

# APPPHYS 217 Thursday 1 April 2010

## Review of Linear Algebra and Ordinary Differential Equations (ODE's) Part 2 of 2

Systems of first-order ODE's

The simplest example an ODE is a first-order, scalar equation:

$$\dot{x} = ax.$$

Here let's assume  $x \in R$ ,  $a \in R$  (nothing really changes if they are complex). In your youth you have see the method of solution,

$$\begin{aligned}\frac{d}{dt}x &= ax \rightarrow \frac{dx}{x} = a dt, \\ \int_{x_i}^{x_f} dx x^{-1} &= a \int_{t_i}^{t_f} dt, \\ \ln\left(\frac{x_f}{x_i}\right) &= a(t_f - t_i), \\ x_f &= x_i \exp(a(t_f - t_i)).\end{aligned}$$

Some simple things to notice at this point are that if  $a < 0$  then  $x$  decreases to zero as  $t_f \rightarrow +\infty$ , and if  $a > 0$  then  $x$  blows up exponentially (exercise: what if  $a$  is complex?).

A higher-order ODE can always be converted into a system of coupled first-order ODE's by expanding the set of variables. For example, if we start with a single second-order equation in  $x$ ,

$$m\ddot{x} + b\dot{x} + kx = 0,$$

we can convert by adding  $\dot{x}$  as a distinct variable. Then,

$$\begin{aligned}\frac{d}{dt}\dot{x} &= -\frac{b}{m}\dot{x} - \frac{k}{m}x, \\ \frac{d}{dt}x &= \dot{x},\end{aligned}$$

which we can write in "state space form" as

$$\frac{d}{dt} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}.$$

(Exercise: can we play the same trick if we start with  $\dot{x} + x^2 = 0$ , by adding  $x^2$  as a new variable?) One often writes,

$$\frac{d}{dt}\vec{x} = A\vec{x}, \quad \vec{x} \equiv \begin{bmatrix} x \\ \dot{x} \end{bmatrix},$$

where  $\vec{x}$  is the "state vector" and  $A$  specifies the dynamics. Noting that the second-order equation we started with is essentially Newton's equation  $F = ma$  for a damped mass-spring system, we see that systems of coupled equations will be ubiquitous in controls problems. Of course, even in problems without "inertia" (acceleration) terms, it is common to consider systems of coupled first-order ODE's from the outset. For example, if our state is the position of a point on the  $R^2$  plane and

our dynamics is a simple rigid rotation about the origin at frequency  $\omega$ , we have

$$\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

(Exercise: using methods from later in today's lecture, verify that this  $A$  matrix gives rise to rotation of the state vector.)

Note that in general if we have the dynamics written in state space form and  $A$  is a diagonal matrix, the ODE's aren't actually coupled to one another:

$$\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_{xx} & 0 \\ 0 & a_{yy} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

is equivalent to

$$\dot{x} = a_{xx}x,$$

$$\dot{y} = a_{yy}y.$$

Obviously we can solve these equations independently to arrive at

$$x(t) = x(0) \exp(a_{xx}t),$$

$$y(t) = y(0) \exp(a_{yy}t).$$

This immediately leads us to one common strategy for integrating systems of first-order ODE's even when the equations are coupled ( $A$  has some non-zero off-diagonal matrix elements), by transforming to the eigenbasis of  $A$ . Applying simple linear algebra concepts, we can see that the state-space form of the dynamical equations

$$\frac{d}{dt} \vec{x} = A \vec{x}$$

is preserved under an invertible linear transformation  $P$  of the state space,

$$\vec{x} \mapsto \vec{x}' \equiv P^{-1} \vec{x}, \quad \vec{x} = P \vec{x}',$$

$$A \mapsto A' \equiv P^{-1} A P, \quad A = P A' P^{-1}.$$

Here's a simple proof:

$$\frac{d}{dt} (P \vec{x}') = P A' P^{-1} (P \vec{x}'),$$

$$P \frac{d}{dt} \vec{x}' = P A' \vec{x}',$$

$$\frac{d}{dt} \vec{x}' = A' \vec{x}'.$$

(Note that it is crucial to assume that  $P^{-1}$  exists.) Hence if  $A$  happens to be diagonalizable, we can choose  $P$  to be the matrix of eigenvectors such that

$$A' = P^{-1} A P = D,$$

where  $D$  is the diagonal matrix of eigenvalues. As a result, we have

$$\frac{d}{dt} \vec{x}' = D \vec{x}'$$

as a set of decoupled equations that we can solve independently. After doing so, we can obtain our solution in the original coordinates by computing

$$\vec{x}(t) = P\vec{x}'(t),$$

which is of course quite simple to do. It follows that for any diagonalizable  $A$ , the solutions  $\vec{x}(t)$  take the form of linear combinations of exponentials of  $t$ . Note that if  $A$  has complex eigenvalues, however, these may combine into trig functions. (Exercise: use this strategy to solve

$$\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

and comment on why the solutions  $x(t)$  and  $y(t)$  are always real (assuming  $x(0)$  and  $y(0)$  are real) even though the eigenvalues of  $A$  are complex.)

What happens if  $A$  is not diagonalizable? For any  $A$  whatsoever, we can always make use of the general closed-form solution

$$\vec{x}(t) = \exp(At)\vec{x}(0),$$

where of course this is to be understood as a matrix exponential. Note how this nicely captures the procedure we followed above, in the case of diagonalizable  $A$ :

$$\begin{aligned} \vec{x}(t) &= \exp(At)\vec{x}(0) \\ &= \exp(PDP^{-1}t)\vec{x}(0) \\ &= \left[ 1 + PDP^{-1}t + \frac{1}{2}(PDP^{-1}t)^2 + \frac{1}{3!}(PDP^{-1}t)^3 + \dots \right] \vec{x}(0) \\ &= \left[ 1 + PDP^{-1}t + P\frac{1}{2}(Dt)^2P^{-1} + P\frac{1}{3!}(Dt)^3P^{-1} + \dots \right] \vec{x}(0) \\ &= P\exp(Dt)P^{-1}\vec{x}(0). \end{aligned}$$

(Make sure you can follow the step from line 3 to line 4.) We can read our eigenbasis procedure from the last line - transform the initial conditions into the eigenbasis, multiply each term by the exponential of an eigenvalue times  $t$ , and then transform back to the original basis. If  $A$  is not diagonalizable, we can still (in principle) use this closed-form solution if we can find some other way to compute the matrix exponential.

