APPPHYS 217 Thursday 8 April 2010

A&M example 6.1: **The double integrator**

Consider the motion of a point particle in 1D, with the applied force as a control input. This is simply Newton's equation $F = ma$ with $F \leftrightarrow u$:

$$\frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u.$$

The set of equilibrium points of this system clearly corresponds to the line $\dot{q} = 0$ (from the equations we see that $u = 0$ is required for equilibrium). With reference to A&M Figure 6.1(b), let us briefly consider how we can use the input u to steer the system, starting from any equilibrium point and ending on any other. It is instructive to visualize this process on the phase portrait.

State-feedback control example: **second-order system**

Consider the driven second-order system

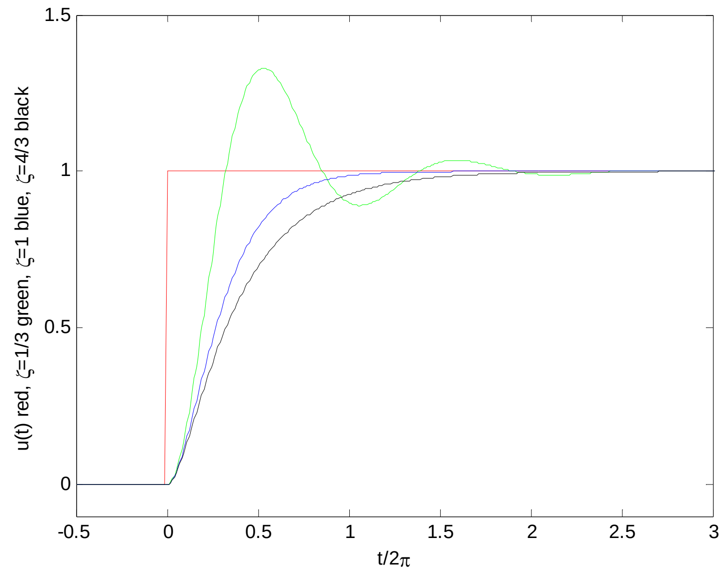
$$\ddot{q} = -2\zeta\omega_0\dot{q} - \omega_0^2 q + u, \quad x_1 \equiv q, \quad x_2 \equiv \dot{q},$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix}.$$

As we have previously discussed, u here could represent an external applied force (in a mechanical mass-spring-damper system) or voltage (in the LCR circuit realization). Anticipating conventional control-theoretic notation that we will introduce below, let us also write this as

$$\frac{d}{dt}\vec{x} = A\vec{x} + Bu, \quad A = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We saw last time that the damping ratio ζ sets important properties of the step (transient) response:



Let us briefly consider the state-feedback control scenario

$$u = -K\vec{x},$$

where (since we have taken u to be scalar)

$$K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}.$$

With this feedback law, we have

$$\frac{d}{dt}\vec{x} = A\vec{x} + Bu = (A - BK)\vec{x},$$

where

$$\begin{aligned} A - BK \rightarrow & \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \\ = & \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix} \\ = & \begin{bmatrix} 0 & 1 \\ -\omega_0^2 - k_1 & -2\zeta\omega_0 - k_2 \end{bmatrix}. \end{aligned}$$

Clearly we can view this as a modified second-order system. If we define

$$\omega_0' = \sqrt{\omega_0^2 + k_1},$$

then apparently we have

$$\begin{aligned} \zeta'\omega_0' &= \zeta\omega_0 - k_2, \\ \zeta' &= \frac{\zeta\omega_0 - k_2}{\sqrt{\omega_0^2 + k_1}}, \end{aligned}$$

and it follows that we can actually set ω_0' to any value we like through choice of the feedback parameter k_1, and once this is set (assuming $\omega_0' \neq 0$) we can adjust ζ' as desired through choice of k_2. Hence if we don't like the transient behavior of our original open-loop system (without feedback), we can in principle modify it however we like by use of *state feedback* as described above. We program the desired behavior by setting the feedback gain K.

