# Non-Archimedean Subjective Probabilities in Decision Theory and Games

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**Abstract**

To allow conditioning on counterfactual events, zero probabilities can be replaced by infinitesimal probabilities that range over a non-Archimedean ordered field. This paper considers a suitable minimal field that is a complete metric space. Axioms similar to those in Anscombe and Aumann (1963) and in Blume, Brandenburger and Dekel (1991) are used to characterize preferences which: (i) reveal unique non-Archimedean subjective probabilities within the field; and (ii) can be represented by the non-Archimedean subjective expected value of any real-valued von Neumann–Morgenstern utility function in a unique cardinal equivalence class, using the natural ordering of the field.

## 1 Introduction and Outline

Following the standard definition due to Kolmogorov (1933), probabilities are usually assumed to be real numbers. For events with positive probabilities, conditional probabilities are found by applying Bayes' updating rule. In game theory, however, as discussed in Section 2, there is often a need to discuss what would happen if a player deviated from a best response. Because the probability of such a deviation is supposed to be zero, game theorists are forced to consider probabilities conditional on zero probability events.

This paper will begin in Section 2 by briefly reviewing some of the main approaches game theorists and others have taken in attempting to

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escape from this and similar impasses.[1] The most interesting ideas appear to involve going beyond Kolmogorov's standard framework, and allowing conditional probabilities to be defined for all (non-empty) events. Some simple attempts founder, however, because they do not allow compound lotteries to be reduced uniquely and unambiguously to simple one-stage lotteries. Section 2 concludes with a simple example illustrating this difficulty.

A more complicated remedy is to introduce the full-blown apparatus of non-standard analysis. This allows probabilities in the form of positive infinitesimals, which are smaller than any positive real number. As discussed by Royden (1968), for instance, the existence of such infinitesimals entails violating the Archimedean axiom which characterizes the real line. Obviously, in non-standard analysis one can attach arbitrary infinitesimal probabilities to deviations from best responses. But Section 3 will argue that the set of all such positive infinitesimal probabilities is excessively rich and hard to interpret.

As discussed in Section 4, the issue has become what the appropriate range of allowable probabilities should be. Using a general non-Archimedean field for the range of allowable probability values, Section 4 sets out axioms similar to those devised by Anscombe and Aumann (1963) and, for non-Archimedean probabilities, by Blume, Brandenburger and Dekel (1991a). Most of these axioms can be given a consequentialist justification along the lines of Hammond (1988, 1998a, 1998b).

In Section 5 it is then proved that the axioms guarantee the existence of a unique cardinal equivalence class of real-valued von Neumann–Morgenstern utility functions (NMUFs) and also, except in the case of universal indifference, unique subjective probabilities belonging to the allowable non-Archimedean range. Then preferences have an expected utility representation, with expected utilities taking values in the non-Archimedean field. Moreover, preferred random consequences have expected utilities that are greater according to the natural ordering of this field.

One of the axioms used in Section 5 is a particular "non-Archimedean continuity" condition that the preference ordering must satisfy. In an attempt to justify it, Section 6 considers a particular non-Archimedean ordered field $\mathbb{R}(\epsilon)$, following Hammond (1994, 1998c). This is the smallest algebraic field that includes both the real line and also at least one positive "basic" infinitesimal $\epsilon$. However, in Section 6 the field $\mathbb{R}(\epsilon)$ will be extended to $\mathbb{R}^\infty(\epsilon)$ so that it becomes a complete metric space.

[1] For a fuller account, see Hammond (1994).

As shown in Section 7, the special field $\mathbb{R}^\infty(\epsilon)$ allows preferences over lotteries to satisfy a suitable extended continuity axiom which, in combination with a non-Archimedean version of a standard continuity axiom, implies the non-Archimedean continuity axiom used in Section 5. In this setting, moreover, applying the natural ordering of $\mathbb{R}^\infty(\epsilon)$ to expected utilities induces a preference ordering which corresponds to a familiar lexicographic expected utility criterion, with a real-valued von Neumann–Morgenstern utility function and non-Archimedean subjective probabilities.

# 2 Background

## 2.1 Counterfactuals in Game Theory

In a normal form game, a Nash equilibrium occurs when each player's strategy is a best response to others' equilibrium strategies. Also, a strategy is rationalizable iff it is a best response to "rationalizable expectations" attaching probability one to the event that all other players choose rationalizable strategies. Thus, the notion of best response is fundamental. Yet, to know whether a strategy is a best response, alternative strategies that are not best responses must be contemplated and the consequences of playing those strategies evaluated. This forces consideration of the counterfactual event that at least one other player chooses what is supposed to be an inferior response. In particular, in games with non-trivial extensive forms, other players may have the opportunity to observe that a supposedly inferior strategy has been played, while remaining unsure what this strategy is. Then the usual rule of calculating conditional probabilities by Bayesian updating is of no use because of the need to condition on a counterfactual event with prior probability zero.

## 2.2 Trembles

Game theorists have resorted to various *ad hoc* procedures to deal with this issue. In proper subgames, Selten's (1965, 1973) criterion of subgame perfection requires consideration of best responses in subgames that are not reached in equilibrium; however, a proper subgame can be analysed as a game in its own right, making it unnecessary in practice to apply Bayesian updating to events with probability zero. More challenging is the case when earlier moves give rise to an "improper subgame" of asymmetric information, where each player remembers any moves he or she has already made, but may be uncertain what moves other players have already chosen. Then, as in

Kreps and Wilson's (1982) theory of "sequential equilibrium," one wants to apply Bayesian updating in order to derive the relevant player's expectations at each information set of the improper subgame. But if the information set is supposed not to be reached when each player chooses best response strategies, what are appropriate expectations and the best response of the player who is to move at that information set?

In order to resolve such questions, Selten (1975) introduced the idea that players would choose their strategies "with a trembling hand," so that even inferior strategies would be chosen with (small) positive probability. In this way, every information set would be reached with positive probability, so Bayesian updating would always be well defined. A "trembling-hand perfect" equilibrium is the limit of Nash equilibria as the largest allowable probability or "tremble" attached to inferior strategies converges to zero. Myerson (1978) refined this concept by requiring in addition that trembles to worse inferior strategies should be much less likely than trembles to better inferior strategies.

## 2.3 Other Extended Probabilities

In their theory of sequential equilibrium, Kreps and Wilson (1982) brought in hierarchies of probability distributions. These allow an ordinary first level probability distribution, then a second level distribution over states (or strategy profiles) whose first level probability is zero, then (if necessary) third level probabilities over states whose first and second level probabilities are both zero, and so on. A definitive treatment of such hierarchies, and their application to derive proper equilibrium, can be found in Blume, Brandenburger and Dekel (1991a, b) — henceforth referred to as BBD.

Not much later, Myerson (1986) provided a rather more formal discussion which involved re-discovering the conditional probability spaces of Rényi (1955, 1956). Like Myerson, I prefer the more evocative name "complete conditional probability systems" (CCPSs). Myerson also linked CCPSs to trembles, in the sense of vanishing sequences of positive probabilities. Earlier Rényi and also Császár (1955) had already related CCPSs to hierarchies of probability distributions. Another similar approach was McLennan's (1989a, b) use of "conditional systems" in the form of logarithmic likelihood ratio functions that are allowed to have values $\pm\infty$ as well as all real values.

