APPPHYS 217 Tuesday 6 April 2010

Stability and input-output performance: second-order systems

Here we present a detailed example to draw connections between today's topics and our prior review of linear algebra and ODE's. Of course, this "example" of second-order systems is actually an important topic in and of itself, as one often gains intuition about more complex systems by noting similarities to and differences from this canonical class of models.

The equation of motion we want to consider is

$$\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2 q = u,$$

where u is an input signal and ω_0, ζ are parameters of the model. This equation of motion could represent, for example, a mass-spring-damper system for which $F = ma$ takes the form

$$m\ddot{x} = -kx - b\dot{x} + F_{ext}, \quad \ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \frac{1}{m}F_{ext},$$

$$\omega_0 \mapsto \sqrt{\frac{k}{m}}, \quad \zeta \mapsto \sqrt{\frac{b^2}{4km}}, \quad u \mapsto \frac{1}{m}F_{ext},$$

where m is the mass, k is the spring constant and b is the damping coefficient. The same form of equation of motion arises for a series LCR circuit (inductor-capacitor oscillator with resistance),

$$V_C = \frac{Q}{C}, \quad V_I = L\frac{dI}{dt} \to L\ddot{Q}, \quad V_R = IR \to \dot{Q}R,$$

$$V_{ext} = L\ddot{Q} + \dot{Q}R + \frac{1}{C}Q, \quad \ddot{Q} + \frac{R}{L}\dot{Q} + \frac{1}{LC}Q - \frac{1}{L}V_{ext} = 0,$$

$$\omega_0 \mapsto \sqrt{\frac{1}{LC}}, \quad \zeta \mapsto \sqrt{\frac{R^2C}{4L}}, \quad u \mapsto \frac{1}{L}V_{ext},$$

where L is the inductance, C the capacitance and R the resistance. Here Q represents the charge on the capacitor.

General solution of the initial value problem for second-order systems

We have already seen how to put this equation of motion into state-space form:

$$\ddot{q} = -2\zeta\omega_0\dot{q} - \omega_0^2 q + u, \quad x_1 \equiv q, \quad x_2 \equiv \dot{q},$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix}.$$

For the moment, let's set u to zero and make sure we understand the behavior of the undriven system. In this case we are looking at $\dot{x} = Ax$ with A the square matrix in the equation above, and we know that we can solve all initial value problems by matrix exponentiation. This in turn tells us that we should look at the eigenvalues of A and check whether it is diagonalizable. Once we find the matrix exponential we can use it to find analytic expressions for the impulse, step and frequency response. We'll do the step response here in class today; the impulse and frequency responses are left as exercises.

Computing the eigenvalues of A for arbitrary ω_0 and ζ, we find

$$\begin{aligned} 0 &= \det(A-\lambda I) = -\lambda(-2\zeta\omega_0 - \lambda) + \omega_0^2, \\ &= \lambda^2 + 2\zeta\omega_0\lambda + \omega_0^2, \\ \lambda &= \frac{-2\zeta\omega_0 \pm \sqrt{4\zeta^2\omega_0^2 - 4\omega_0^2}}{2} = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2-1}\,. \end{aligned}$$

Assuming $\omega_0 > 0$ and $\zeta \geq 0$, we can then identify three distinct "cases" for ζ :

$$\begin{aligned} \zeta < 1: &\quad \zeta^2 - 1 < 0 \Rightarrow \lambda_+ = \bar{\lambda}_- \in C, \\ \zeta = 1: &\quad \lambda_+ = \lambda_- = -\omega_0, \\ \zeta > 1: &\quad \zeta^2 - 1 > 0 \Rightarrow 0 > \lambda_+ > \lambda_- \in R. \end{aligned}$$

Traditionally one refers to the $\zeta < 1$ case as being *underdamped*, $\zeta = 1$ as *critically damped*, and $\zeta > 1$ as *overdamped*. Let's try to figure out why.