Note that if the components of $\vec{x}$ are available as electronic signals, it is straightforward to produce the feedback signal u with an op-amp circuit. (LCR example)

The fact that we have full power to reprogram the oscillation frequency and damping ratio in this state feedback scenario can be made obvious if we look at the dynamics with feedback, transformed back to a single second-order ODE:

$$\frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 - k_1 & -2\zeta\omega_0 - k_2 \end{bmatrix}\begin{bmatrix} q \\ \dot{q} \end{bmatrix},$$

$$0 = \ddot{q} + (2\zeta\omega_0 + k_2)\dot{q} + (\omega_0^2 + k_1)q.$$

Hence k_1 is associated with a modification of the harmonic restoring force while k_2 directly and independently modifies the velocity-damping.

Just for fun, let us calculate the eigenvalues of the feedback-modified A matrix:

$$0 = \det(A - BK - I\lambda) \to \begin{bmatrix} -\lambda & 1 \\ -\omega_0^2 - k_1 & -2\zeta\omega_0 - k_2 - \lambda \end{bmatrix},$$

$$\begin{aligned} 0 &= \lambda(2\zeta\omega_0 + k_2 + \lambda) + \omega_0^2 + k_1 \\ &= \lambda^2 + (2\zeta\omega_0 + k_2)\lambda + \omega_0^2 + k_1. \end{aligned}$$

Considering the solutions given by the quadratic equation,

$$\lambda = \frac{1}{2}\left(-2\zeta\omega_0 - k_2 \pm \sqrt{(2\zeta\omega_0 + k_2)^2 - 4\omega_0^2 - 4k_1}\right),$$

we see that we can achieve any desired pair of eigenvalues $\{\lambda_+, \lambda_-\}$ by setting

$$\begin{aligned} \lambda_+ + \lambda_- &= -2\zeta\omega_0 - k_2, \\ k_2 &= -2\zeta\omega_0 - \lambda_+ - \lambda_-, \end{aligned}$$

and

$$\begin{aligned} \lambda_+ - \lambda_- &= \sqrt{(2\zeta\omega_0 + k_2)^2 - 4\omega_0^2 - 4k_1}, \\ k_1 &= \frac{1}{4}(2\zeta\omega_0 + k_2)^2 - \omega_0^2 - \frac{1}{4}(\lambda_+ - \lambda_-)^2 \\ &= \frac{1}{4}(\lambda_+ + \lambda_-)^2 - \omega_0^2 - \frac{1}{4}(\lambda_+ - \lambda_-)^2 \\ &= \lambda_+\lambda_- - \omega_0^2. \end{aligned}$$

Hence, we can arbitrarily program the eigenvalues of the closed-loop dynamics matrix.

Before turning to some generalities, we note that the second-order system with the control structure we are considering can be pushed into arbitrary states in a similar manner as the double-integrator. Suppose we are starting at the origin at $t = 0$ and want to reach $\begin{bmatrix} q & \dot{q} \end{bmatrix}'$ at a target time T. If we use the input u and allow our input signals to be unbounded, we can in principle take the following approach. First, give the system a momentum impulse at time $t = 0$ that will cause the system to pass through position q at time T. To convince ourselves that this is really possible, we could look at the matrix exponential we calculated last time and show that we can always solve for $\dot{q}_0$ such that

$$\begin{bmatrix} q \\ \cdot \end{bmatrix} = \exp(AT)\begin{bmatrix} 0 \\ \dot{q}_0 \end{bmatrix}.$$

In the overdamped case, for example, this leads to

$$\dot{q}_0 = \frac{\omega_0\sqrt{\zeta^2 - 1}}{e^{-\omega_0\zeta T}\sinh\left(\omega_0 T\sqrt{\zeta^2 - 1}\right)}q.$$

Then, at time T, apply a second momentum impulse that “corrects” the final velocity $\dot{q}$ to its desired value. Note that even with this rather singular strategy, in which we rely on our ability to apply controls so strong that they overwhelm most of the system’s natural dynamics (harmonic restoring force and velocity-damping), we still rely on the inherent integrator structure

$$\frac{d}{dt}q = \dot{q}$$

that allows us to use an input to $\dot{q}$ only to affect the position q.