In earlier work (Hammond, 1994), I have shown that Rényi's and McLennan's formulations are equivalent, at least for finite state spaces of the kind that arise naturally in games with a finite number of players who each have a finite strategy set. Also equivalent are hierarchies of probability distributions when the different distributions are required to have pairwise disjoint supports. Somewhat more general are hierarchies of distributions whose supports may overlap, as considered by BBD.

### 2.4 Non-Reduction of Compound Lotteries

For dealing with counterfactuals, Rényi's CCPS formulation has considerable intuitive appeal. It simply requires $P(E'|E)$ to be specified whenever $E' \subset E$, even if event $E$ has probability zero. The obvious interpretation is: "I believe that $E$ cannot occur, but in the extremely unlikely event that it does, then my new beliefs about $E'$ will be described by $P(E'|E)$." Such beliefs seem to be exactly what is needed for game theory when $E$ is the event that a player deviates from a presumed best response.

A crucial hypothesis of von Neumann and Morgenstern is that it should always be enough to analyse the normal form of a game, because the outcome should be invariant to alterations in the extensive form that leave the normal form unaffected. If one is content to consider games in extensive form, and not impose this normal form invariance hypothesis, then it may well be unnecessary to go beyond the elegant device of CCPSs. But the invariance hypothesis is actually fundamental to orthodox game theory, and even more to its foundations in consequentialist single person decision theory. So we should see how far it is possible to proceed within the normal form framework.

For this invariance hypothesis to hold, lotteries whose prizes are tickets for other lotteries must be reducible to equivalent simple lotteries merely by multiplying probabilities. That is, the fundamental and often implicit *reduction of compound lotteries* postulate of single-person decision theory must hold. Unfortunately, where CCCPs are concerned, this reduction property may fail because there can be infinitely many different ways of compounding two lotteries which both involve zero probability events. A simple example to show this is illustrated in Fig. 1.

In this tree, the conditional probabilities associated with the first chance move at the initial node $n_0$ are assumed to satisfy $P(\{n\}|\{\,c, n\,\}) = 1$, implying that $P(\{\,a, b\,\}|\{\,a, b, c\,\}) = 1$, whereas the conditional probabilities associated with the second chance move at node $n$ are assumed to satisfy $P(\{a\}|\{\,a, b\,\}) = 1$. Of course, then the other conditional probabilities given $\{\,c, n\,\}$ and $\{\,a, b\,\}$ must be 0. Compounding these conditional probabilities evidently implies that

$$P(\{a\}|\{\,a, b, c\,\}) = P(\{a\}|\{\,a, b\,\})\,P(\{\,a, b\,\}|\{\,a, b, c\,\}) = 1$$

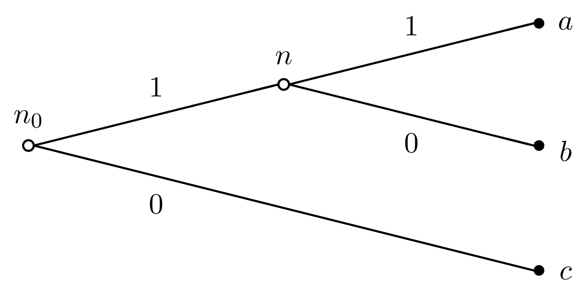


Figure 1: Non-Reduction of Compound Conditional Probabilities

Also, because $P(\{a\}|\{a,b,c\}) = P(\{a\}|\{a,c\})\,P(\{a,c\}|\{a,b,c\}) = 1$ it must be true that $P(\{a\}|\{a,c\}) = 1$. Of course, then all other conditional probabilities given the non-trivial events $\{a,b\}$, $\{a,c\}$ and $\{a,b,c\}$ must be 0. But so far nothing determines the relative likelihoods of the two different "trembles" which result in states $b$ and $c$ respectively. Accordingly, the two other conditional probabilities $P(\{b\}|\{b,c\})$ and $P(\{c\}|\{b,c\})$ could be any real numbers in the interval $[0,1]$ which sum to 1. Hence, reduction of compound lotteries can only occur in a richer space allowing more information to be provided about the relative likelihood of the two trembles.

# 3 Non-Archimedean Probabilities

## 3.1 Infinitesimals

Selten (1975) and many successors have modelled a tremble as a vanishing sequence of positive probabilities — or, equivalently, as vanishing perturbations of the players' (expected) payoff functions. Such vanishing sequences are obviously closely related to the "infinitesimals" used in the early development of "infinitesimal calculus" by Leibniz and many successors. Formally, a positive infinitesimal is smaller than any positive rational number and so, because the rationals are dense in the reals, smaller than any positive real number. This obviously violates the "Archimedean" property of the real line, requiring that for every $r > 0$, no matter how small, there should exist an integer $n$ such that $r\,n > 1$. So infinitesimals are only allowed in suitable non-Archimedean structures. Evidently, infinitesimals are not real in the mathematical sense.

Such infinitesimals have become incorporated into modern mathematics as a result of the development of "non-standard analysis" by Abraham Robinson and others. They have also been used in utility theory by Chipman

(1960, 1971a, b), Richter (1971), Skala (1975), etc. As for non-Archimedean probabilities, they were briefly mentioned by Chernoff (1954) and then by BBD. They appeared more prominently in an earlier unpublished paper by Blume on his own. Finally, LaValle and Fishburn (1996) have recently investigated several variant forms of non-Archimedean expected utility, especially those involving "matrix" probabilities.

### 3.2 Non-Standard Analysis

One way of introducing non-Archimedean probabilities that this earlier work suggests is to admit all members of $^*\mathbb{R}$, the "hyperreal" line. This allows one to use all the results of non-standard analysis, especially the "transfer principle" establishing a correspondence between results in standard analysis and results in non-standard analysis concerning so-called "internal" mathematical objects like sets, functions, binary relations, etc. As Anderson (1991, p. 2161) puts it, "any theorem about the standard world which has a non-standard proof is guaranteed to have a standard proof." Perhaps considering all of $^*\mathbb{R}$ is what we need to do. However, $^*\mathbb{R}$ is an incredibly complicated space many of whose members do not easily meet our intuitive notions of infinitesimal or of vanishingly small probability.

In fact, the definition of $^*\mathbb{R}$ involves Zorn's Lemma or the axiom of choice in a rather essential and awkward way in order to construct a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, the set of natural numbers or positive integers. The existence of such a free ultrafilter is equivalent to the existence of a finitely additive measure $\mu : \mathbb{N} \to \{0, 1\}$. That is, $\mu$ must give every set measure 0 or 1. In fact, it is required that every finite set be given measure 0, implying that every co-finite set has measure 1. The ultrafilter $\mathcal{U}$ then consists of all sets having measure 1. Note that if $S_i$ ($i = 1$ to $m$) is any partition of $\mathbb{N}$ into $m$ disjoint sets, then exactly one set of the finite collection satisfies $\mu(S_i) = 1$; all other sets of the collection have measure 0.