### Exponentiation of non-diagonalizable matrices

Consider the case  $F = ma$  with  $F = 0$  (a free particle). Then

$$\ddot{x} = 0,$$

which in state space form reads

$$\frac{d}{dt} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}.$$

Then our closed-form solution reads,

$$\vec{x}(t) = \exp(At)\vec{x}(0), \quad At = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}.$$

Note that this is a nilpotent matrix! Explicitly,

$$At = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}, \quad (At)^2 = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} = 0.$$

Hence,

$$\begin{aligned} \exp(At) &= 1 + At + \frac{1}{2}(At)^2 + \dots \\ &\rightarrow 1 + At = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

As a result  $\vec{x}(t) = (1 + At)\vec{x}(0)$ , which reads componentwise

$$x(t) = x(0) + \dot{x}(0)t,$$

$$\dot{x}(t) = \dot{x}(0).$$

This agrees with what we would write down based on freshman physics. Right? Note that the  $A$  matrix in this case is not diagonalizable and that the closed-form solution for  $\vec{x}(t)$  does not have the form of a linear combination of exponentials of  $t$ .

Things will not be so simple in general. However, here's a trick that can often be used to simplify calculation of the matrix exponential. Suppose  $A$  is not diagonalizable, but that we can see a way to split it into a symmetric part and a nilpotent part:

$$A = S + N, \quad S = S^T, \quad N^r = 0.$$

For example,

$$\begin{aligned} A &= \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix} = S + N = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \\ N^2 &= \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} = 0. \end{aligned}$$

If furthermore we can verify that  $S$  and  $N$  commute,

$$[S, N] = SN - NS = 0,$$

(as they do in our simple example) then computation of the matrix exponential is simplified as follows:

$$\begin{aligned} \exp(At) &= \exp(St + Nt) \\ &= \exp(St) \exp(Nt) \\ &= P \exp(Dt) P^{-1} \left( \sum_{j=1}^{r-1} \frac{1}{j!} (Nt)^j \right). \end{aligned}$$

Note that the first step (breaking up the exponential) relies on the assumption that  $S$  and  $N$  commute. Also note that the symmetry of  $S$  guarantees that we can find an invertible  $P$  to diagonalize it. In principle, we can always transform to a basis in which  $A'$  splits up into a diagonal matrix and nilpotent matrix - the Jordan canonical form:

$$\begin{aligned}
A &= TJT^{-1} = T(D + N)T^{-1}, \\
\exp(At) &= T\exp(Dt + Nt)T^{-1} \\
&= T\exp(Dt)\exp(Nt)T^{-1}.
\end{aligned}$$

Note that the detailed structure of the Jordan form guarantees that  $D$  and  $N$  here commute. But finding the invertible linear transformation that does this is not always so easy. (Exercise: above we saw that diagonalizable  $A$  give rise to integrated solutions  $\vec{x}(t)$  that are sums of exponentials, whereas our free particle  $F = ma$  example gave us a linear dependence on  $t$ ; what is the most general form of dependence on  $t$  for the integrated solutions of  $\frac{d}{dt}\vec{x} = A\vec{x}$ ?)

### Inhomogeneous and driven systems

Sometimes one encounters inhomogeneous linear ODE's:

$$\dot{x} = ax + b,$$

where  $\{a, b\} \in \mathbb{R}$ . A simple way to solve this scalar case is to change to a new variable,

$$\begin{aligned}
x' &\equiv x + \frac{b}{a}, \\
\dot{x}' &= \dot{x}.
\end{aligned}$$

Then

$$\begin{aligned}
\dot{x}' &= ax + b = a\left(x' - \frac{b}{a}\right) + b \\
&= ax',
\end{aligned}$$

which can be solved by simple exponentiation. Similarly, we may encounter an inhomogeneous system of coupled ODE's

$$\frac{d}{dt}\vec{x} = A\vec{x} + \vec{b},$$

with  $\{\vec{x}, \vec{b}\} \in \mathbb{R}^n$ . In order to play the analogous, trick we need to find a solution  $\vec{c}$  to the equation

$$A\vec{c} = -\vec{b}.$$

If we can do that, then

$$\begin{aligned}
\vec{x}' &\equiv \vec{x} + \vec{c}, \\
\frac{d}{dt}\vec{x}' &= \frac{d}{dt}\vec{x} = A\vec{x} + \vec{b} = A(\vec{x}' - \vec{c}) + \vec{b} = A\vec{x}'.
\end{aligned}$$

This transformation strategy is often referred to as 'translating the equilibrium to the origin,' since we note that the state  $\vec{x} = -\vec{c}$  is a stationary solution of the dynamics:

$$\frac{d}{dt}\vec{x} \mapsto A(-\vec{c}) + \vec{b} = 0.$$

If  $A$  is not invertible, however, a different procedure must be used.