Reachability rank condition

We say that a linear system

$$\dot{x} = Ax + Bu$$

is *reachable* if for any initial state $x(0) = x_0$, desired final state x_f and ‘target time’ T it is possible to find a control input $u(t),\ t \in [0, T]$ that steers the system to reach $x(T) = x_f$. There is a theorem (see A&M pp. 167-170) that says that a system is reachable if its *reachability matrix*

$$W_r \equiv [B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B]$$

has full rank (here $x \in R^n$). If a system is reachable, it is furthermore possible to solve the *pole placement* problem, in which we want to design a state feedback law

$$u = -Kx$$

such that we can pick any eigenvalues we want for the controlled dynamics

$$\begin{aligned}\dot{x} &= Ax + Bu \\ &= (A - BK)x.\end{aligned}$$

Here A and B are given, and we must find a K to achieve the desired eigenvalues for $(A - BK)$.

In the case of our second-order system with

$$A = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

we have

$$AB = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2\zeta\omega_0 \end{bmatrix}, \quad W_r = \begin{bmatrix} 0 & 1 \\ 1 & -2\zeta\omega_0 \end{bmatrix},$$

and clearly W_r has full rank as $\det(W_r) = -1$.

Before moving on, let's look at a simple example (from Åström and Murray, Ex. 5.3) of a system that is *not* reachable:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}u.$$

Here we can easily compute

$$W_r = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix},$$

which clearly has determinant zero. This can be understood by noting the complete 'symmetry' of the way that u modifies the evolution of x_1 and x_2. For example, if $x_1(0) = x_2(0)$ there is no way to use u to achieve $x_1(T) \neq x_2(T)$ at any later time.

Predator-Prey example

We have the equations of motion

$$\dot{H} = f_H(H,L) = rH\left(1 - \frac{H}{k}\right) - \frac{aHL}{c+H},$$
$$\dot{L} = f_L(H,L) = b\frac{aHL}{c+H} - dL.$$

First we find the equilibrium points. For steady-state of the lynx equation, we have

$$0 = b\frac{aHL}{c+H} - dL$$
$$L = \frac{abH}{d(c+H)}L,$$

which has a trivial solution $L = 0$ as well as

$$d(c+H) = abH,$$
$$cd = (ab-d)H,$$
$$H = H_{eq} \equiv \frac{cd}{ab-d}.$$

Then going to the hares equation, if $L = 0$ then we need

$$0 = rH\left(1 - \frac{H}{k}\right),$$

which has solutions $H = 0$ and $H = k$. Thus we have two distinct equilibria so far, $\{L = 0, H = 0\}$ and $\{L = 0, H = k\}$. If we set $H = H_{eq}$ however then we need

$$\begin{aligned}\frac{aH_{eq}L}{c+H_{eq}} &= rH_{eq}\left(1-\frac{H_{eq}}{k}\right),\\ L = L_{eq} \equiv \frac{c+H_{eq}}{a}r\left(1-\frac{H_{eq}}{k}\right) &= \frac{r}{ak}(c+H_{eq})(k-H_{eq})\\ &= \frac{r}{ak}\left(c+\frac{cd}{ab-d}\right)\left(k-\frac{cd}{ab-d}\right)\\ &= \frac{r}{ak}\left(\frac{abc-cd+cd}{ab-d}\right)\left(\frac{abk-dk-cd}{ab-d}\right)\\ &= \frac{r}{k}\left(\frac{bc}{ab-d}\right)\left(\frac{abk-dk-cd}{ab-d}\right)\\ &= \frac{bcr(abk-dk-cd)}{(ab-d)^2k}.\end{aligned}$$

This gives us a third equilibrium point $\{L = L_{eq}, H = H_{eq}\}$.