Let $\mathbb{R}^\infty$ denote the Cartesian product space of all infinite sequences of real numbers, with typical member $\mathbf{r} = \langle r_n \rangle_{n \in \mathbb{N}}$. After the measure $\mu$ or the equivalent free ultrafilter $\mathcal{U}$ has been constructed, define the equivalence relation $\sim$ on $\mathbb{R}^\infty$ so that for every pair $\mathbf{a}, \mathbf{b} \in \mathbb{R}^\infty$ one has

$$\mathbf{a} \sim \mathbf{b} \iff \mu(\{ n \in \mathbb{N} \mid a_n = b_n \}) = 1$$

Hence, $\mathbf{a}$ and $\mathbf{b}$ are regarded as equivalent iff their corresponding elements $a_n$ and $b_n$ agree on a subset of $\mathbb{N}$ having measure 1. Now $^*\mathbb{R}$ can be defined as the quotient space $\mathbb{R}^\infty / \sim$ of equivalence classes in $\mathbb{R}^\infty$. Let $^*\mathbf{r}$ denote the unique equivalence class in $^*\mathbb{R}$ containing the element $\mathbf{r} \in \mathbb{R}^\infty$.

The space ${}^*\mathbb{R}$ is an obvious extension of $\mathbb{R}$ because each real number $r \in \mathbb{R}$ can be associated with the unique equivalence class ${}^*(r\,\mathbf{1})$ of all sequences $\mathbf{r}$ such that $r_n = r$ on a set of measure 1. Let ${}^*r$ denote this equivalence class.

Note next that any two elements ${}^*\mathbf{a}$ and ${}^*\mathbf{b}$ of ${}^*\mathbb{R}$ have:

1. a sum ${}^*\mathbf{a} + {}^*\mathbf{b} = {}^*(\mathbf{a} + \mathbf{b})$ where $\mathbf{a} + \mathbf{b} = \langle a_n + b_n \rangle_{n \in \mathbb{N}}$;

2. a difference ${}^*\mathbf{a} - {}^*\mathbf{b} = {}^*(\mathbf{a} - \mathbf{b})$ where $\mathbf{a} - \mathbf{b} = \langle a_n - b_n \rangle_{n \in \mathbb{N}}$;

3. a product ${}^*\mathbf{a} \cdot {}^*\mathbf{b} = {}^*(\mathbf{a} \cdot \mathbf{b})$ where $\mathbf{a} \cdot \mathbf{b} = \langle a_n \cdot b_n \rangle_{n \in \mathbb{N}}$;

4. and also a quotient ${}^*\mathbf{a}/{}^*\mathbf{b} = {}^*(\mathbf{a}/\mathbf{b})$ where $\mathbf{a}/\mathbf{b} = \langle a_n/b_n \rangle_{n \in \mathbb{N}}$, provided of course that ${}^*\mathbf{b} \neq {}^*0$, implying that $b_n \neq 0$ for all $n$ after changing $\mathbf{b}$ if necessary on a set of measure 0.

Thus, ${}^*\mathbb{R}$ has the structure of an algebraic field with zero element ${}^*0$ and unit element ${}^*1$. Note too that for each pair $\mathbf{a}, \mathbf{b} \in \mathbb{R}^\infty$, the sets $\{ n \mid a_n > b_n \}$, $\{ n \mid a_n = b_n \}$ and $\{ n \mid a_n < b_n \}$ form a partition of $\mathbb{N}$. Hence, exactly one of these three sets must have measure 1. So there is a total ordering ${}^*\!>$ on ${}^*\mathbb{R}$ defined by

$$ {}^*\mathbf{a} \ \ {}^*\!> \ \ {}^*\mathbf{b} \iff \mu(\{ n \in \mathbb{N} \mid a_n > b_n \}) = 1 $$

This makes ${}^*\mathbb{R}$ an ordered field.

The field ${}^*\mathbb{R}$ violates the Archimedean axiom because it has positive infinitesimal elements, for example $\langle 1/n \rangle_{n \in \mathbb{N}}$, which are less than any positive real ${}^*r$. The reciprocals of these positive infinitesimal elements are positive infinite elements of ${}^*\mathbb{R}$, for example $\langle n \rangle_{n \in \mathbb{N}}$, which are greater than any (finite) real ${}^*r$.

The trouble with this ultrafilter construction is that the associated finitely additive measure $\mu$ has strongly counter-intuitive properties. It has already been remarked how, in a partition of $\mathbb{N}$ into finitely many subsets, exactly one set has measure 1. Hence, for each $k \in \mathbb{N}$ there must exist an integer $m_k \in \{ 1, 2, \ldots, k \}$ for which the set $S_k := \{ m_k + n\,k \mid n = 0, 1, 2, \ldots \}$ has measure 1. This is true even though $S_k$ becomes arbitrarily sparse for $k$ large enough. Thus, while non-standard analysis is no doubt a useful device for proving theorems in pure mathematics via the transfer principle, I venture to suggest that for applications to decision and game theory, we need a more intuitively appealing framework.

# 4 Non-Archimedean Expected Utility

## 4.1 What Range of Probabilities?

Originally probabilities were rational numbers, usually with small denominators. Every monetary bet that has ever been made and settled on the precise agreed terms has involved a rational odds ratio. More sophisticated mathematical models of probability involve continuous distributions, which necessitate irrational but real probabilities. To date, real-valued probability measures have served us rather well. Yet the previous discussion suggests the need for infinitesimal probabilities in game theory. This forces us to face the question: How rich should the range of allowable probabilities be?

My provisional answer is: As small as possible, provided that some essential requirements are met. If the transfer principle of non-standard analysis is not an essential requirement, then there is no good reason to use every possible "hyperreal" between 0 and 1, as non-standard analysis seems to require. But the range must include some infinitesimal elements, it seems, as well as all real numbers in the interval $[0, 1]$.

Probabilities need to be added, because of finite additivity. They must also be multiplied in order to compound lotteries. And they should be divided in order to calculate conditional probabilities by Bayesian updating. This makes it natural to impose the algebraic structure of a field — or, more precisely, an appropriate positive cone within such a field.[2] Finally, since the expectation of a real-valued von Neumann–Morgenstern utility function lies within the same field, and expected utilities should represent a complete preference ordering, there should be a linear or total order on the field.[3]

## 4.2 General Non-Archimedean Ordered Fields

An *ordered field* $\langle \mathbb{F}, +, \cdot, 0, 1, > \rangle$ is a set $\mathbb{F}$ together with:

1. the two algebraic operations $+$ (addition) and $\cdot$ (multiplication);

2. the two corresponding identity elements 0 and 1;

3. the binary relation $>$ which is a linear order on $\mathbb{F}$ satisfying $1 > 0$.

[2] Rather counter-intuitively, some calculations in quantum mechanics involve negative probabilities at intermediate steps, though these are always later cancelled out by the addition of larger positive probabilities (Feynman, 1987). Such negative probabilities have no role in decision or game theory, however.

[3] A binary relation $>$ is a *linear* (or total) order on a domain $X$ if it is transitive and, for every disjoint pair $x, y \in X$, one has either $x > y$ or $y > x$.

