APPPHYS 217 Tuesday 20 April 2010

A&M 7.4 - Bicycle Dynamics

The linearized model for a bicycle is given in equation (3.5), which has the form

$$J\frac{d^2\varphi}{dt^2} - \frac{Dv_0}{b}\frac{d\delta}{dt} = mgh\varphi + \frac{mv_0^2h}{b}\delta,$$

where φ is the tilt of the bicycle and δ is the steering angle. Give conditions under which the system is observable and explain any special situations where it loses observability.

We first transform the model to state-space form, by choosing

$$x_1 = \varphi, \quad x_2 = \dot{\varphi}, \quad u_1 = \delta, \quad u_2 = \dot{\delta},$$

although we should be careful to keep in mind that u_1 and u_2 are not independent signals. Then the equation (3.5) gives us

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{mgh}{J} & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{mv_0^2h}{bJ} & \frac{Dv_0}{bJ} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Note that the eigenvalues and eigenvectors of A are (using $J \approx mh^2$)

$$\begin{bmatrix} \sqrt{\frac{h}{g}} \\ 1 \end{bmatrix} \leftrightarrow \sqrt{\frac{g}{h}}, \quad \begin{bmatrix} -\sqrt{\frac{h}{g}} \\ 1 \end{bmatrix} \leftrightarrow -\sqrt{\frac{g}{h}},$$

which are just the (inverted) pendulum modes of the bicycle. If we parametrize

$$C = \begin{bmatrix} C_1 & C_2 \end{bmatrix},$$

then

$$CA = \begin{bmatrix} C_1 & C_2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ \frac{mgh}{J} & 0 \end{bmatrix} = \begin{bmatrix} \frac{mgh}{J}C_2 & C_1 \end{bmatrix},$$

the observability matrix is

$$W_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \\ \frac{mgh}{J}C_2 & C_1 \end{bmatrix}.$$

The determinant vanishes if

$$C_1^2 = \frac{mgh}{J}C_2^2,$$

that is if

$$y = c\left[\sqrt{\frac{mgh}{J}}\,\varphi + \dot{\varphi}\right].$$

How shall we interpret this? We note that with this output signal, y would vanish *at all times* for a trajectory

$$\varphi(t) = \varphi_0 \exp\left(-\sqrt{\frac{mgh}{J}}\, t\right),$$

which is in fact a valid solution for the equation of motion with $\delta(t) = 0$, since

$$J\frac{d^2}{dt^2}\varphi(t) = mgh\varphi(t).$$

As noted in the text, $J \approx mh^2$ and thus $\sqrt{mgh/J} \approx \sqrt{g/h}$, which approximates the natural frequency of the (inverted) pendulum of the bicycle frame. Hence the "invisible" $\varphi(t)$ trajectory we noted above would in fact correspond to an approach to the saddle equilibrium point along its stable manifold; we cannot determine where we are along the stable manifold from an output signal y of the given form.

Incidentally we note that this system likewise has an input zero. If we set

$$\delta(t) = \delta_0 e^{st},$$

we have

$$Bu = \begin{bmatrix} 0 & 0 \\ \frac{mv_0^2h}{bJ} & \frac{Dv_0}{bJ} \end{bmatrix}\begin{bmatrix} \delta_0 e^{st} \\ s\delta_0 e^{st} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{mv_0^2h}{bJ} + s\frac{Dv_0}{bJ} \end{bmatrix}\delta_0 e^{st},$$

which vanishes if

$$s = -\frac{bJ}{Dv_0}\frac{mv_0^2h}{bJ} = -\frac{mv_0h}{D}.$$

Hence a steering policy of

$$\delta(t) = \delta_0 \exp(-mv_0ht/D)$$

will have no effect whatsoever on the dynamical variables $\{\varphi, \dot{\varphi}\}$.