In order to determine the stability of these equilibria, we compute the derivative

$$f \equiv \begin{pmatrix} f_H \\ f_L \end{pmatrix} = \begin{pmatrix} rH\left(1-\frac{H}{k}\right) - \frac{aHL}{c+H} \\ b\frac{aHL}{c+H} - dL \end{pmatrix} = \begin{pmatrix} rH - \frac{r}{k}H^2 - aHL(c+H)^{-1} \\ abHL(c+H)^{-1} - dL \end{pmatrix}$$

$$df(\cdot) = \begin{pmatrix} \frac{\partial f_H}{\partial H} & \frac{\partial f_H}{\partial L} \\ \frac{\partial f_L}{\partial H} & \frac{\partial f_L}{\partial L} \end{pmatrix} = \begin{pmatrix} r - \frac{2r}{k}H - aL(c+H)^{-1} + aHL(c+H)^{-2} & -aH(c+H)^{-1} \\ abL(c+H)^{-1} - abHL(c+H)^{-2} & abH(c+H)^{-1} - d \end{pmatrix}.$$

Evaluating this at each equilibrium point:

$$\begin{pmatrix} H \\ L \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} : \quad df \to \begin{pmatrix} r & 0 \\ 0 & -d \end{pmatrix},$$

$$\begin{pmatrix} H \\ L \end{pmatrix} = \begin{pmatrix} k \\ 0 \end{pmatrix} : \quad df \to \begin{pmatrix} -r & \frac{ak}{c+k} \\ 0 & \frac{abk}{c+k} - d \end{pmatrix},$$

$$\begin{pmatrix} H \\ L \end{pmatrix} = \begin{pmatrix} H_{eq} \\ L_{eq} \end{pmatrix} : \quad df \to \begin{pmatrix} r - \frac{2r}{k}H_{eq} - aL_{eq}(c+H_{eq})^{-1} + aH_{eq}L_{eq}(c+H_{eq})^{-2} & -aH_{eq}(c+H_{eq})^{-1} \\ abL_{eq}(c+H_{eq})^{-1} - abH_{eq}L_{eq}(c+H_{eq})^{-2} & abH_{eq}(c+H_{eq})^{-1} - \end{pmatrix}$$

By computing eigenvalues it follows that all three equilibrium points are unstable, with the first two being saddles and the third a source.

Following Example 6.5 from A&M, we now consider state-feedback stabilization of the non-trivial equilibrium point $\begin{bmatrix} H_{eq} & L_{eq} \end{bmatrix}'$, where once again

$$H_{eq} = \frac{cd}{ab-d}, \quad L_{eq} = \frac{bcr(abk-dk-cd)}{(ab-d)^2k}.$$

The feedback strategy will be to modulate the net birth rate of hares (which could be done for example by modulating their food supply), $r \to r+u$, so that

$$\dot{H} = f_H(H,L) = (r+u)H\left(1 - \frac{H}{k}\right) - \frac{aHL}{c+H},$$
$$\dot{L} = f_L(H,L) = b\frac{aHL}{c+H} - dL,$$

Using the derivative $df(\bullet)$ that we derived above, we can linearize the controlled dynamics (assuming $u \ll r$) as

$$\frac{d}{dt}\begin{bmatrix} \delta H \\ \delta L \end{bmatrix} = \begin{bmatrix} r - \frac{2r}{k}H_{eq} - aL_{eq}(c+H_{eq})^{-1} + aH_{eq}L_{eq}(c+H_{eq})^{-2} & -aH_{eq}(c+H_{eq})^{-1} \\ abL_{eq}(c+H_{eq})^{-1} - abH_{eq}L_{eq}(c+H_{eq})^{-2} & abH_{eq}(c+H_{eq})^{-1} - d \end{bmatrix}\begin{bmatrix} \delta H \\ \delta L \end{bmatrix}$$
$$+ \begin{bmatrix} H_{eq}\left(1 - \frac{H_{eq}}{k}\right) \\ 0 \end{bmatrix}u.$$