The easiest case to analyze is $\zeta = 0$ (the *undamped* case). Then

$$A = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix}, \quad \lambda_+ = i\omega_0 \leftrightarrow \frac{1}{\sqrt{1+\frac{1}{\omega_0^2}}}\begin{bmatrix} -\frac{i}{\omega_0} \\ 1 \end{bmatrix}, \quad \lambda_- = -i\omega_0 \leftrightarrow \frac{1}{\sqrt{1+\frac{1}{\omega_0^2}}}\begin{bmatrix} \frac{i}{\omega_0} \\ 1 \end{bmatrix},$$

so we have

$$P = \frac{1}{\sqrt{1+\frac{1}{\omega_0^2}}}\begin{bmatrix} -\frac{i}{\omega_0} & \frac{i}{\omega_0} \\ 1 & 1 \end{bmatrix}, \quad P^{-1} = \sqrt{1+\frac{1}{\omega_0^2}}\begin{bmatrix} \frac{i\omega_0}{2} & \frac{1}{2} \\ -\frac{i\omega_0}{2} & \frac{1}{2} \end{bmatrix},$$

$$\begin{aligned} \exp(At) &= P\begin{bmatrix} e^{i\omega_0 t} & 0 \\ 0 & e^{-i\omega_0 t} \end{bmatrix}P^{-1} = \begin{bmatrix} \frac{1}{2}e^{-it\omega_0} + \frac{1}{2}e^{it\omega_0} & \frac{i}{2\omega_0}e^{-it\omega_0} - \frac{i}{2\omega_0}e^{it\omega_0} \\ -\frac{i\omega_0}{2}e^{-it\omega_0} + \frac{i\omega_0}{2}e^{it\omega_0} & \frac{1}{2}e^{-it\omega_0} + \frac{1}{2}e^{it\omega_0} \end{bmatrix} \\ &= \begin{bmatrix} \cos(\omega_0 t) & \frac{1}{\omega_0}\sin(\omega_0 t) \\ -\omega_0\sin(\omega_0 t) & \cos(\omega_0 t) \end{bmatrix}. \end{aligned}$$

We see from this that for any non-zero initial condition, the integrated trajectory will consist of undamped oscillations.

If $\zeta > 0$ we can still find general expressions for the eigenvectors. In unnormalized form,

$$\lambda_+ = -\omega_0\zeta + \omega_0\sqrt{\zeta^2-1} \leftrightarrow \begin{bmatrix} -\frac{1}{\omega_0}\left(\zeta + \sqrt{\zeta^2-1}\right) \\ 1 \end{bmatrix},$$

$$\lambda_- = -\omega_0\zeta - \omega_0\sqrt{\zeta^2-1} \leftrightarrow \begin{bmatrix} -\frac{1}{\omega_0}\left(\zeta - \sqrt{\zeta^2-1}\right) \\ 1 \end{bmatrix}.$$

These two eigenvectors are independent as long as $\zeta \neq 1$, so let's assume that for now. Then

$$\exp(At) = P\begin{bmatrix} \exp(\lambda_+ t) & 0 \\ 0 & \exp(\lambda_- t) \end{bmatrix} P^{-1}$$

$$= \frac{1}{2\sqrt{\zeta^2-1}}\begin{bmatrix} \zeta E_- + \sqrt{\zeta^2-1}\,E_+ & \frac{1}{\omega_0}E_- \\ -\omega_0 E_- & -\zeta E_- + \sqrt{\zeta^2-1}\,E_+ \end{bmatrix},$$

where

$$E_- \equiv \exp(\lambda_+ t) - \exp(\lambda_- t),$$
$$E_+ \equiv \exp(\lambda_+ t) + \exp(\lambda_- t).$$

Note that if $0 < \zeta < 1,$ then

$$\lambda_+ = -\omega_0\zeta + \omega_0\sqrt{\zeta^2-1} = -\gamma + i\nu, \quad \gamma \equiv \omega_0\zeta, \quad \nu \equiv \omega_0\sqrt{1-\zeta^2}, \quad \lambda_- = -\gamma - i\nu,$$

$$E_- = e^{-\gamma t}(e^{i\nu t} - e^{-i\nu t}) = 2ie^{-\gamma t}\sin(\nu t), \quad E_+ = 2e^{-\gamma t}\cos(\nu t),$$

so

$$\exp(At) \to \frac{e^{-\gamma t}}{\sqrt{1-\zeta^2}}\begin{bmatrix} \zeta\sin(\nu t) + \sqrt{1-\zeta^2}\cos(\nu t) & \frac{1}{\omega_0}\sin(\nu t) \\ -\omega_0\sin(\nu t) & -\zeta\sin(\nu t) + \sqrt{1-\zeta^2}\cos(\nu t) \end{bmatrix}.$$