In fact we can derive the Laplace-domain transfer function from δ to φ,

$$\tilde{\varphi}(s) = G_{\varphi\delta}(s)\,\tilde{\delta}(s),$$

via

$$\begin{aligned} G_{\varphi\delta}(s) &= C(sI - A)^{-1}B\begin{bmatrix} 1 \\ s \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}\left(\begin{bmatrix} s & -1 \\ -\frac{mgh}{J} & s \end{bmatrix}\right)^{-1}\begin{bmatrix} 0 & 0 \\ \frac{mv_0^2h}{bJ} & \frac{Dv_0}{bJ} \end{bmatrix}\begin{bmatrix} 1 \\ s \end{bmatrix} \\ &= \frac{1}{Js^2 - ghm}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} Js & J \\ ghm & Js \end{bmatrix}\begin{bmatrix} 0 & 0 \\ \frac{mv_0^2h}{bJ} & \frac{Dv_0}{bJ} \end{bmatrix}\begin{bmatrix} 1 \\ s \end{bmatrix} \\ &= \frac{1}{Js^2 - ghm}\begin{bmatrix} \frac{1}{b}hmv_0^2 & \frac{1}{b}Dv_0 \end{bmatrix}\begin{bmatrix} 1 \\ s \end{bmatrix} \\ &= \frac{hmv_0^2 + Dv_0s}{b(Js^2 - ghm)}. \end{aligned}$$

We see that it has poles at

$$s_\pm = \pm\sqrt{\frac{ghm}{J}} \approx \pm\sqrt{\frac{g}{h}},$$

as well as a zero at

$$s_0 = -\frac{hmv_0}{D}.$$

We have already seen what this zero means; the poles coincide with the eigenvalues of the A matrix. We'll talk a lot more about transfer functions next week as we start to move into frequency-domain control theory.

A&M 7.10 - Observer design for motor drive

Consider the normalized model of the motor drive in Exercise 2.10 where the open loop system has the eigenvalues $0, 0, -0.05 \pm i$. A state feedback that gave a closed loop system with eigenvalues in $-2, -1$ and $-1 \pm i$ was designed in Exercise 6.11. Design an observer for the system that has eigenvalues $-4, -2$ and $-2 \pm 2i$. Combine the observer with the state feedback from Exercise 6.11 to obtain an output feedback and simulate the complete system.

We have the motor drive model (2.39)

$$J_1 \frac{d^2\varphi_1}{dt^2} + c\left(\frac{d\varphi_1}{dt} - \frac{d\varphi_2}{dt}\right) + k(\varphi_1 - \varphi_2) = k_I I,$$

$$J_2 \frac{d^2\varphi_2}{dt^2} + c\left(\frac{d\varphi_2}{dt} - \frac{d\varphi_1}{dt}\right) + k(\varphi_2 - \varphi_1) = T_d.$$

Introducing the normalized state variables

$$x_1 = \varphi_1, \quad x_2 = \varphi_2, \quad x_3 = \dot{\varphi}_1/\omega_0, \quad x_4 = \dot{\varphi}_2/\omega_0,$$

$$\omega_0 \equiv \sqrt{k(J_1 + J_2)/(J_1 J_2)},$$

we have

$$J_1\omega_0 \frac{d}{dt} x_3 = k_I I - c\left(\frac{d\varphi_1}{dt} - \frac{d\varphi_2}{dt}\right) - k(\varphi_1 - \varphi_2),$$

$$J_2\omega_0 \frac{d}{dt} x_4 = T_d - c\left(\frac{d\varphi_2}{dt} - \frac{d\varphi_1}{dt}\right) - k(\varphi_2 - \varphi_1),$$

so

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \omega_0 & 0 \\ 0 & 0 & 0 & \omega_0 \\ -k/(J_1\omega_0) & k/(J_1\omega_0) & -c/J_1 & c/J_1 \\ k/(J_2\omega_0) & -k/(J_2\omega_0) & c/J_2 & -c/J_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ k_I/(J_1\omega_0) & 0 \\ 0 & 1/(J_2\omega_0) \end{bmatrix}\begin{bmatrix} I \\ T_d \end{bmatrix}$$