Hence in the control-theoretic notation we have been using,

$$\frac{d}{dt}\begin{bmatrix} \delta H \\ \delta L \end{bmatrix} = A\begin{bmatrix} \delta H \\ \delta L \end{bmatrix} + Bu,$$

and inserting parameter values as in A&M,

$$a = 3.2, \quad b = 0.6, \quad c = 50,$$
$$d = 0.56, \quad k = 125, \quad r = 1.6,$$

we numerically evaluate

$$A \approx \begin{bmatrix} 0.13 & -0.93 \\ 0.57 & 0 \end{bmatrix}, \quad B \approx \begin{bmatrix} 17.2 \\ 0 \end{bmatrix} \equiv \begin{bmatrix} b \\ 0 \end{bmatrix}.$$

The open-loop eigenvalues of A are then $0.0631 \pm 0.7254i$, unstable as claimed. We can now try to find a feedback law

$$u = -K\begin{bmatrix} \delta H \\ \delta L \end{bmatrix},$$

such that the closed-loop eigenvalues become real and negative, for example $\lambda \to \{-0.1, -0.2\}$. Using the Matlab function **`place`**, we find

$$K = \begin{bmatrix} 0.0248 & -0.0522 \end{bmatrix} \equiv \begin{bmatrix} k_1 & k_2 \end{bmatrix}.$$

This state-feedback law leads to new linearized dynamics

$$\frac{d}{dt}\begin{bmatrix} \delta H \\ \delta L \end{bmatrix} = (A - BK)\begin{bmatrix} \delta H \\ \delta L \end{bmatrix},$$
$$A - BK \approx \begin{bmatrix} -0.3 & -0.0352 \\ 0.568 & 0 \end{bmatrix}.$$

Assuming the state-feedback law is implemented linearly, we can in fact write the nonlinear dynamics with feedback as

$$\dot{H} = f_H(H,L) = (r - k_1(H - H_{eq}) - k_2(L - L_{eq}))H\left(1 - \frac{H}{k}\right) - \frac{aHL}{c+H},$$
$$\dot{L} = f_L(H,L) = b\frac{aHL}{c+H} - dL.$$

Using **pplane8** we can investigate the phase portrait of the closed-loop system in the vicinity of the open-loop equilibrium point.

State feedback versus output feedback

Note that in our discussion of stabilization and pole-placement so far, we have assumed that it makes sense to design a control law of the form

$$u = -Kx.$$

This is called a 'state feedback' law since in order to determine the control input $u(t)$ at time t, we generally need to have full knowledge of the state $x(t)$. In practice this is often not possible, and thus we usually specify the available output signals when defining a control design problem:

$$\dot{x} = Ax + Bu,$$
$$y = Cx.$$

Here the output signal $y(t)$, which can in principle be a vector of any dimension, represents the information about the evolving system state that is made available to the controller via sensors. An 'output feedback' law must take the form

$$u(t) = f[y(\tau \leq t)],$$

where, in general, we can allow $u(t)$ to depend on the entire history of $y(\tau)$ with $\tau \leq t$ (more on this below and later in the course). Output feedback is a natural setting for practical applications. For example, if we are talking about cruise control for an automobile, x may represent a complex set of variables having to do with the internal state of the engine, wheels and chassis while y is only a readout from the speedometer. Hopefully it will seem natural that it is usually prohibitively difficult to install a sensor to monitor every coordinate of the system's state space, and also that it will often be unnecessary to do so (cruise control electronics can function quite well with just the car's speed).