For example, we might consider  $F = ma$  with constant  $F = f$ . In state space form,

$$\frac{d}{dt}\vec{x} = A\vec{x} + \vec{b}, \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 0 \\ f/m \end{bmatrix}.$$

Looking at the second row,

$$\frac{d}{dt}\dot{x} = f/m,$$

we can easily integrate this to find

$$\dot{x}(t) = \dot{x}(0) + \frac{f}{m}t.$$

We can then substitute this into the first row to find the scalar ‘driven’ ODE,

$$\frac{d}{dt}x = \dot{x} = \dot{x}(0) + \frac{f}{m}t.$$

We can in turn integrate this to find

$$\int_{x(0)}^{x(t)} dx = \int_0^t dt \left( \dot{x}(0) + \frac{f}{m}t \right),$$

$$x(t) = x(0) + \dot{x}(0)t + \frac{f}{2m}t^2.$$

It is straightforward to convince yourself that a solution like this cannot correspond to

$$\frac{d}{dt}\vec{x}' = A\vec{x}'$$

with  $\vec{x}' \in R^2 = \vec{x} + \vec{c}$  for constant  $\vec{c}$  (do this as an exercise; think about what you can and cannot do with nilpotent  $2 \times 2$  matrices).

This brings us finally to a general consideration of linear systems with additive time-dependent driving terms:

$$\frac{d}{dt}\vec{x} = A\vec{x} + \vec{b}(t),$$

where  $\vec{x} \in R^n$  and  $\vec{b}(t) \in R^n$  is an arbitrary time-dependent driving term. A general solution for the initial value problem can be written as

$$\vec{x}(t) = \exp(At)\vec{x}(0) + \exp(At) \int_0^t ds \exp(-As)\vec{b}(s)$$

$$= \exp(At)\vec{x}(0) + \int_0^t ds \exp(A(t-s))\vec{b}(s).$$

We can gain some intuition regarding the form of this solution by thinking about the superposition principle for linear dynamics. Note that if

$$\vec{x}(0) = \vec{x}_a(0) + \vec{x}_b(0),$$

then

$$\vec{x}(t) = \exp(At)\vec{x}(0) = \exp(At)\vec{x}_a(0) + \exp(At)\vec{x}_b(0).$$

Obviously this works for any number of additive terms in the initial condition. Likewise, for any given value of the state vector at time  $t$ , we can split  $\vec{x}(t)$  up into any number of parts and think of each one as having ‘come from’ some corresponding component of the initial condition:

$$\vec{x}(t) = \sum_j \vec{x}_j(t) = \sum_j \exp(At)\vec{x}_j(0),$$

where

$$\vec{x}_j(0) \equiv \exp(-At)\vec{x}_j(t).$$

Note that this works even if  $A$  itself is not invertible. Hence, rewriting the first form of the general solution above as

$$\vec{x}(t) = \exp(At) \left\{ \vec{x}(0) + \int_0^t ds \exp(-As)\vec{b}(s) \right\},$$

we can interpret the integral as signifying that the effects of the driving term are incorporated by adding terms  $\sim \exp(-As)\vec{b}(s)ds$  to the 'effective' initial condition for the state vector. Each of these terms looks like the vector obtained by evolving  $\vec{b}(s)ds$  backwards in time back to 0.

The general input-output state-space form for linear dynamics is conventionally written

$$\begin{aligned} \frac{d}{dt}\vec{x} &= A\vec{x} + B\vec{u}, \\ \vec{y} &= C\vec{x}. \end{aligned}$$

Here  $\vec{u}(t)$  is a completely arbitrary function of time, and we can identify  $B\vec{u}(t)$  with the driving term  $\vec{b}(t)$  considered above.

### Integrating ODE's in Matlab

The general state-space form of dynamics is

$$\frac{d}{dt}\vec{x} = f(x, t),$$

where  $f(x, t)$  may be a nonlinear function. In Matlab, one can numerically solve the initial value problem for such a system using 'ODE solvers' such as `ode45`. In order to do this, you first need to write a Matlab function that evaluates  $f(x, t)$  and pass a handle to that function file as an argument to `ode45` (see `help ode45` from the Matlab prompt). Note that there are other Matlab ODE solver such as `ode15s`, which are tailored to handle stiff differential equations (involving a very wide range of timescales). (Exercise: give this a try for  $\dot{x} = \log(\sqrt{x+1})$  for  $t_i = 0$ ,  $t_f = 10$  and  $x(0) = 0$ ).

Of course, for linear systems you can always use the closed-form solutions described above and the built-in matrix exponentiation routine `expm`.

### Linearization of nonlinear systems near equilibrium points

Consider an open-loop system with dynamics written in state-space form,

$$\dot{x} = f(x),$$

with  $x, f(x) \in R^n$ . The equilibrium points  $x_0$  satisfy

$$f(x_0) = 0,$$

meaning that in the absence of external perturbations the solution to the initial value problem with  $x(0) = x_0$  is simply  $x(t) = x_0$ .

We define the derivative of  $f$  as the square matrix

$$Df(\cdot) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}.$$

The notation here is meant to emphasize the fact that the derivative of  $f$  is still a function of  $x$ , now an  $n \times n$  matrix rather than a vector.

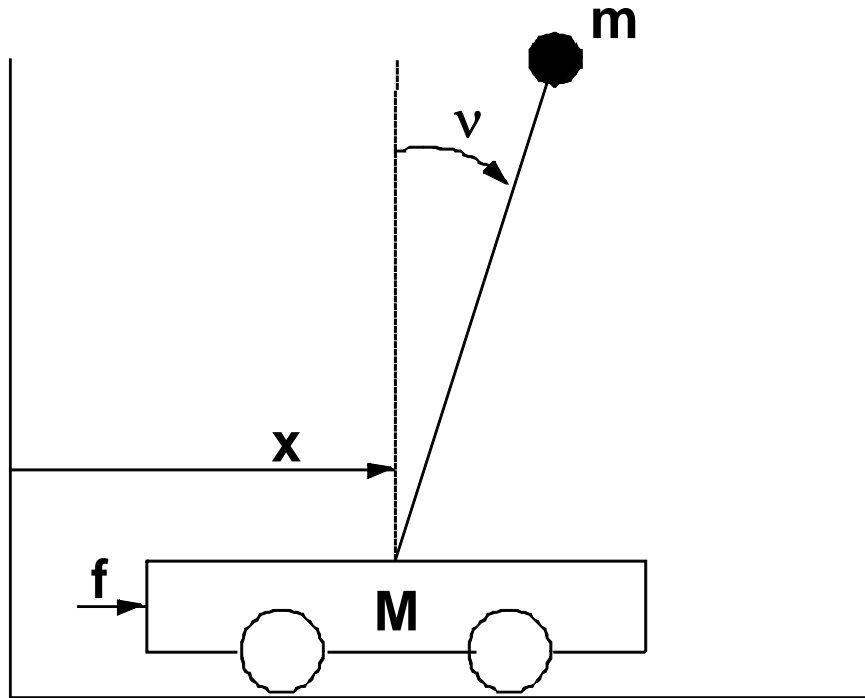
The derivative of  $f$  evaluated at an equilibrium point  $x_0$  can be used to define a new linear dynamical system called the linearization of  $f$  at  $x_0$  :