In Exercise 6.11 we are told to set

$$J_1 = 10/9, \quad J_2 = 10, \quad c = 0.1, \quad k = 1, \quad k_I = 1,$$

from which it follows also that

$$\omega_0 = \sqrt{k(J_1 + J_2)/(J_1 J_2)} = \sqrt{\left(\frac{10}{9} + \frac{90}{9}\right)\frac{9}{100}} = 1,$$

so in open loop

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -0.9 & 0.9 & -0.09 & 0.09 \\ 0.1 & -0.1 & 0.01 & -0.01 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix},$$

and the eigenvalues are indeed $0, 0, -0.05 \pm i$. Taking I to be the input and assuming $T_d = 0$, we have

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -0.9 & 0.9 & -0.09 & 0.09 \\ 0.1 & -0.1 & 0.01 & -0.01 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0.9 \\ 0 \end{bmatrix},$$

and using

```
K=place(A,B,[-2 -1 -1+i -1-i]),
```

we obtain the state-feedback gain matrix

$$K = \begin{bmatrix} 8.9333 & 35.5111 & 5.4444 & 101.2222 \end{bmatrix}.$$

Assuming we implement a state-feedback control law of the form

$$u = -K\vec{x} + k_r r,$$

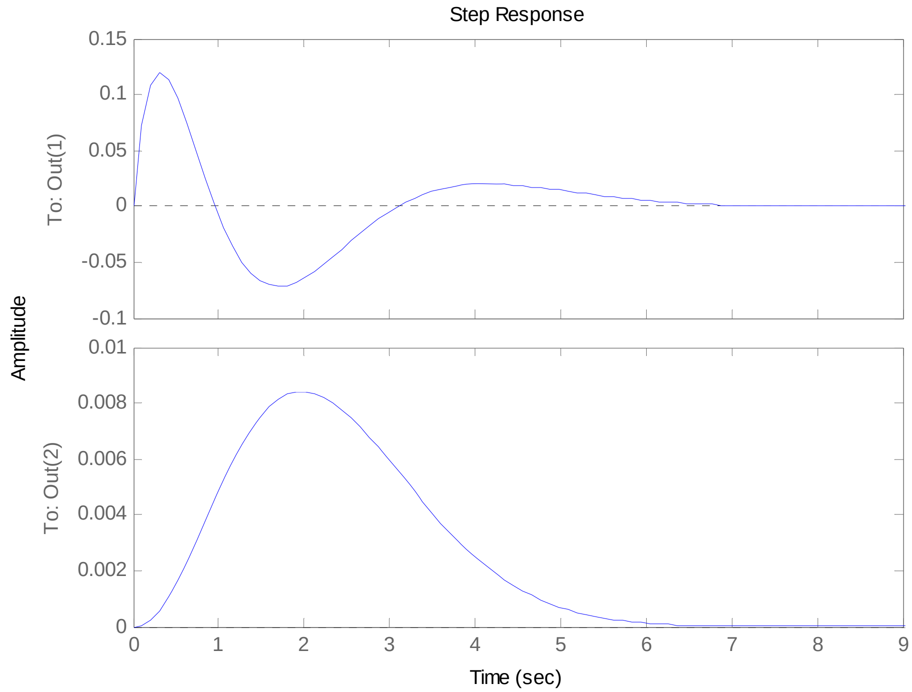
we then have

$$\frac{d}{dt}\vec{x} = (A - BK)\vec{x} + k_r B r.$$

The response to a step in the command input r is obtained by

```
motdrv = ss(A-B*K,B,[0 0 1 0;0 0 0 1],[]),
step(motdrv),
```