One simple example of a system in which full state knowledge is clearly not necessary is (asymptotic) stabilization of a simple harmonic oscillator. If the natural dynamics of the plant is

$$m\ddot{x} = -kx,$$

and our actuation mechanism is to apply forces directly on the mass, then the control system looks like

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-k}{m} & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u,$$

(where x_1 is now the position and x_2 the velocity). We can clearly make the equilibrium point at the origin asymptotically stable via the feedback law

$$u = -bx_2 = \begin{bmatrix} 0 & -b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

which makes the overall equation of motion

$$\ddot{x}_1 = -\frac{k}{m}x_1 - b\dot{x}_1,$$

which we recognize as a damped harmonic oscillator. Thus it is clear that the controller only needs to know the velocity of the oscillator in order to implement a successful feedback strategy. So even if we go to a SISO (single-input single-output) formulation of this problem,

$$\dot{x} = Ax + Bu,$$

$$y = Cx,$$

we are obviously fine for any C of the form $(\alpha \neq 0)$

$$C = \begin{bmatrix} 0 & \alpha \end{bmatrix},$$

since $x_2 = y/\alpha$ and we can implement an output-feedback law of the form

$$u = -bx_2 = -\frac{b}{\alpha}y.$$

In contrast to this, consider the analogous stabilization problem for the predator-prey system that we discussed above. Even if our attention to the immediate vicinity of the natural equilibrium point and assume that a linearized model is sufficient for the design, it is not clear that we could succeed without requiring knowledge of both the lynx and hare populations.

Clearly, if C is a square matrix and y has the same dimension as $x,$ everything will be easy if C is invertible. As a generalization of what we did for the simple harmonic oscillator above, we could just design a state feedback controller $K,$ set

$$\hat{x} = C^{-1}y,$$

and apply feedback

$$u = -K\hat{x} = -KC^{-1}y.$$

However this is a special case and not the sort of convenience we want to count on!

State estimation

At this point it might seem like we might need completely new theorems about reachability and pole-placement for output-feedback laws, when $u(t)$ is only allowed to depend on $y(\tau < t).$ However, it turns out that we can build naturally on our previous results by appealing to a *separation method*. The basic idea is that we will try to construct a procedure for processing the data $y(\tau < t)$ to obtain an estimate $\hat{x}(t)$ of the true system state $x(t)$, and then apply a feedback law $u = -K\hat{x}$ based on this estimate. This can be possible even when C is not invertible (not even square). The controller thus assumes the structure of a dynamical system itself, with $y(t)$ as its input, $u(t)$ as its output and $\hat{x}(t)$ as its internal state. There are various ways of designing 'state estimators' to extract $\hat{x}(t)$ from $y(\tau < t)$, of which we will discuss two, and there is also a convenient procedure for determining whether or not y contains enough information to make full state reconstruction possible in principle. The latter test looks a lot like the test for reachability, for not accidental reasons.

Let's start by thinking about the simple harmonic oscillator again. We noted that in order to asymptotically stabilize the equilibrium point at the origin, it would be most convenient to have an output signal that told us directly about its velocity x_2. However, you may have already realized that in a scenario with $(\alpha \neq 0)$

$$C = \begin{bmatrix} \alpha & 0 \end{bmatrix},$$

$$y = \alpha x_1,$$

it should be simple to obtain a good estimate of x_2 via

$$\hat{x}_2 = \frac{d}{dt}\alpha^{-1}y.$$

This is certainly a valid procedure for estimating x_2, although in practice one should be wary of taking derivatives of measured data since that tends to accentuate high-frequency noise.