$$Df(x_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} |_{x_0} & \frac{\partial f_1}{\partial x_2} |_{x_0} & \cdots & \frac{\partial f_1}{\partial x_n} |_{x_0} \\ \frac{\partial f_2}{\partial x_1} |_{x_0} & \frac{\partial f_2}{\partial x_2} |_{x_0} & \cdots & \frac{\partial f_2}{\partial x_n} |_{x_0} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} |_{x_0} & \frac{\partial f_n}{\partial x_2} |_{x_0} & \cdots & \frac{\partial f_n}{\partial x_n} |_{x_0} \end{bmatrix} \equiv A,$$

$$\tilde{x} \equiv (x - x_0) \Rightarrow \frac{d}{dt} \tilde{x} = f(\tilde{x} + x_0) \approx A \tilde{x}.$$

Roughly speaking, we think of  $A \tilde{x}$  as being the leading term in a multivariable Taylor expansion of  $f(\cdot)$  about  $x_0$ . Note that for some functions  $f$ , the linearization  $A$  can be identically zero in which case one may say that the linearization is trivial or doesn't exist.





Let's consider the example of an inverted pendulum on a cart. With state-space variables  $x, \dot{x}, \theta, \dot{\theta}$  one can use  $F = ma$  or a Lagrangian approach to derive the nonlinear equations of motion

$$(M + m)\ddot{x} + ml\ddot{\theta} \cos(\theta) = -b\dot{x} + ml\dot{\theta}^2 \sin(\theta),$$

$$(J + ml^2)\ddot{\theta} + ml\ddot{x} \cos(\theta) = -mgl \sin(\theta),$$

where  $M, m, J, l,$  and  $b$  are parameters of the model ( $g$  is the local gravitational acceleration). In order to put these into state-space form one needs to do a fair bit of rearranging. Simply by moving terms to the left-hand sides we can get

$$\ddot{x} = -\frac{ml\ddot{\theta} \cos(\theta)}{M + m} - \frac{b\dot{x}}{M + m} + \frac{ml\dot{\theta}^2 \sin(\theta)}{M + m},$$

$$\ddot{\theta} = -\frac{ml\ddot{x} \cos(\theta)}{J + ml^2} - \frac{mgl \sin(\theta)}{J + ml^2},$$

but then we end up with an unwanted  $\ddot{\theta}$  in the expression for  $\frac{d}{dt}\dot{x}$  (and similarly an  $\ddot{x}$  in the expression for  $\frac{d}{dt}\dot{\theta}$ ). In order to get rid of this we can plug the second equation back into the first,

$$\begin{aligned} \ddot{x} &= -\frac{ml \cos(\theta)}{M + m} \left( -\frac{ml\ddot{x} \cos(\theta)}{J + ml^2} - \frac{mgl \sin(\theta)}{J + ml^2} \right) - \frac{b\dot{x}}{M + m} + \frac{ml\dot{\theta}^2 \sin(\theta)}{M + m} \\ &= \frac{m^2 l^2 \cos^2(\theta)}{(M + m)(J + ml^2)} \ddot{x} + \frac{m^2 g l^2 \sin(\theta) \cos(\theta)}{(M + m)(J + ml^2)} - \frac{b\dot{x}}{M + m} + \frac{ml\dot{\theta}^2 \sin(\theta)}{M + m}, \end{aligned}$$

from which we can get

$$\ddot{x} \left( 1 - \frac{m^2 l^2 \cos^2(\theta)}{(M+m)(J+ml^2)} \right) = \frac{m^2 g l^2 \sin(\theta) \cos(\theta)}{(M+m)(J+ml^2)} - \frac{b\dot{x}}{M+m} + \frac{ml\dot{\theta}^2 \sin(\theta)}{M+m},$$

$$\ddot{x} = \left( 1 - \frac{m^2 l^2 \cos^2(\theta)}{(M+m)(J+ml^2)} \right)^{-1} \left( \frac{m^2 g l^2 \sin(\theta) \cos(\theta)}{(M+m)(J+ml^2)} - \frac{b\dot{x}}{M+m} + \frac{ml\dot{\theta}^2 \sin(\theta)}{M+m} \right)$$

$$= (JM + Jm + mMI^2 + m^2 l^2 \sin^2(\theta))^{-1}$$

$$\times (m^2 g l^2 \sin(\theta) \cos(\theta) - (J + ml^2)b\dot{x} + (J + ml^2)ml\dot{\theta}^2 \sin(\theta)).$$