Moreover, the set $\mathbb{F}$ must be closed under the two algebraic operations. The usual properties of real number arithmetic also have to be satisfied — i.e., addition and multiplication both have to be commutative and associative, the distributive law must hold, and every element $x \in \mathbb{F}$ must have both an additive inverse $-x$ and a multiplicative inverse $1/x$, except that $1/0$ is undefined. The order $>$ must be such that $y > z \iff y - z > 0$. Also, the set $\mathbb{F}_+$ of positive elements in $\mathbb{F}$ must be closed under both addition and multiplication. Both the real line and the rationals are important and obvious examples of ordered fields.

Any ordered field $\mathbb{F}$ has positive integer elements $n = 1, 2, \ldots$ defined as the sums $n = 1 + 1 + \ldots + 1$ of $n$ copies of the element $1 \in \mathbb{F}$. Then $\mathbb{F}$ is said to be "Archimedean" if, given any $x \in \mathbb{F}_+$, there exists such a positive integer $n$ for which $n\,x > 1$. For any $x \in \mathbb{F}$, let $|x|$ denote $x$ if $x \geq 0$ and $-x$ if $x < 0$. Say that any $x \in \mathbb{F} \setminus \{0\}$ is *infinitesimal* iff $|x| < 1/n$ for every (large) positive integer $n$. Evidently, a field is Archimedean iff it has no infinitesimal elements. Say that $x$ is *finite* if $|x| < n$ for some (large enough) integer $n$. Any $x \in \mathbb{F}$ is said to be *infinite* iff it is not finite. Any non-zero $x \in \mathbb{F}$ is therefore infinitesimal iff $1/x$ is infinite.

In the rest of the paper, $\mathbb{F}$ will denote any non-Archimedean ordered field that extends the real line $\mathbb{R}$. Then, for any positive $r \in \mathbb{R}$, there exists an ordinary positive integer $n \in \mathbb{N}$ satisfying $1/n < r$. Any positive infinitesimal $\epsilon \in \mathbb{F}_+$ must therefore satisfy $\epsilon \leq 1/n < r$, and so must be smaller than any positive real number. So the set of all positive infinitesimals is bounded above by any positive real number, but not by any infinitesimal. It follows that there is no least upper bound. In fact, as pointed out by Royden (1968), if any ordered field contains all the rational numbers and has the property that every bounded set includes both a supremum and an infimum, then the field is isomorphic to the real line and so is Archimedean.

Also, in the ensuing analysis, it will be convenient to use the notation $(0,1)_{\mathbf{F}}$ to indicate the interval $\{\, x \in \mathbb{F} \mid 0 < x < 1 \,\}$. This notation helps to distinguish the non-Archimedean interval from the usual real interval $(0,1)$, which will often be denoted by $(0,1)_{\mathbf{R}}$.

### 4.3 Consequences and States of the World

Let $Y$ be a fixed domain of possible *consequences*, and $S$ a fixed finite set of possible *states of the world*. No probability distribution over $S$ is specified. An *act*, according to Savage (1954), is a mapping $a : S \to Y$ specifying what consequence results in each possible state. Inspired by the Arrow (1953, 1964) and Debreu (1959) device of "contingent" securities or commodities in general equilibrium theory, I refer instead of *contingent consequence functions*, or CCFs for short. Also, each CCF will be considered as a list $y^S = \langle y_s \rangle_{s \in S}$ of contingent consequences in the Cartesian product space $Y^S := \prod_{s \in S} Y_s$, where each set $Y_s$ is a copy of the consequence domain $Y$.

Anscombe and Aumann (1963) allowed subjective probabilities for the outcomes of "horse lotteries" or CCFs to be inferred from expected utility representations of preferences over compounds of horse and "roulette lotteries". Formally, Savage's (1954) framework is extended to allow preferences over not only CCFs of the form $y^S \in Y^S$, but also over $\Delta(Y^S)$, the space of all (finitely supported) simple roulette lotteries defined on $Y^S$. Each $\lambda^S \in \Delta(Y^S)$ specifies a real number $\lambda^S(y^S)$ as the objective probability that the CCF is $y^S \in Y^S$. This implies that the collection of random variables $y_s$ $(s \in S)$ has a multivariate distribution with probabilities $\lambda^S(y^S)$.

This paper considers instead the space $\Delta(Y^S; \mathbb{F})$ of non-Archimedean $\mathbb{F}$-valued simple lotteries on $Y^S$. Each $\lambda^S \in \Delta(Y^S; \mathbb{F})$ satisfies $\lambda^S(y^S) = 0$ iff $y^S$ is outside the finite *support* $F \subset Y^S$ of $\lambda^S$. On $F$ each probability $\lambda^S(y^S) \in \mathbb{F}_+$, and $\sum_{y^S \in F} \lambda^S(y^S) = 1$. That is, $\lambda^S(y^S) > 0$ for all $y^S \in Y^S$, even though $\lambda^S(y^S)$ could be infinitesimal.

### 4.4 Axioms for Objective Expected Utility

Anscombe and Aumann directly assumed expected utility maximization for roulette lotteries, and then imposed extra conditions for preferences over horse lotteries guaranteeing the existence of subjective probabilities. BBD, like Fishburn (1970) and many others, laid out axioms implying expected utility maximization rather than assuming it directly. Apart from being in an obvious sense more fundamental, here such an approach is essential because non-Archimedean expected utility maximization should be deduced before non-Archimedean subjective probabilities are inferred. Accordingly, I assume:

(O) *Ordering.* There exists a (reflexive, complete and transitive) preference ordering $\succsim^S$ on $\Delta(Y^S; \mathbb{F})$.

(I*) *Strong Independence.* If $\lambda^S, \mu^S, \nu^S \in \Delta(Y^S; \mathbb{F})$ and $\alpha \in (0,1)_{\mathbf{F}}$, then

$$\lambda^S \succsim^S \mu^S \iff \alpha\lambda^S + (1-\alpha)\nu^S \succsim^S \alpha\mu^S + (1-\alpha)\nu^S$$

(NAC) *Non-Archimedean Continuity.* For all lotteries $\lambda^S, \mu^S, \nu^S \in \Delta(Y^S; \mathbb{F})$ satisfying $\lambda^S \succ^S \mu^S$ and $\mu^S \succ^S \nu^S$, there exists $\alpha \in (0,1)_{\mathbf{F}}$ such that $\alpha\lambda^S + (1-\alpha)\nu^S \sim^S \mu^S$.

Of these, axioms (O) and (I*) are obvious extensions to $\mathbb{F}$-valued probability distributions in $\Delta(Y^S;\mathbb{F})$ of standard conditions for real-valued probability distributions in $\Delta(Y^S)$. Even the real-valued version of the third axiom (NAC) has appeared as an assumption in the literature. However, when probabilities are real-valued, an equivalent condition is usually derived from more fundamental continuity assumptions on preferences or behaviour — see, for example, Hammond (1998a, b) for further discussion. For this reason, condition (NAC) is discussed further in the special framework of Sections 6 and 7.

The following two Lemmas and their proofs are simple adaptations to non-Archimedean probabilities of results which are familiar for real-valued probabilities. Accordingly, the proofs will not be provided here. For details see, for example, Fishburn (1970) or Hammond (1998a).

