -2x_2 + x_2x_3 - x_1^3,\\ \dot{x}_2 &= x_1 - x_1x_3 - x_2^3,\\ \dot{x}_3 &= x_1x_2 - x_3^3.\end{aligned}$$

Here the origin is clearly an equilibrium point, but the linearization there is

$$Df = \begin{pmatrix} -3x_1^2 & -2 + x_3 & x_2 \\ 1 - x_3 & 3x_2^2 & -x_1 \\ x_2 & x_1 & -3x_3^2 \end{pmatrix} \to \begin{pmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which has eigenvalues 0 and $\pm\sqrt{2}\,i$. The function

$$V(x) = x_1^2 + 2x_2^2 + x_3^2$$

vanishes at the origin but is positive everywhere else, and thus provides a candidate Lypapunov function. We can compute

$$\begin{aligned}\dot{V} &= 2x_1\dot{x}_1 + 4x_2\dot{x}_2 + 2x_3\dot{x}_3\\ &= 2x_1(-2x_2 + x_2x_3 - x_1^3) + 4x_2(x_1 - x_1x_3 - x_2^3) + 2x_3(x_1x_2 - x_3^3)\\ &= -2x_1^4 - 4x_1x_2 + 2x_1x_2x_3 + 4x_1x_2 - 4x_1x_2x_3 - 4x_2^4 + 2x_1x_2x_3 - 2x_3^4\\ &= -2x_1^4 - 4x_2^4 - 2x_3^4,\end{aligned}$$

which is negative everywhere except the origin and vanishes at the origin. Hence by the theorem, the origin is asymptotically stable.

Lyapunov functions can sometimes be used in controller design. In a nonlinear state feedback setting, for example, if there is a candidate Lyapunov function one can try to apply feedback that maintains $\dot{V} < 0$ at all times in order to asymptotically stabilize a point x_0.