This shows us that for arbitrary initial conditions, the integrated trajectories look like exponentially damped oscillations. Likewise if $\zeta > 1$, then

$$\lambda_+ = -\omega_0\zeta + \omega_0\sqrt{\zeta^2-1} = -\gamma + \delta, \quad \gamma \equiv \omega_0\zeta, \quad \delta \equiv \omega_0\sqrt{\zeta^2-1}, \quad \lambda_- = -\gamma - \delta,$$

$$E_- = e^{-\gamma t}(e^{\delta t} - e^{-\delta t}) = 2e^{-\gamma t}\sinh(\delta t), \quad E_+ = 2e^{-\gamma t}\cosh(\delta t),$$

so

$$\exp(At) \to \frac{e^{-\gamma t}}{\sqrt{\zeta^2-1}}\begin{bmatrix} \zeta\sinh(\delta t) + \sqrt{\zeta^2-1}\cosh(\delta t) & \frac{1}{\omega_0}\sinh(\delta t) \\ -\omega_0\sinh(\delta t) & -\zeta\sinh(\delta t) + \sqrt{\zeta^2-1}\cosh(\delta t) \end{bmatrix}.$$

Taking into account the fact that $\delta < \zeta,$ we see that the integrated trajectories are exponentially damped without any oscillating factors.

Finally, we consider the case $\zeta = 1.$ Now

$$A = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\omega_0 \end{bmatrix}, \quad \lambda_+ = \lambda_- = -\omega_0.$$

Looking at the eigenvalue equation,

$$0 = (A - \lambda I)x,$$
$$= \begin{bmatrix} \omega_0 & 1 \\ -\omega_0^2 & -\omega_0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

we see that the only solutions are of the form $x_2 = -\omega_0 x_1$ and are thus all linearly dependent (correspond to a single eigenvector). Hence we are in the case where A is not diagonalizable. As mentioned in previous lecture notes, however, we can find a decomposition into Jordan form:

$$A = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\omega_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\omega_0 & 1 \end{bmatrix}\begin{bmatrix} -\omega_0 & 1 \\ 0 & -\omega_0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \omega_0 & 1 \end{bmatrix} \equiv TJT^{-1},$$

where

$$J = \begin{bmatrix} -\omega_0 & 1 \\ 0 & -\omega_0 \end{bmatrix} = \begin{bmatrix} -\omega_0 & 0 \\ 0 & -\omega_0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \equiv S + N,$$

and clearly $[S,N] = 0$ since S is proportional to the identity. Then

$$\begin{aligned} \exp(At) &= T\exp(Jt)T^{-1} = T\exp(St)\exp(Nt)T^{-1} \\ &= \begin{bmatrix} 1 & 0 \\ -\omega_0 & 1 \end{bmatrix}\begin{bmatrix} e^{-\omega_0 t} & 0 \\ 0 & e^{-\omega_0 t} \end{bmatrix}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \omega_0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{-\omega_0 t} + t\omega_0 e^{-\omega_0 t} & te^{-\omega_0 t} \\ -\omega_0 e^{-\omega_0 t} + \omega_0(e^{-\omega_0 t} - t\omega_0 e^{-\omega_0 t}) & e^{-\omega_0 t} - t\omega_0 e^{-\omega_0 t} \end{bmatrix}. \end{aligned}$$

Hence we see that the integrated trajectories will contain exponentially damped terms as well as terms that intially $(t \ll 1/\omega_0)$ grow linearly with time but then are dominanted by the $e^{-\omega_0 t}$ factors as well.

Step response of second-order systems

Our solutions to the initial value problem can directly be used to solve for the step response, once we introduce a small trick.

Remember that the step response is defined as the system output corresponding to an input signal of the form

$$u(t) = 0, \quad t < 0,$$

$$u(t) = u_0, \quad t \geq 0,$$

assuming the state variables are at equilibrium prior to $t = 0$. Looking at the second-order equation of motion in its original form,

$$\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2 q = u_0,$$

we see that $(q,\dot{q}) = 0$ is the desired equilibrium for $t < 0$, and thus that the step response of the state space variables simply corresponds to an initial value problem,

$$\begin{aligned} \ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2 q &= u_0, \\ q(0) = \dot{q}(0) &= 0. \end{aligned}$$