**Lemma 1** *Suppose that axioms (O) and (I*) are satisfied on* $\Delta(Y^S;\mathbb{F})$*. Then, for any pair of lotteries* $\lambda^S, \mu^S \in \Delta(Y^S;\mathbb{F})$ *with* $\lambda^S \succ^S \mu^S$ *and any* $\alpha', \alpha'' \in (0,1)_{\mathbf{F}}$ *with* $\alpha' > \alpha''$*, one has*

$$\lambda^S \succ^S \alpha'\lambda^S + (1-\alpha')\mu^S \succ^S \alpha''\lambda^S + (1-\alpha'')\mu^S \succ^S \mu^S$$

Say that the the utility function $U^S : \Delta(Y^S;\mathbb{F}) \to \mathbb{F}$ *represents* $\succsim^S$ on $\Delta(Y^S;\mathbb{F})$ if for every pair of lotteries $\lambda^S, \mu^S \in \Delta(Y^S;\mathbb{F})$ one has

$$\lambda^S \ \succsim^S \ \mu^S \iff U^S(\lambda^S) \geq U^S(\mu^S)$$

Say also that $U^S$ satisfies the *non-Archimedean mixture preservation* condition (NAMP) provided that, whenever $\alpha \in (0,1)_{\mathbf{F}}$, then

$$U^S(\alpha\lambda^S + (1-\alpha)\mu^S) = \alpha\, U^S(\lambda^S) + (1-\alpha)\, U^S(\mu^S)$$

Note how property (NAMP) implies that for every $\lambda^S \in \Delta(Y^S;\mathbb{F})$ one can write $U^S(\lambda^S)$ in the *expected utility* form

$$U^S(\lambda^S) = \sum\nolimits_{y^S \in Y^S} \lambda^S(y^S)\, v(y^S)$$

where $v(y^S) := U^S(1_{y^S})$ is the utility of the degenerate lottery $1_{y^S}$ which attaches probability 1 to the particular CCF $y^S \in Y^S$.

Finally, say that the two functions $U^S, V^S : \Delta(Y^S;\mathbb{F}) \to \mathbb{F}$ are *cardinally equivalent* provided there exist constants $\rho \in \mathbb{F}_+$ and $\delta \in \mathbb{F}$ such that $V^S(\lambda^S) = \delta + \rho U^S(\lambda^S)$ for all $\lambda^S \in \Delta(Y^S;\mathbb{F})$.

**Lemma 2** *Suppose that the three axioms (O), (I*), and (NAC) are all satisfied on* $\Delta(Y^S;\mathbb{F})$*. Then there exists a unique cardinal equivalence class of utility functions* $U^S : \Delta(Y^S;\mathbb{F}) \to \mathbb{F}$ *which represent* $\succsim^S$ *and satisfy (NAMP).*

# 5 Non-Archimedean Subjective Probability

## 5.1 The Non-Archimedean SEU Hypothesis

The first aim of this section is to provide sufficient conditions like those of Anscombe and Aumann for the preference ordering $\precsim^S$ on $\Delta(Y^S; \mathbb{F})$ to have a *subjective expected utility* (SEU) representation which, for each $\lambda^S \in \Delta(Y^S; \mathbb{F})$, takes the form

$$U^S(\lambda^S) = \sum\nolimits_{y^S \in Y^S} \lambda^S(y^S) \sum\nolimits_{s \in S} p_s \, v(y_s) \tag{1}$$

for a unique cardinal equivalence class of NMUFs $v : Y \to \mathbb{F}$, and for suitable unique *subjective probabilities* $p_s \in \mathbb{F}_+$ satisfying $\sum_{s \in S} p_s = 1$. That is, the subjective probability distribution $p = \langle p_s \rangle_{s \in E}$ must belong to $\Delta^0(S; \mathbb{F})$, the set of all distributions having support equal to the whole of $S$. The difference from Anscombe and Aumann is that the subjective probabilities $p_s$ and utilities $v(y)$ may be non-Archimedean, as may the objective probabilities $\lambda^S(y^S)$ determining the lottery $\lambda^S \in \Delta(Y^S; \mathbb{F})$. In addition, the subjective probabilities $p_s$ must be positive rather than merely non-negative. This particular characterization of the preference ordering $\precsim^S$ will be called the *non-Archimedean SEU hypothesis*.

A second aim is to ensure that the utility function $v$ is real-valued, making use of a new state-independent continuity axiom (SIC). Then only the probabilities, and not the utilities, are allowed to be non-Archimedean.

## 5.2 Reversal of Order

In order to derive such subjective probabilities, Anscombe and Aumann added three more axioms to their basic hypothesis that "roulette lotteries" would be chosen to maximize objective expected utility.

For each $y \in Y$ and $s \in S$, define $Y^S_s(y) := \{ y^S \in Y^S \mid y_s = y \}$ as the set of CCFs yielding the particular consequence $y$ in state $s$. Then, given any lottery $\lambda^S \in \Delta(Y^S; \mathbb{F})$, any state $s \in S$ and any consequence $y \in Y$, let

$$\lambda_s(y) := \sum\nolimits_{y^S \in Y^S_s(y)} \lambda^S(y^S)$$

denote the *marginal probability* that $y$ occurs in $s$. Then the probabilities $\lambda_s(y)$ $(y \in Y)$ specify the *marginal distribution* in state $s \in S$.

The first of the three additional axioms I shall present is:

(RO) *Reversal of Order.* Whenever $\lambda^S, \mu^S \in \Delta(Y^S)$ have marginal distributions satisfying $\lambda_s = \mu_s$ for all $s \in S$, then $\lambda^S \sim \mu^S$.

This condition owes its name to the fact that there is indifference between: (i) having the roulette lottery $\lambda^S$ determine the random CCF $y^S$ before the horse lottery that resolves which state $s \in S$ and which ultimate consequence $y_s$ occur; and (ii) resolving the horse lottery first, before its outcome $s \in S$ determines which marginal roulette lottery $\lambda_s$ generates the ultimate consequence $y$.

In particular, let $\mu^S := \prod_{s \in E} \lambda_s$ denote the *product lottery* defined, for all $y^S = \langle y_s \rangle_{s \in S} \in Y^S$, by $\mu^S(y^S) := \prod_{s \in S} \lambda_s(y_s)$. Thus, the different random consequences $y_s$ ($s \in S$) become independently distributed. Then condition (RO) requires $\lambda^S$ to be treated as equivalent to $\mu^S$, whether or not the different consequences $y_s$ ($s \in S$) are correlated random variables when the joint distribution is $\lambda^S$. Only marginal distributions matter. So any $\lambda^S \in \Delta(Y^S)$ can be regarded as equivalent to the list $\langle \lambda_s \rangle_{s \in S}$ of corresponding marginal distributions. This has the effect of reducing the space $\Delta(Y^S)$ to the Cartesian product space $\prod_{s \in S} \Delta(Y_s)$, with $Y_s = Y$ for all $s \in S$.