Likewise,

$$\ddot{\theta} = -\frac{ml \cos(\theta)}{J + ml^2} \left( -\frac{ml\ddot{\theta} \cos(\theta)}{M+m} - \frac{b\dot{x}}{M+m} + \frac{ml\dot{\theta}^2 \sin(\theta)}{M+m} \right) - \frac{mgl \sin(\theta)}{J + ml^2}$$

$$= \frac{m^2 l^2 \cos^2(\theta)}{(J + ml^2)(M+m)} \ddot{\theta} + \frac{ml \cos(\theta)b\dot{x}}{(J + ml^2)(M+m)} - \frac{m^2 l^2 \sin(\theta) \cos(\theta)\dot{\theta}^2}{(J + ml^2)(M+m)} - \frac{mgl \sin(\theta)}{J + ml^2},$$

$$\ddot{\theta} = \left( 1 - \frac{m^2 l^2 \cos^2(\theta)}{(J + ml^2)(M+m)} \right)^{-1} \left( \frac{ml \cos(\theta)b\dot{x}}{(J + ml^2)(M+m)} - \frac{m^2 l^2 \sin(\theta) \cos(\theta)\dot{\theta}^2}{(J + ml^2)(M+m)} - \frac{mgl \sin(\theta)}{J + ml^2} \right)$$

$$= \left( \frac{ml}{JM + Jm + mMI^2 + m^2 l^2 \sin^2(\theta)} \right) (\cos(\theta)b\dot{x} - ml \sin(\theta) \cos(\theta)\dot{\theta}^2 - (M+m)g \sin(\theta)).$$

At this point we are technically in state-space form.

Now we can compute the derivative of  $f$ , but let's skip the really messy terms in  $\frac{\partial}{\partial \theta}$ . In general,

$$Df = \begin{bmatrix} \frac{\partial f_x}{\partial x} & \frac{\partial f_x}{\partial \dot{x}} & \frac{\partial f_x}{\partial \theta} & \frac{\partial f_x}{\partial \dot{\theta}} \\ \frac{\partial f_{\dot{x}}}{\partial x} & \frac{\partial f_{\dot{x}}}{\partial \dot{x}} & \frac{\partial f_{\dot{x}}}{\partial \theta} & \frac{\partial f_{\dot{x}}}{\partial \dot{\theta}} \\ \frac{\partial f_{\theta}}{\partial x} & \frac{\partial f_{\theta}}{\partial \dot{x}} & \frac{\partial f_{\theta}}{\partial \theta} & \frac{\partial f_{\theta}}{\partial \dot{\theta}} \\ \frac{\partial f_{\dot{\theta}}}{\partial x} & \frac{\partial f_{\dot{\theta}}}{\partial \dot{x}} & \frac{\partial f_{\dot{\theta}}}{\partial \theta} & \frac{\partial f_{\dot{\theta}}}{\partial \dot{\theta}} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-(J+ml^2)b}{JM+Jm+mMI^2+m^2l^2 \sin^2(\theta)} & ? & \frac{2(J+ml^2)ml\dot{\theta} \sin(\theta)}{JM+Jm+mMI^2+m^2l^2 \sin^2(\theta)} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{mlb \cos(\theta)}{JM+Jm+mMI^2+m^2l^2 \sin^2(\theta)} & ? & -\frac{2m^2 l^2 \sin(\theta) \cos(\theta)\dot{\theta}}{JM+Jm+mMI^2+m^2l^2 \sin^2(\theta)} \end{bmatrix}.$$

At any equilibrium point with  $\theta = \pi$ ,

$$Df \mapsto \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-(J+ml^2)b}{JM+Jm+mMI^2} & ? & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-mlb}{JM+Jm+mMI^2} & ? & 0 \end{bmatrix}.$$

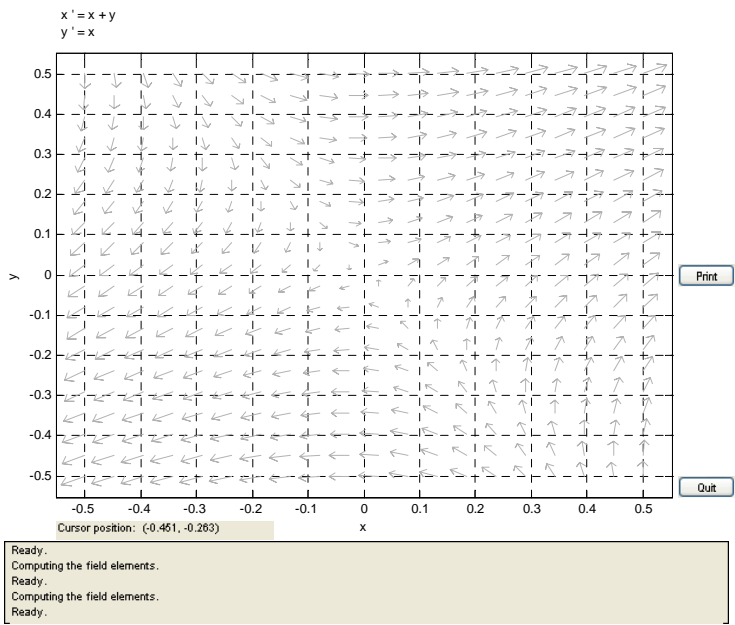
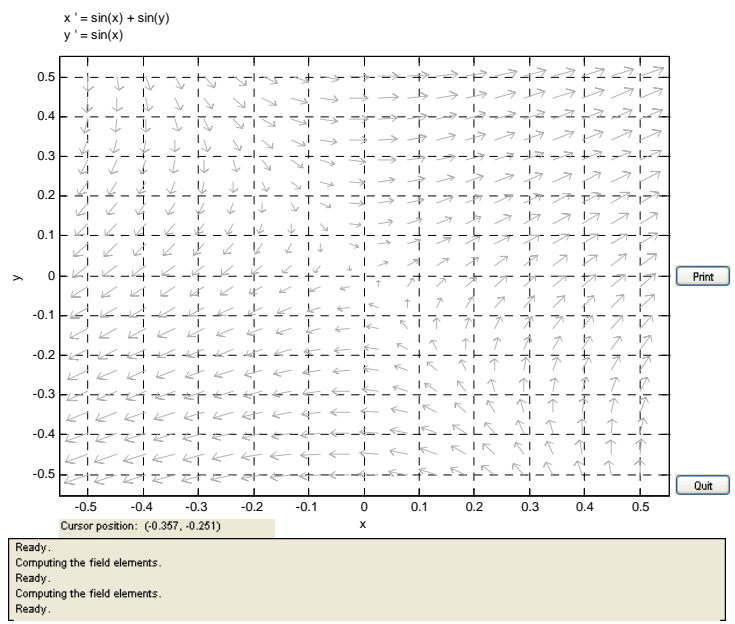
### Inferring stability from the linearized dynamics

If the eigenvalues of the linearization **all** have non-zero real part, they determine the local stability of the equilibrium point under the full nonlinear dynamics. If any eigenvalues have zero real part, the linearization is inconclusive.