If we can solve for $q(t)$ and $\dot{q}(t)$, we can use them to compute any desired output signal

$$y(t) \equiv C\begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix}.$$

At this point we note that the initial value problem we want to consider can be transformed by a simple change of variables,

$$q \mapsto q' = q - \frac{u_0}{\omega_0^2}, \quad \dot{q}' = \dot{q}, \quad \ddot{q}' = \ddot{q},$$

into

$$\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2 q - u_0 = 0,$$

$$\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2\left(q - \frac{u_0}{\omega_0^2}\right) = 0,$$

$$\ddot{q}' + 2\zeta\omega_0\dot{q}' + \omega_0^2 q' = 0,$$

with initial condition

$$q'(0) = -\frac{u_0}{\omega_0^2}, \quad \dot{q}'(0) = 0.$$

Hence, we can simply use the solutions we derived above for $u = 0$ with an initial condition corresponding to the desired step size, and then transform the integrated solution according to

$$q(t) = q'(t) + \frac{u_0}{\omega_0^2}, \quad \dot{q}'(t) = \dot{q}'(t).$$

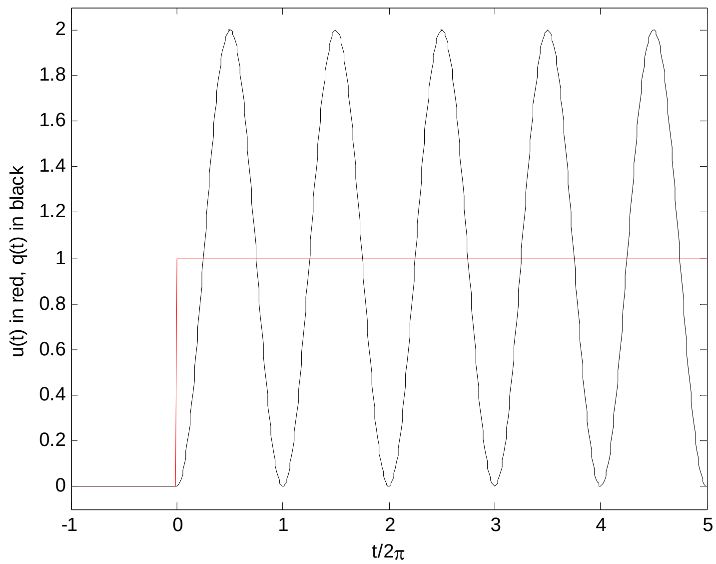
In the undamped case we have

$$\begin{bmatrix} q'(t) \\ \dot{q}'(t) \end{bmatrix} = \begin{bmatrix} \cos(\omega_0 t) & \frac{1}{\omega_0}\sin(\omega_0 t) \\ -\frac{1}{\omega_0}\sin(\omega_0 t) & \cos(\omega_0 t) \end{bmatrix}\begin{bmatrix} -\frac{u_0}{\omega_0^2} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{u_0}{\omega_0^2}\cos(\omega_0 t) \\ \frac{u_0}{\omega_0^3}\sin(\omega_0 t) \end{bmatrix},$$

so

$$\begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix} = \frac{u_0}{\omega_0^2}\begin{bmatrix} 1 - \cos(\omega_0 t) \\ \frac{1}{\omega_0}\sin(\omega_0 t) \end{bmatrix}.$$

To help visualize this we can make a plot for $\omega_0 = u_0 = 1$:



Note the serious overshoot and lack of any settling whatsoever!

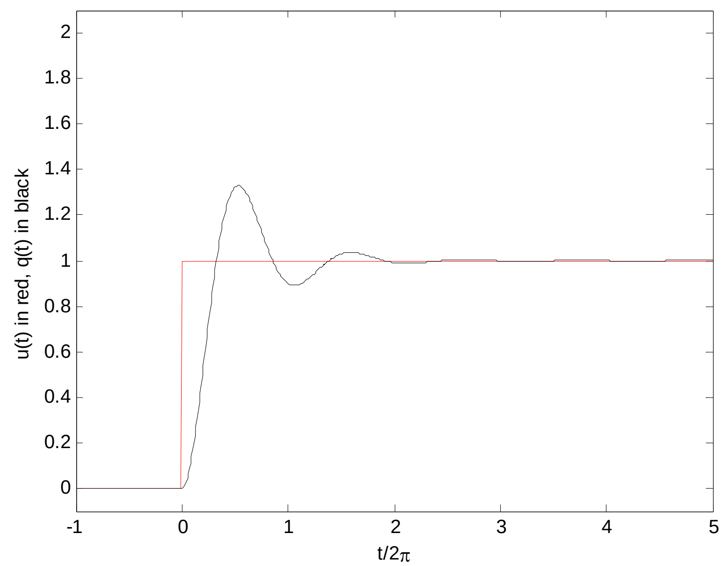
In the underdamped case we have

$$\begin{bmatrix} q'(t) \\ \dot{q}'(t) \end{bmatrix} = -\frac{u_0}{\omega_0^2}\frac{e^{-\gamma t}}{\sqrt{1-\zeta^2}}\begin{bmatrix} \zeta\sin(\nu t) + \sqrt{1-\zeta^2}\,\cos(\nu t) \\ -\omega_0\sin(\nu t) \end{bmatrix},$$