### 5.3 The Sure Thing Principle

The second of Anscombe and Aumann's additional axioms concerns any event $E \subset S$, together with the product space $Y^E := \prod_{s \in E} Y_s$ of *contingent* CCFs taking the form $y^E = \langle y_s \rangle_{s \in E} \in Y^E$, and the existence of an associated *contingent preference ordering* $\succsim^E$. Here it is natural to assume that $\succsim^E$ is defined on $\Delta(Y^E; \mathbb{F})$, the space of non-Archimedean probability distributions, instead of only on $\Delta(Y^E)$, the space of real-valued probability distributions. So the second extra axiom becomes:

(STP) *Sure Thing Principle.* Given any event $E \subset S$, there exists a contingent preference ordering $\succsim^E$ on $\Delta(Y^E; \mathbb{F})$ satisfying

$$\lambda^E \succsim^E \mu^E \iff (\lambda^E, \nu^{S \setminus E}) \succsim (\mu^E, \nu^{S \setminus E})$$

for all $\lambda^E, \mu^E \in \Delta(Y^E; \mathbb{F})$ and $\nu^{S \setminus E} \in \Delta(Y^{S \setminus E}; \mathbb{F})$, where $(\lambda^E, \nu^{S \setminus E})$ denotes the combination of the conditional lottery $\lambda^E$ if $E$ occurs with $\nu^{S \setminus E}$ if $S \setminus E$ occurs, and similarly for $(\mu^E, \nu^{S \setminus E})$.

However, following an idea due originally to Raiffa (1961) and then used by BBD, it is easy to show that (STP) is implied by axioms (O), (I*) and (RO):

**Lemma 3** *Suppose that the three axioms (O), (I*), and (RO) are all satisfied on $\Delta(Y^S; \mathbb{F})$. Then so is (STP).*

**Proof:** Consider any event $E \subset S$ and also any lotteries $\lambda^E, \mu^E \in \Delta(Y^E; \mathbb{F})$, $\bar{\nu}^{S\setminus E} \in \Delta(Y^{S\setminus E}; \mathbb{F})$ satisfying $(\lambda^E, \bar{\nu}^{S\setminus E}) \succsim^S (\mu^E, \bar{\nu}^{S\setminus E})$. For any other lottery $\nu^{S\setminus E} \in \Delta(Y^{S\setminus E}; \mathbb{F})$, axioms (I*) and (RO) respectively imply that

$$\begin{aligned} \tfrac{1}{2}(\lambda^E, \nu^{S\setminus E}) + \tfrac{1}{2}(\lambda^E, \bar{\nu}^{S\setminus E}) \quad &\succsim^S \quad \tfrac{1}{2}(\lambda^E, \nu^{S\setminus E}) + \tfrac{1}{2}(\mu^E, \bar{\nu}^{S\setminus E}) \\ &\sim^S \quad \tfrac{1}{2}(\mu^E, \nu^{S\setminus E}) + \tfrac{1}{2}(\lambda^E, \bar{\nu}^{S\setminus E}) \end{aligned}$$

By transitivity of $\succsim^S$ and axiom (I*), one has $(\lambda^E, \nu^{S\setminus E}) \succsim^S (\mu^E, \nu^{S\setminus E})$. This confirms condition (STP), asserting the existence of a contingent preference ordering $\succsim^E$ on $\Delta(Y^E; \mathbb{F})$ that is independent of $\nu^{S\setminus E}$. ∎

Because of this result, (STP) will not be imposed as an axiom, but it will often be used in the ensuing proofs.

### 5.4 State Independence

The last of Anscombe and Aumann's axioms can now be stated. It relates to the fact that the NMUF $v : Y \to \mathbb{F}$ is independent of the state $s$. Now, for each $s \in S$, condition (STP) ensures the existence of a contingent preference ordering $\succsim^{\{s\}}$ on $\Delta(Y_s; \mathbb{F}) = \Delta(Y; \mathbb{F})$ that is represented by the expected value of $v(y_s)$, and so is independent of $s$. So the last axiom is:

(SI) *State Independence.* Given any state $s \in S$, the contingent preference ordering $\succsim^{\{s\}}$ over $\Delta(Y; \mathbb{F})$ is independent of $s$.

Let $\succsim^*$ denote this state-independent preference ordering. When $\succsim^S$ on $\Delta(Y^S; \mathbb{F})$ satisfies conditions (O), (I*) and (NAC), so too must $\succsim^*$ on $\Delta(Y; \mathbb{F})$, because of (STP).

### 5.5 Subjective Probabilities

The five axioms (O), (I*), (NAC), (RO), and (SI) are assumed throughout the following, as is condition (STP).

**Lemma 4** *(a) Suppose that $E \subset S$ is any event and that $\lambda^E, \mu^E \in \Delta(Y^E; \mathbb{F})$ satisfy $\lambda_s \succsim^* \mu_s$ for all $s \in E$. Then $\lambda^E \succsim^E \mu^E$.*

*(b) If in addition $\lambda^E \sim^E \mu^E$, then $\lambda_s \sim^* \mu_s$ for every state $s \in E$.*

**Proof:** There is an obvious proof by induction on $m$, the number of states in $E$. ∎

Suppose it were true that $\lambda \sim^* \mu$ for all pure roulette lotteries $\lambda, \mu \in \Delta(Y; \mathbb{F})$. Because $S$ is finite, Lemma 4 would then imply that $\lambda^S \sim^S \mu^S$ for all $\lambda^S, \mu^S \in \Delta(Y^S; \mathbb{F})$. However, the ordering $\precsim^S$ could then be represented by the trivial subjective expected utility function $\sum_{s \in E} p_s \, U^*(\lambda_s)$ for arbitrary subjective probabilities $p_s$ and any constant utility function $U^* : \Delta(Y; \mathbb{F}) \to \{c\} \subset \mathbb{F}$. So from now on, exclude the trivial case of universal indifference by assuming throughout that there exist two pure roulette lotteries $\bar{\lambda}, \underline{\lambda} \in \Delta(Y; \mathbb{F})$ with $\bar{\lambda} \succ^* \underline{\lambda}$.

The key idea of the following proof involves the $\mathbb{F}$-valued NMUF whose existence was claimed in Section 4.4. Because $\precsim^*$ satisfies conditions (O), (I*) and (NAC), Lemma 2 can be applied. Therefore, $\precsim^*$ can be represented by a non-Archimedean expected utility function $U^* : \Delta(Y) \to \mathbb{F}$ which is normalized so that

$$U^*(\underline{\lambda}) = 0 \quad \text{and} \quad U^*(\bar{\lambda}) = 1 \tag{2}$$

while also satisfying the mixture preservation property (NAMP).

Next, given any event $E \subset S$ and any lottery $\lambda \in \Delta(Y; \mathbb{F})$, let $\lambda 1^E$ denote the lottery in $\Delta(Y^E; \mathbb{F})$ whose marginal distribution in each state $s \in E$ is $\lambda_s = \lambda$, independent of $s$.

**Lemma 5** *The ordering* $\precsim^*$ *on* $\Delta(Y; \mathbb{F})$ *is represented by a utility function* $U^* : \Delta(Y; \mathbb{F}) \to \mathbb{F}$ *satisfying* $U^*(\lambda) = U^S(\lambda \, 1^S)$ *for all* $\lambda \in \Delta(Y; \mathbb{F})$.