-Lin/nonlin phase portraits for stable and unstable systems with non-zero eigenvalues

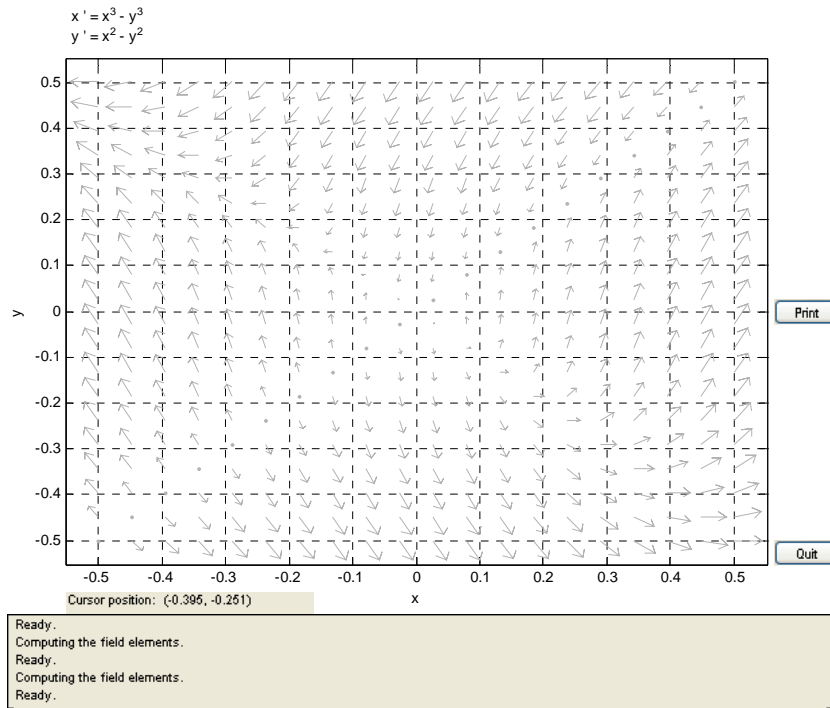
$$\dot{x} = \sin(x) + \sin(y), \quad \dot{y} = \sin(x),$$

$$Df = \begin{bmatrix} \cos(x) & \cos(y) \\ \cos(x) & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad \lambda = \frac{1}{2} \pm \frac{1}{2} \sqrt{5},$$

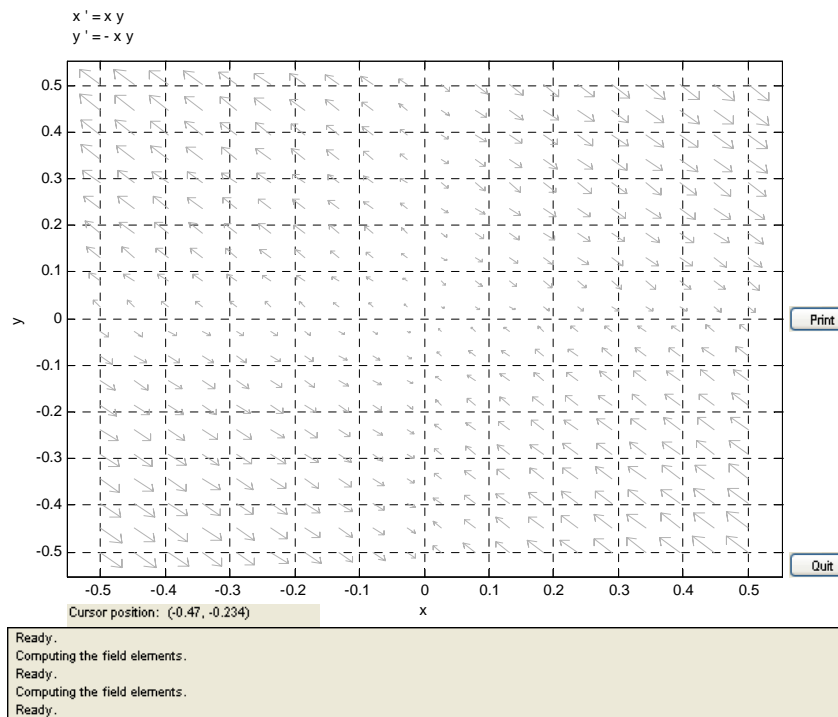


-Examples of systems for which the linearization vanishes

$$\dot{x} = x^3 - y^3, \quad \dot{y} = x^2 - y^2, \quad Df = \begin{bmatrix} 3x^2 & -3y^2 \\ 2x & -2y \end{bmatrix} \mapsto \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$