In a similar spirit, we note that for any dynamical system

$$\dot{x} = Ax + Bu,$$

$$y = Cx,$$

if we hold u at zero we can make use of the general relations

$$\begin{aligned}
\dot{y} &= C\dot{x} = CAx, \\
\ddot{y} &= C\ddot{x} = C\frac{d}{dt}\dot{x} = C\frac{d}{dt}Ax = CA\dot{x} = CA^2x, \\
&\vdots \\
\frac{d^n}{dt^n}y &= CA^nx.
\end{aligned}$$

If we look at how this applies to our modified simple harmonic oscillator example with

$$C = \begin{bmatrix} \alpha & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix},$$

$$y = C\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \alpha x_1,$$

we have

$$CA = \begin{bmatrix} \alpha & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \alpha \end{bmatrix},$$

$$\dot{y} = CA\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \alpha x_2,$$

and we start to get a sense for how the natural dynamics A can move information about state space variables into the 'support' of C. Hopefully it should thus seem reasonable that in order for a system to be *observable*, we require that the observability matrix

$$W_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

have full rank. Informally, if a system is observable then we are guaranteed that we can design a procedure (such as the derivatives scheme above) to extract a faithful estimate $\hat{x}$ from y. However, it will generally be necessary to monitor y for some time (and with good accuracy) before the estimation error

$$\tilde{x}(t) \equiv x(t) - \hat{x}(t)$$

can be made small. In the derivatives scheme, for instance, we can't estimate high derivatives of $y(t)$ until we see enough of it to get an accurate determination of its slope, curvature, etc.

State observer with innovations

A more common (and more robust) method for estimating x from y is to construct a state observer that applies corrections to an initial guess $\hat{x}$ until $C\hat{x}$ becomes an accurate predictor of y.

Suppose that at some arbitrary point in time t we have an estimate $\hat{x}(t)$. How should we update this estimate to generate estimates of the state $x(t')$ with $t' > t$? Most simply, we could integrate

$$\frac{d}{dt}\hat{x} = A\hat{x} + Bu,$$

assuming we know A and B for the plant. It is generally assumed that we know u since this signal is under our control! Then we notice that the estimation error $\tilde{x}$ evolves as

$$\begin{aligned} \frac{d}{dt}\tilde{x} &= \frac{d}{dt}(x - \hat{x}) \\ &= (Ax + Bu) - (A\hat{x} + Bu) \\ &= A(x - \hat{x}) \\ &= A\tilde{x}. \end{aligned}$$

Hence, this strategy has the nice feature that if A is stable,

$$\lim_{t \to \infty} \tilde{x} = 0,$$

meaning that our estimate will eventually converge to the true system state. Note that this works even if B and u are non-zero.

What if we are not so lucky as to have sufficiently stable natural dynamics A? As mentioned above, a good strategy is to try to apply corrections to $\hat{x}$ at every time step, in proportion to the so-called innovation,

$$w \equiv y - C\hat{x}.$$

Here $y - C\hat{x}$ is the error we make in predicting $y(t)$ on the basis of $\hat{x}(t)$. Clearly when $\tilde{x}$ is small, so is w. A 'Luenberger state observer' can thus be constructed as

$$\frac{d}{dt}\hat{x} = A\hat{x} + Bu + L(y - C\hat{x}),$$

where L is a 'gain' matrix that is left to our design. This observer equation results in

$$\begin{aligned}\frac{d}{dt}\tilde{x} &= \dot{x} - \frac{d}{dt}\hat{x} \\ &= (Ax - Bu) - (A\hat{x} + Bu + L(y - C\hat{x})) \\ &= A(x - \hat{x}) - L(y - C\hat{x}) \\ &= A(x - \hat{x}) - LC(x - \hat{x}) \\ &= (A - LC)\tilde{x}.\end{aligned}$$

Hence we see that our design task should be to choose L, given A and C, such that $(A - LC)$ has nice stable eigenvalues.