**Proof:** Because $\precsim^S$ on $\Delta(Y^S; \mathbb{F})$ and $\precsim^*$ on $\Delta(Y; \mathbb{F})$ both satisfy conditions (O), (I*) and (NAC), Lemma 2 of Section 4.2 implies that they can be represented by normalized utility functions $U^S : \Delta(Y^S; \mathbb{F}) \to \mathbb{F}$ and $U^* : \Delta(Y; \mathbb{F}) \to \mathbb{F}$ satisfying (NAMP) and also

$$U^S(\underline{\lambda} \, 1^S) = 0, \quad U^S(\bar{\lambda} \, 1^S) = 1, \quad \text{and} \quad U^*(\underline{\lambda}) = 0, \quad U^*(\bar{\lambda}) = 1 \tag{3}$$

Next, Lemma 4(a) implies that $\lambda \precsim^* \mu \Longrightarrow \lambda \, 1^S \precsim^S \mu \, 1^S$. On the other hand, Lemma 4(b) implies that $\mu \succ^* \lambda \Longrightarrow \mu \, 1^S \succ \lambda \, 1^S$. Because $\precsim^*$ and $\precsim^S$ are complete orderings, the reverse implication $\lambda \, 1^S \precsim^S \mu \, 1^S \Longrightarrow \lambda \precsim^* \mu$ follows. Hence, $\lambda \, 1^S \precsim^S \mu \, 1^S \iff \lambda \precsim^* \mu$. So $U^S(\lambda \, 1^S)$ and $U^*(\lambda)$ must be cardinally equivalent functions of $\lambda$ on the domain $\Delta(Y; \mathbb{F})$. Because of the two normalizations in (3), the result follows immediately. ∎

Next, define the functions $g_s : \Delta(Y; \mathbb{F}) \to \mathbb{R}$ and constants $q_s$ (all $s \in S$) by

$$g_s(\lambda) := U^S(\underline{\lambda} \, 1^{S \setminus \{s\}}, \lambda) \quad \text{and} \quad q_s := g_s(\bar{\lambda}) \tag{4}$$

Evidently, because of the normalization (3), it must be true that

$$g_s(\underline{\lambda}) = 0 \tag{5}$$

**Lemma 6** *For each $\lambda^S \in \Delta(Y^S; \mathbb{F})$ one has*

$$U^S(\lambda^S) = \sum_{s \in S} q_s \, U^*(\lambda_s) \tag{6}$$

*where $q_s := U^S(\underline{\lambda}\, 1^{S \setminus \{s\}}, \bar{\lambda}) \in \mathbb{F}_+$ for all $s \in S$, implying that $\sum_{s \in S} q_s = 1$.*

**Proof:** (cf. Fishburn, 1970) Let $m$ be the number of elements in the finite set $S$. Note that

$$\sum_{s \in S} \frac{1}{m} \, (\underline{\lambda}\, 1^{S \setminus \{s\}}, \lambda_s) = \frac{m-1}{m} \, \underline{\lambda}\, 1^S + \frac{1}{m} \, \lambda^S \tag{7}$$

for all $\lambda^S \in \Delta(Y^S; \mathbb{F})$. Because $U^S$ satisfies (NAMP), applying $U^S$ to the mixtures on each side of (7) gives

$$\sum_{s \in S} \frac{1}{m} \, U^S(\underline{\lambda}\, 1^{S \setminus \{s\}}, \lambda_s) = \frac{m-1}{m} \, U^S(\underline{\lambda}\, 1^S) + \frac{1}{m} \, U^S(\lambda^S) \tag{8}$$

But $U^S(\underline{\lambda}\, 1^S) = 0$ by (3), so (8) and definition (4) imply that

$$U^S(\lambda^S) = \sum_{s \in S} g_s(\lambda_s) \tag{9}$$

Then (STP) and (4) jointly imply that $q_s := g_s(\bar{\lambda}) \in \mathbb{F}_+$.

Because the function $U^S$ satisfies (NAMP), equation (4) and Lemma 5 evidently imply that the functions $g_s$ ($s \in S$) and $U^*$ do the same. Also, by (STP), $g_s(\lambda)$ and $U^*(\lambda)$ both represent $\precsim^{\{s\}}$ on $\Delta(Y; \mathbb{F})$ while satisfying (NAMP), so they must be cardinally equivalent utility functions. By (2) and (5), $U^*(\underline{\lambda}) = g_s(\underline{\lambda}) = 0$. Hence, there exists $\rho > 0$ for which

$$g_s(\lambda) \equiv \rho \, U^*(\lambda) \tag{10}$$

By (2), $U^*(\bar{\lambda}) = 1$. By (4), putting $\lambda = \bar{\lambda}$ in (10) yields $q_s = g_s(\bar{\lambda}) = \rho$. Therefore (10) becomes $g_s(\lambda) \equiv q_s \, U^*(\lambda)$. Substituting this into (9) gives (6). Finally, (3), (6) and (2) jointly imply that

$$1 = U^S(\bar{\lambda}\, 1^S) = \sum_{s \in S} q_s \, U^*(\bar{\lambda}) = \sum_{s \in S} q_s$$

which completes the proof. ∎

**Lemma 7** *There exists a unique cardinal equivalence class of NMUFs $v : Y \to \mathbb{F}$ and, unless there is universal indifference, unique subjective probabilities $p_s$ ($s \in S$) such that the ordering $\precsim^S$ on $\Delta(Y^S; \mathbb{F})$ is represented by the subjective expected utility function*

$$U^S(\lambda^S) \equiv \sum_{s \in S} p_s \sum_{y \in Y} \lambda_s(y) \, v(y) \tag{11}$$

**Proof:** By Lemma 6, with $p_s$ replacing $q_s$ in (6), one has

$$U^S(\lambda^S) = \sum\nolimits_{s \in S} p_s \, U^*(\lambda_s) = \sum\nolimits_{s \in S} p_s \sum\nolimits_{y \in Y} \lambda_s(y) \, v^*(y) \qquad (12)$$

where $v^*(y) := U^*(1_y)$ for all $y \in Y$, and the second equality in (12) follows because $\lambda_s$ is the (finite) mixture $\sum_{y \in Y} \lambda_s(y) \, 1_y$ and $U^*$ satisfies (NAMP). As in Lemma 2, the NMUF $v^*$ could be replaced by any cardinally equivalent $v : Y \to \mathbb{F}$. But this is equivalent to replacing $U^S$ by a cardinally equivalent $V^S : \Delta(Y^S; \mathbb{F}) \to \mathbb{F}$. Any such transformation leaves the ratio

$$p_s = \frac{U^S(\underline{\lambda} \, 1^{S \setminus \{s\}}, \bar{\lambda}) - U^S(\underline{\lambda} \, 1^S)}{U^S(\bar{\lambda} \, 1^S) - U^S(\underline{\lambda} \, 1^S)}$$

of expected utility differences unaffected. This ensures that the subjective probabilities are unique. ∎

### 5.6 Real-Valued Utility

By axiom (SI), there is a state-independent contingent preference ordering $\succsim^*$ on $\Delta(Y; \mathbb{F})$. Now, in game theory, relevant states of the world include profiles of other players' strategies. The motivation for non-Archimedean probabilities is to allow infinitesimal subjective probabilities to be attached to strategy profiles in which some players deviate from their presumed best responses. Where there is no uncertainty of this kind, but only risk in the form of specified objective probabilities, there is no good reason to depart from classical expected utility theory, which requires preferences over real-valued probability distributions to be continuous, and expected utility to be real-valued. So it will be assumed that $\succsim^*$ on $\Delta(Y)$ satisfies a standard continuity axiom for real-valued probability distributions. Hence:

(SIC) *State Independent Continuity.*[4] For all $\lambda, \mu, \nu \in \Delta(Y)$ with $\lambda \succ^* \mu \succ^* \nu$, there exist $\alpha', \alpha'' \in (0, 1)$ such that

$$\alpha' \lambda + (1 - \alpha') \nu \succ^* \mu \quad \text{and} \quad \mu \succ^* \alpha'' \lambda + (1 - \alpha'') \nu$$

Finally:

[4] This is sometimes called the *Archimedean axiom* — see, for instance, Karni and Schmeidler (1991, p. 1769). For obvious reasons I avoid this name here.