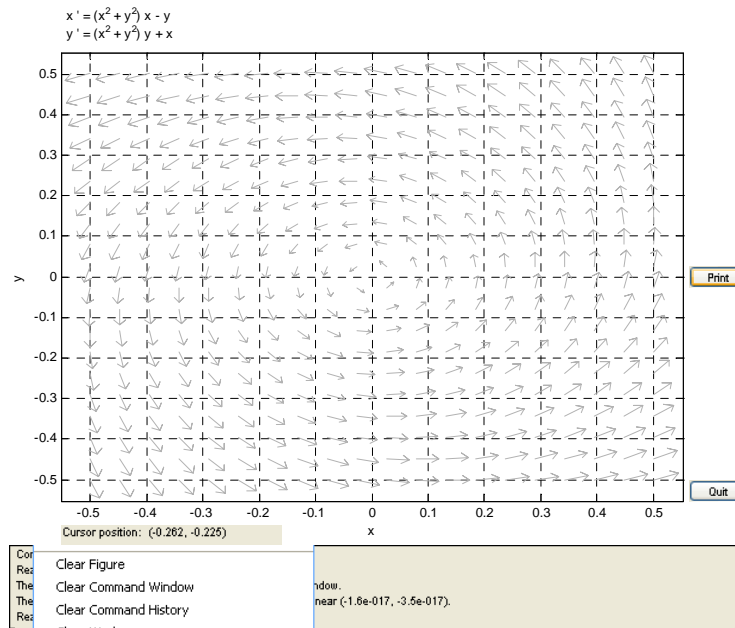
$$\dot{x} = xy, \quad \dot{y} = -xy, \quad Df = \begin{bmatrix} y & x \\ -y & -x \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$



-Phase portraits for stable and unstable systems with imaginary eigenvalues

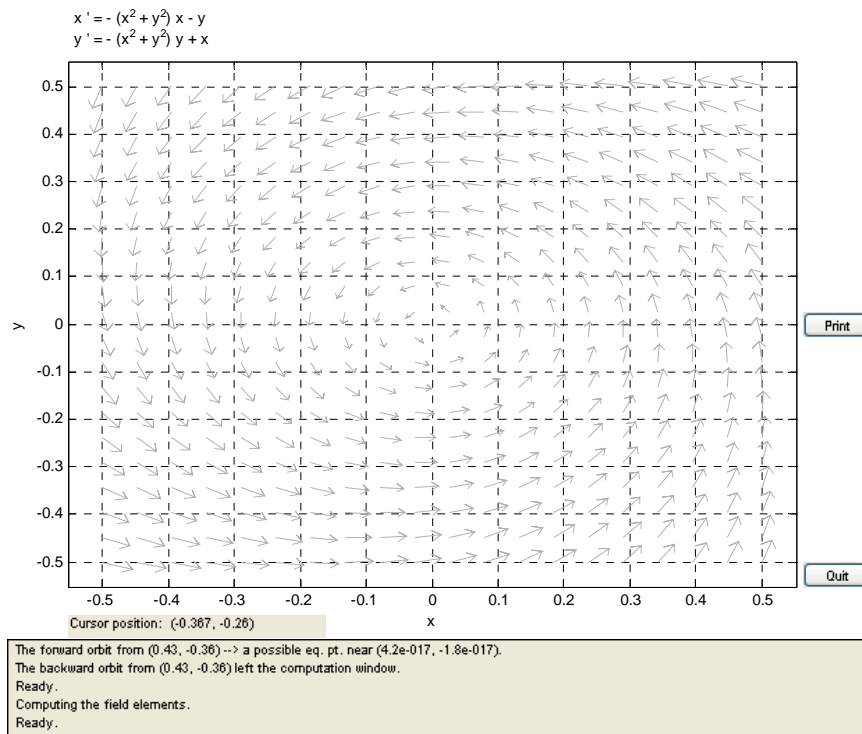
$$\dot{r} = r^3, \quad \dot{\theta} = 1, \quad \dot{x} = (x^2 + y^2)x - y, \quad \dot{y} = (x^2 + y^2)y + x,$$

$$Df = \begin{bmatrix} 3x^2 + y^2 & 2xy - 1 \\ 2xy + 1 & x^2 + 3y^2 \end{bmatrix} \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \lambda = \pm i,$$



$$\dot{r} = -r^3, \quad \dot{\theta} = 1, \quad \dot{x} = -(x^2 + y^2)x - y, \quad \dot{y} = -(x^2 + y^2)y + x,$$

$$Df = \begin{bmatrix} -3x^2 - y^2 & 2xy - 1 \\ -2xy + 1 & -x^2 - 3y^2 \end{bmatrix} \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \lambda = \pm i,$$



Exercises:

(a) Use linearization (or phase portraits if necessary) to determine the stability of the equilibrium points of the system

$$\ddot{x} = -4x^3 + 2x.$$

(side note: There is a theorem that says the equilibrium points of a Hamiltonian system are either saddles or centers.)

(b) Use linearization to determine the stability of the equilibrium points of

$$\dot{x} = y + x^2z + 3,$$

$$\dot{y} = -e^x + e^z,$$

$$\dot{z} = \sin(x + z).$$

### Linearization of an input-output system

Now that we get the basic idea let's consider an input-output system based on a familiar example, the damped pendulum. Here the state-space is 2D with variables  $\theta$  (which we choose as the angle of the pendulum from vertical with  $\theta = 0$  for the inverted configuration and  $\theta$  increasing clockwise) and  $\dot{\theta}$ , and equation of motion

$$mI\ddot{\theta} = mg\sin(\theta) - b\dot{\theta} + \tau,$$

where  $\tau \equiv \tau_0 u(t)$  is to be considered an input signal (external torque). In state-space form,

$$\frac{d}{dt} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \frac{g}{l} \sin(\theta) - \frac{b}{ml} \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\tau_0}{ml} \end{bmatrix} u,$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix},$$

where we are also choosing an output signal corresponding to the angle.

Clearly  $(\theta, \dot{\theta}) = (\pi, 0)$  is an equilibrium point of the open-loop dynamics. The derivative of the open-loop dynamics evaluated at this point is

$$D \begin{bmatrix} \dot{\theta} \\ \frac{g}{l} \sin(\theta) - \frac{b}{ml} \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} \cos(\theta) & -\frac{b}{ml} \end{bmatrix} \mapsto \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{b}{ml} \end{bmatrix},$$

hence the linearized dynamics may be written

$$\frac{d}{dt} \begin{bmatrix} \tilde{\theta} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{b}{ml} \end{bmatrix} \begin{bmatrix} \tilde{\theta} \\ \dot{\theta} \end{bmatrix},$$

where  $\tilde{\theta} \equiv \theta - \pi$ . Now as long as we are willing to restrict our attention to input signals  $u(t)$  that do not cause the system to stray far from this equilibrium point, we can approximate

$$\frac{d}{dt} \begin{bmatrix} \tilde{\theta} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{b}{ml} \end{bmatrix} \begin{bmatrix} \tilde{\theta} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\tau_0}{ml} \end{bmatrix} u,$$

$$\tilde{y} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\theta} \\ \dot{\theta} \end{bmatrix}$$

for the input-output system as well (where  $\tilde{y} \equiv y - \pi$ ). Let's look at some numerical examples to explore the range of validity of this approximation.