so

$$\begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix} = \frac{u_0}{\omega_0^2}\begin{bmatrix} 1 - \frac{e^{-\zeta\omega_0 t}}{\sqrt{1-\zeta^2}}\left(\zeta\sin\left(\omega_0\sqrt{1-\zeta^2}\,t\right) + \sqrt{1-\zeta^2}\,\cos\left(\omega_0\sqrt{1-\zeta^2}\,t\right)\right) \\ \frac{\omega_0 e^{-\zeta\omega_0 t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_0\sqrt{1-\zeta^2}\,t\right) \end{bmatrix}.$$

For an example to plot we can choose $\omega_0 = u_0 = 1$ and take $\zeta = 1/3$:



Here the overshoot is greatly reduced and the system settles within a time $\sim 4\pi/\omega_0$.

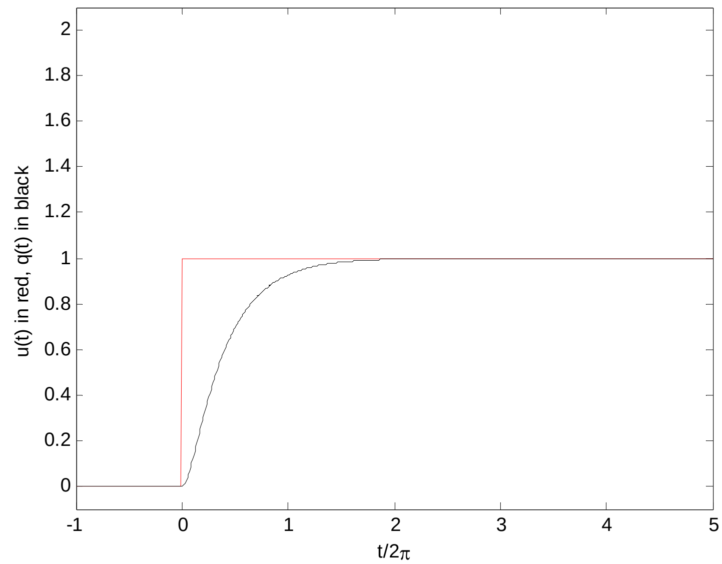
Next the overdamped case,

$$\begin{bmatrix} q'(t) \\ \dot{q}'(t) \end{bmatrix} = -\frac{u_0}{\omega_0^2} \frac{e^{-\gamma t}}{\sqrt{\zeta^2 - 1}} \begin{bmatrix} \zeta \sinh(\delta t) + \sqrt{\zeta^2 - 1}\cosh(\delta t) \\ -\omega_0 \sinh(\delta t) \end{bmatrix}$$

so

$$\begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix} = \frac{u_0}{\omega_0^2} \begin{bmatrix} 1 - \frac{e^{-\zeta\omega_0 t}}{\sqrt{\zeta^2-1}} \zeta \sinh\left(\omega_0 \sqrt{\zeta^2 - 1}\, t\right) + \sqrt{\zeta^2 - 1}\cosh\left(\omega_0 \sqrt{\zeta^2 - 1}\, t\right) \\ \omega_0 \frac{e^{-\zeta\omega_0 t}}{\sqrt{\zeta^2-1}} \sinh\left(\omega_0 \sqrt{\zeta^2 - 1}\, t\right) \end{bmatrix}.$$