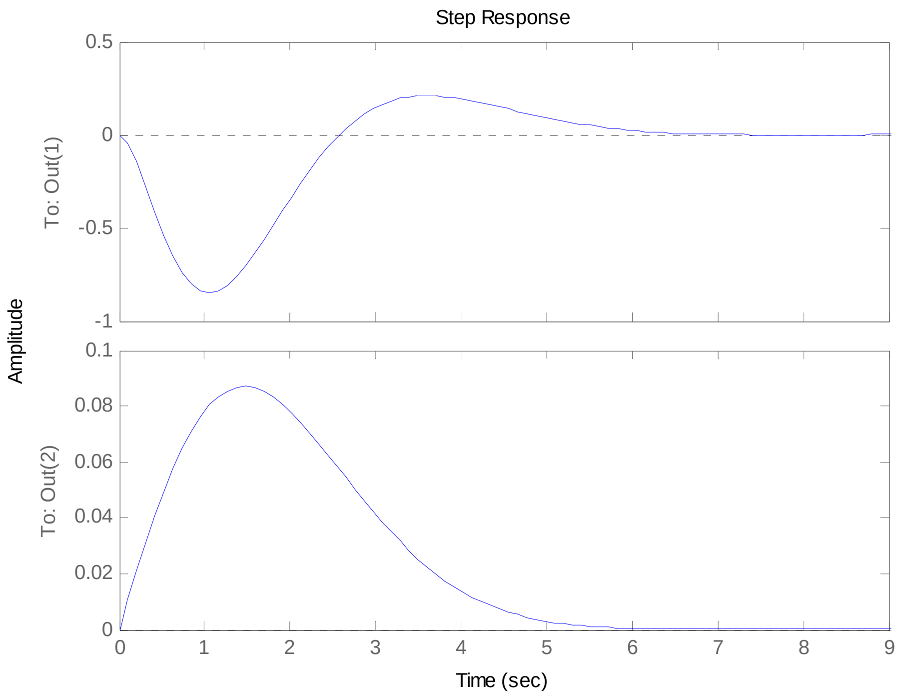
yielding



The response to a step in the disturbance torque T_d is obtained by

```
motdrv = ss(A-B*K,[0;0;0;0.1],[0 0 1 0;0 0 0 1],[]),
step(motdrv),
```

yielding



Next we are supposed to design an observer, but A&M do not tell us what to assume for C. Using the observability matrix

$$W_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix},$$

and Matlab's **cond** routine, it is easy to establish that measurements of the angular velocities are not so appropriate. We therefore choose

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix},$$

corresponding to a measurement of the angular position of the motor drive shaft. Then using

```
L=place(A',C',[-4 -2 -2+2*i -2-2*i])',
```

we find

$$L = \begin{bmatrix} 9.9 \\ 80.6778 \\ 38.01 \\ 66.8878 \end{bmatrix}.$$

Hence the observer evolves according to

$$\frac{d}{dt}\hat{x} = A\hat{x} + Bu + L(y - C\hat{x}),$$

and our output feedback law is

$$u = -K\hat{x} + k_r r.$$

To simulate the output feedback system we note that we can write

$$\begin{aligned}
\frac{d}{dt}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} &= \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} u + \begin{bmatrix} 0 \\ L \end{bmatrix}\begin{bmatrix} C & -C \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} \\
&= \left(\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} + \begin{bmatrix} 0 \\ L \end{bmatrix}\begin{bmatrix} C & -C \end{bmatrix}\right)\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix}\begin{bmatrix} 0 & -K \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} k_r r \\
&= \left(\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix}\begin{bmatrix} 0 & -K \end{bmatrix} - \begin{bmatrix} 0 \\ L \end{bmatrix}\begin{bmatrix} C & -C \end{bmatrix}\right)\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} k_r r \\
&= \begin{bmatrix} A & -BK \\ LC & A - BK - LC \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} k_r r.
\end{aligned}$$

Hence in Matlab we can easily do simulations of the output-feedback system in closed loop. For example let us set $r = 0$ and simulate the behavior of the system under the influence of an (unobserved) disturbance torque:

$$\frac{d}{dt}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & A - BK - LC \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} T_d.$$