This should remind you immediately of the pole-placement problem in state feedback, in which we wanted to choose K, given A and B, such that $(A - BK)$ had desired eigenvalues. Indeed, one can map between the two problems by noting that the transpose of a matrix M^T has the same eigenvalues as M. Thus we can view our observer design problem as being the choice of L^T such that

$$(A - LC)^T = A^T - C^T L^T$$

has nice stable eigenvalues, and this now has precisely the same structure as before. Indeed, there is a complete 'duality' between state feedback and observer design, with correspondences

$$A \leftrightarrow A^T, \quad B \leftrightarrow C^T, \quad K \leftrightarrow L^T, \quad W_r \leftrightarrow W_o^T.$$

Hence it should be clear, for example, how Matlab's **`place`** function can be used for observer design. And as long as the observability matrix has full rank, we are guarenteed to be able to find an L such that $(A - LC)$ has arbitrary desired eigenvalues.

Pole-placement with output feedback

As discussed in section 7.3 of Åström and Murray, the following theorem holds (here we simplify to the $r = 0$ case):

For a system

$$\dot{x} = Ax + Bu,$$

$$y = Cx,$$

the controller described by

$$u = -K\hat{x},$$

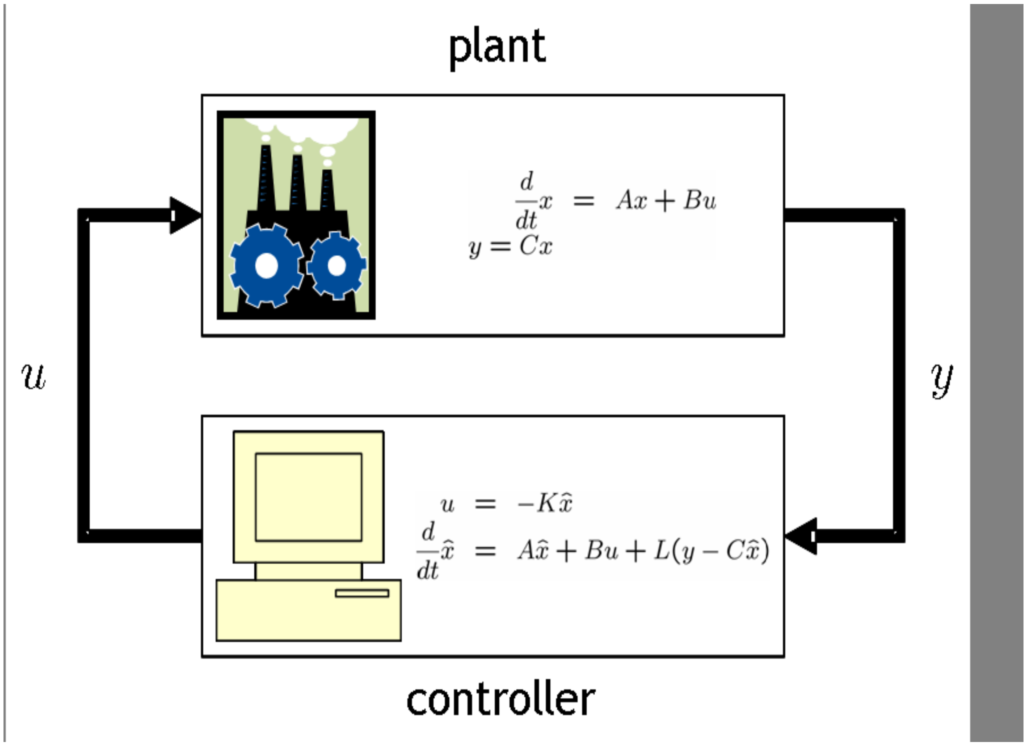
$$\frac{d}{dt}\hat{x} = A\hat{x} + Bu + L(y - C\hat{x})$$

gives a closed-loop system with the characteristic polynomial

$$\det(sI - A + BK)\det(sI - A + LC).$$

This polynomial can be assigned arbitrary roots if the system is observable and reachable.

The overall setup is summarized in the following cartoon:



Next time we'll have a look at how this sort of strategy performs in the presence of noise.