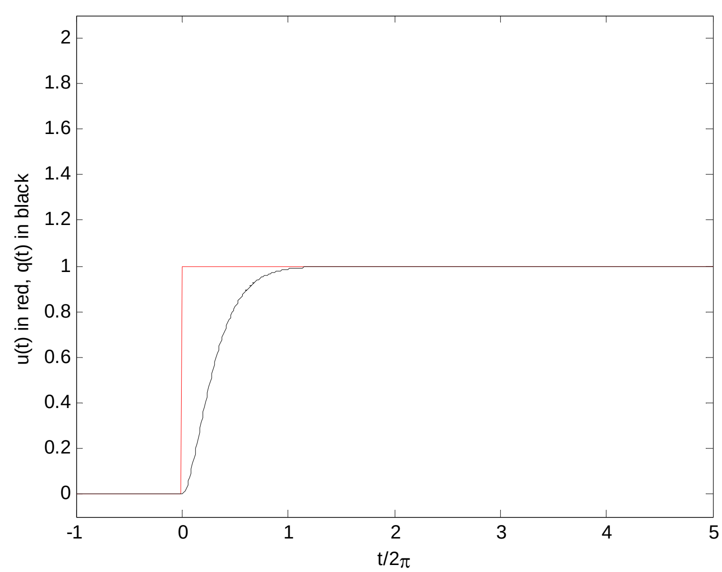
Again taking $\omega_0 = u_0 = 1$ and now $\zeta = 4/3$, we can plot:



Here there is no overshoot at all and settling takes $\sim 3\pi/\omega_0$. Finally we consider the critically damped case,

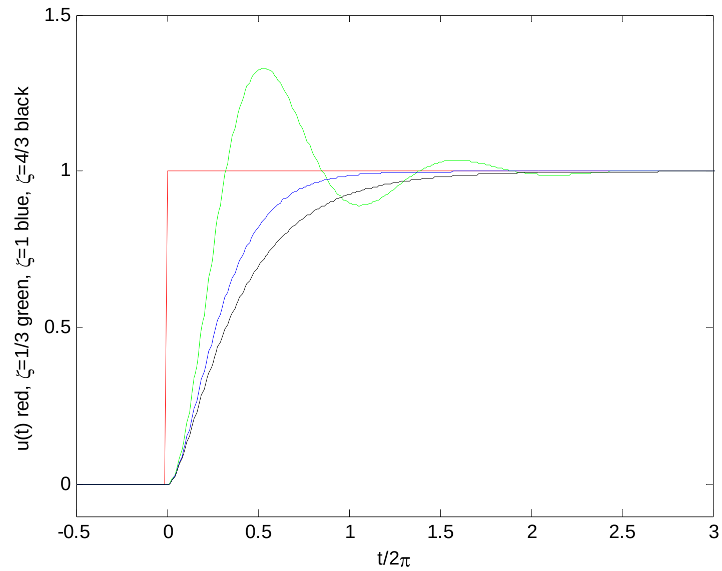
$$\begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix} = \frac{u_0}{\omega_0^2} \begin{bmatrix} 1 - e^{-\omega_0 t} - t\omega_0 e^{-\omega_0 t} \\ \omega_0 e^{-\omega_0 t} - \omega_0 (e^{-\omega_0 t} - t\omega_0 e^{-\omega_0 t}) \end{bmatrix},$$

which for $\omega_0 = u_0 = 1$ looks like



Here again there is no overshoot and settling occurs within $\sim 2\pi/\omega_0$.

Putting our $\zeta > 0$ cases together into one plot:



Exercise 1: How does the rise time vary with ζ? Consider all values of $\zeta \geq 0$.
Exercise 2: How does the settling time vary with ζ? Consider all values of $\zeta \geq 0$.
Exercise 3: Consider the following second-order control system:

$$\frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_0^2 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 1 & \frac{1}{\omega_0} \end{bmatrix}\begin{bmatrix} q \\ \dot{q} \end{bmatrix},$$

where $u(t)$ is a scalar input signal and $y(t)$ is a scalar output signal. For a feedback law of the form

$$u(t) = -Ky(t),$$

explain the dependence of the closed-loop step response on the gain K for $K > 0$.

Transient versus steady-state response
Looking at the general solution of a driven linear system that we discussed last week,

$$\vec{x}(t) = \exp(At)\vec{x}(0) + \int_0^t ds \exp(A(t-s))\vec{b}(s),$$

we can now garner some insight as to transient versus steady-state response. For our second-order systems with $\zeta > 0$ (*i.e.*, any case but the completely undamped case), we know that $\exp(A\tau)$ is exponentially damping on long timescales. Hence, for $t \gg 0$ the initial conditions will be "forgotten" (will no longer make a significant contribution to $\vec{x}(t)$). The driving input signal $\vec{b}(s)$ at early times such that $(t-s) \gg 0$ will also be forgotten. Hence, if $\vec{b}(t)$ is a repetitive signal we can expect that $\vec{x}(t)$ settles into a steady-state regime in which $\vec{x}(0)$ is forgotten and it doesn't matter exactly how $\vec{b}$ was "switched on."