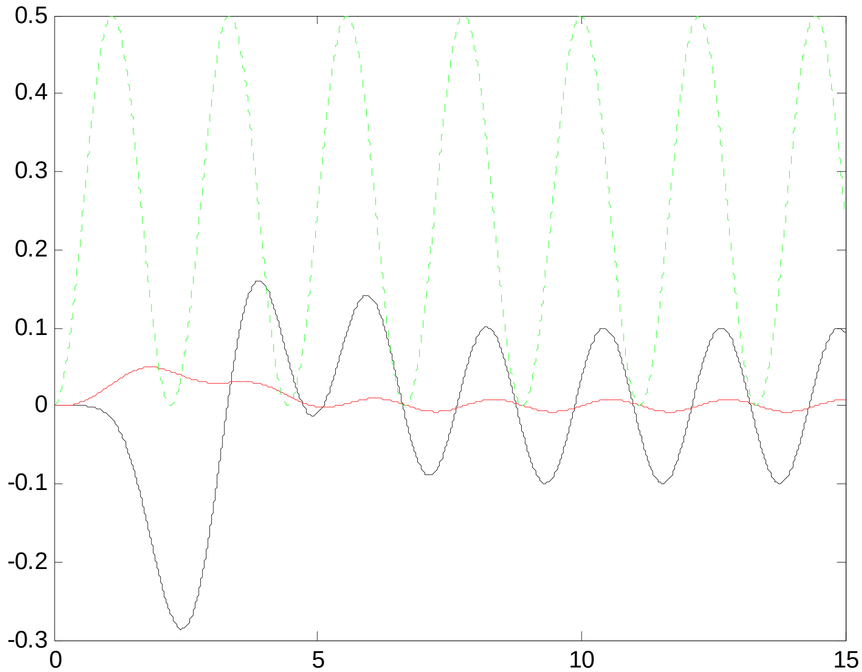
If we take for example

$$T_d(t) = \frac{1}{2}\sin^2\left(\sqrt{2}\,t\right),$$

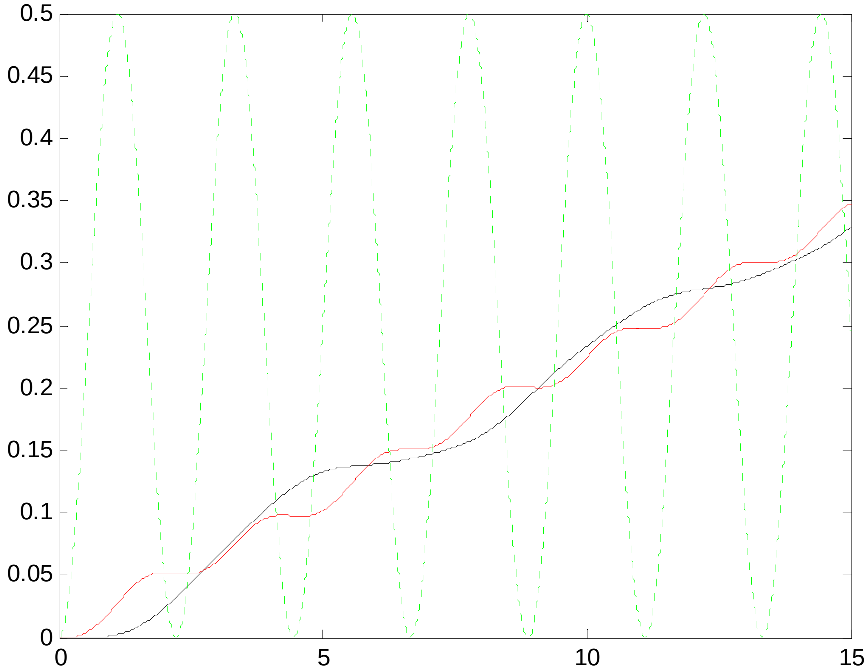
we can run the Matlab code

```
AA=[A -B*K; L*C A-B*K-L*C],
Nt=30000; t=linspace(0,15,Nt); dt=t(2)-t(1);
x=zeros(8,Nt); Td=0.5*sin(sqrt(2)*t).^2;
for ii=2:Nt,
 x(:,ii) = x(:,ii-1) + AA*x(:,ii-1)*dt +
[0;0;0;0.1;0;0;0;0]*Td(ii)*dt;
end;
figure; plot(t,x(3,:),'k-',t,x(4,:),'r-',t,Td,'g:');
```

to generate the plot



Contrast this with the open-loop behavior, obtained by setting $K = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$ and re-running the above Matlab code:


0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
5
10
15