Our linearization is actually a second-order system with

$$\omega_0^2 = \frac{g}{l}, \quad \zeta = \sqrt{\frac{b^2}{4m^2gl}}.$$

For numerical convenience let's assume that we are underdamped ( $\zeta = 0.1$ ) and let us choose time units such that  $g/l = 1$  and set  $\tau_0 = ml$ . Then

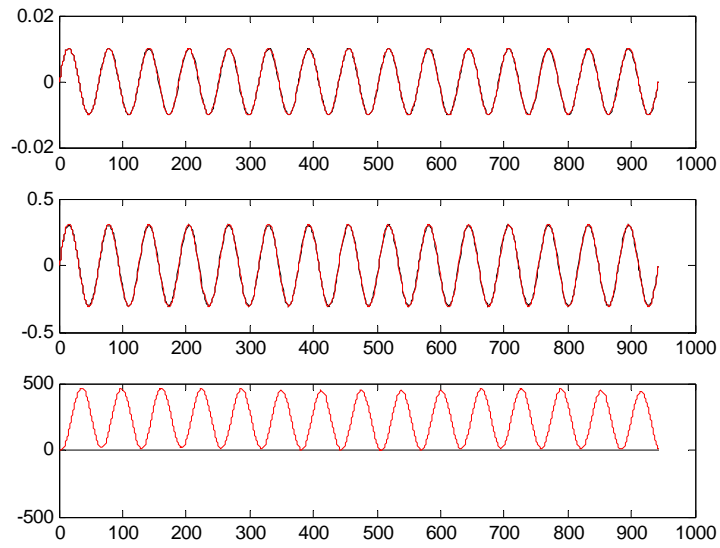
$$\ddot{\theta} = -\tilde{\theta} - 0.2\dot{\theta} + u$$

in the linearization and

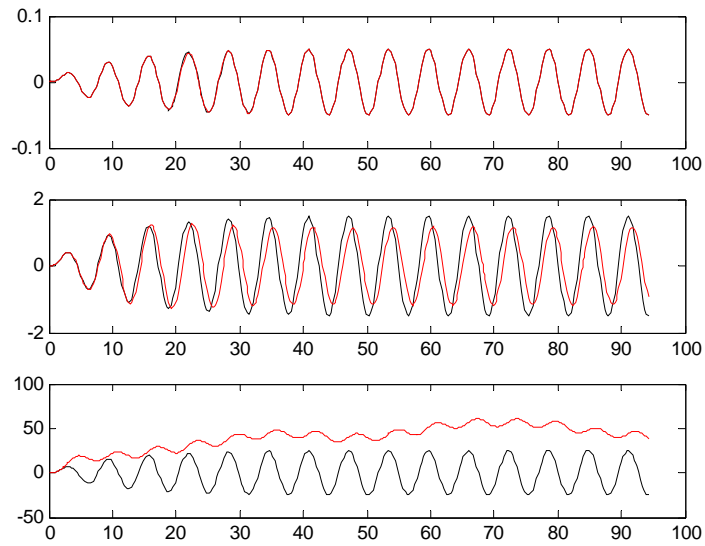
$$\ddot{\theta} = \sin(\tilde{\theta} + \pi) - 0.2\dot{\theta} + u$$

for the full nonlinear dynamics. Let  $u = A \sin(\omega t)$ .

For  $\omega = 0.1 \ll \omega_0$ , ( $A = 0.01, 0.3, 5$ ):

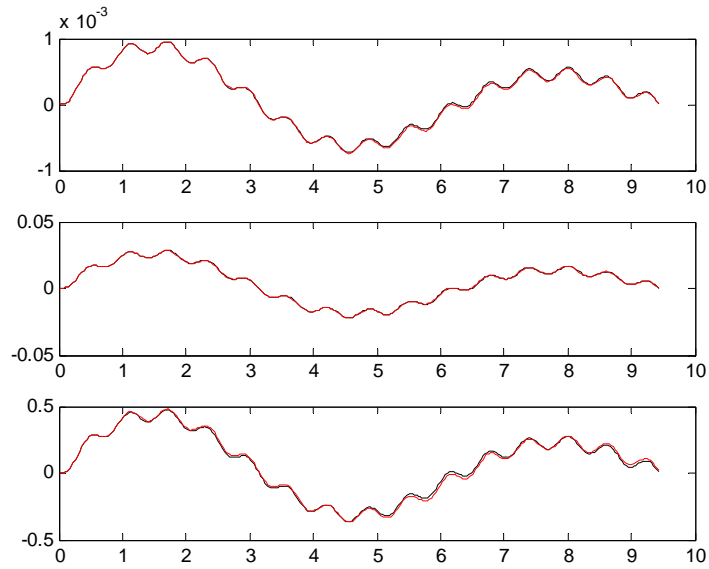


For  $\omega = 1 = \omega_0$ , ( $A = 0.01, 0.3, 5$ ):



For  $\omega = 10 \gg \omega_0$ , ( $A = 0.01, 0.3, 5$ ):





Hence it is apparent that the linearization is a good approximation only for  $A, \omega$  such that the steady-state solutions have small amplitude. It is often possible to use feedback to ensure that a system stays close to some equilibrium point, in which case the linearization can be a good approximation even for "large disturbances."