**Theorem 8** *Suppose that the six axioms (O), (I*), (NAC), (RO), (SI) and (SIC) are all satisfied on $\Delta(Y^S; \mathbb{F})$. Then, unless there is universal indifference, there exist unique subjective probabilities $p_s \in \Delta(S; \mathbb{F}_+)$ and a unique cardinal equivalence class of* **real-valued** *NMUFs $v : Y \to \mathbb{R}$ such that*

$$\lambda^S \succsim^S \mu^S \iff \sum_{s\in S} p_s \sum_{y\in Y} \lambda_s(y)\, v(y) \geq \sum_{s\in S} p_s \sum_{y\in Y} \mu_s(y)\, v(y)$$

**Proof:** The existence of unique subjective probabilities $p_s$ and of a suitable NMUF $v : Y \to \mathbb{F}$ are direct implications of Lemma 7, as is the uniqueness of $v$ up to cardinal transformations.

Because there is not universal indifference, the NMUF $v : Y \to \mathbb{F}$ cannot be constant. So there must exist $\bar{y}, \underline{y} \in Y$ such that $v(\bar{y}) > v(\underline{y})$, implying that the NMUF $v : Y \to \mathbb{F}$ can be normalized to satisfy $v(\bar{y}) = 1$ and $v(\underline{y}) = 0$.

Next, let $\succsim^*_R$ denote the restriction of the ordering $\succsim^*$ to the space $\Delta(Y)$ of real-valued probability distributions. Note that $\succsim^*_R$ satisfies axioms (O), (I*), and (SIC). Hence, a standard result in (real-valued) expected utility theory implies that $\succsim^*_R$ can be represented by a real-valued utility function $U^*_R : \Delta(Y) \to \mathbb{R}$ which satisfies the traditional real-valued mixture preservation property (MP). Also, both $U^*_R$ and the associated NMUF $v^*_R : Y \to \mathbb{R}$ defined by $v^*_R(y) \equiv U^*_R(1_y)$ can be normalized to satisfy $v^*_R(\bar{y}) = 1$ and $v^*_R(\underline{y}) = 0$. But now, because the expected values of both $v$ and $v^*_R$ represent $\succsim^*_R$ on $\Delta(Y)$, these two NMUFs are cardinally equivalent. Therefore, because of the common normalization, they must be identical. That is, $v(y) = v^*_R(y) \in \mathbb{R}$ for all $y \in Y$, implying that $v : Y \to \mathbb{R}$. ∎

# 6 The Ordered Field $\mathbb{R}^\infty(\epsilon)$ as a Complete Metric Space

## 6.1 The Ordered Field $\mathbb{R}(\epsilon)$

In Hammond (1994, 1998c), the search for a minimal suitable range of probabilities led me to consider the particular ordered field $\mathbb{R}(\epsilon)$, originally described by Robinson (1973). This field contains a basic infinitesimal element $\epsilon \in \mathbb{F}_+$ and so is non-Archimedean. One can regard $\epsilon$ as a positive infinitesimal element of $^*\mathbb{R}$. Or, probably more intuitively, as any vanishing sequence $\langle \epsilon_n \rangle_{n=1}^\infty$ of positive real numbers. Then $\mathbb{R}(\epsilon)$ is the smallest field that includes all the real line $\mathbb{R}$ as well as $\epsilon$. Each of its elements can be written as a "rational expression" $f(\epsilon) = a(\epsilon)/b(\epsilon)$, where $a(\epsilon)$ and $b(\epsilon)$ are both "polynomial expressions" involving powers of $\epsilon$, with $b(\epsilon) \neq 0$. That is, if $\epsilon$ were replaced by a real variable $r$, then $f(r)$ would be the ratio of two polynomials, with the denominator not identically equal to zero.

In fact, the typical element of $\mathbb{R}(\epsilon)$ can be expressed in the normalized form

$$f(\epsilon) = \frac{a_k\,\epsilon^k + \sum_{i=k+1}^{n} a_i\,\epsilon^i}{1 + \sum_{j=1}^{m} b_j\,\epsilon^j} \tag{13}$$

for some unique integer $k$ and some unique leading coefficient $a_k$. Note that $a_k \neq 0$ unless $f(\epsilon) = 0$.

Define the ordering $>$ on $\mathbb{R}(\epsilon)$ so that $f(\epsilon) > 0$ whenever $f(r) > 0$ for all small enough real $r > 0$. It follows from (13) that $f(\epsilon) > 0$ iff $a_k > 0$. This makes $>$ a lexicographic linear ordering on $\mathbb{R}(\epsilon)$, in effect. It also makes $\mathbb{R}(\epsilon)$ a non-Archimedean ordered field.

### 6.2 A Metric

In Sections 4 and 5, the existence of subjective probabilities relied on preferences over lotteries satisfying the non-Archimedean continuity axiom (NAC) of Section 4.2. Section 7 will be concerned with finding weaker sufficient conditions for this axiom to be satisfied when probabilities are allowed to range over $\mathbb{R}(\epsilon)$. For this reason, the space $\mathbb{R}(\epsilon)$ needs to be given a topology. One suitable topology for $\mathbb{R}(\epsilon)$ is based on a metric that has been suggested by Lightstone and Robinson (1975).

Given the normalized form (13) of $f(\epsilon)$, define $k$ as the *infinitesimal order* $v[f(\epsilon)]$ of $f(\epsilon)$, with $v[0] := \infty$ for the zero element of $\mathbb{R}(\epsilon)$. If $f(\epsilon) \neq 0$, note that:

1. if $k > 0$, then $f(\epsilon)$ is an infinitesimal of order $k$;
2. if $k < 0$, then $f(\epsilon)$ is really an infinite number of order $-k$, which can be regarded as an infinitesimal of negative order;
3. if $k = 0$, then $f(\epsilon)$ is infinitesimally different from a non-zero real number, in which case it is said to be of infinitesimal order 0.

Note too that, for all non-zero pairs $f(\epsilon), g(\epsilon) \in \mathbb{R}(\epsilon)$, one has

$$v[f(\epsilon) \cdot g(\epsilon)] = v[f(\epsilon)] + v[g(\epsilon)] \quad \text{and} \quad v[f(\epsilon) + g(\epsilon)] \geq \min\{\, v[f(\epsilon)], v[g(\epsilon)]\,\}$$

It follows that the infinitesimal order is an instance of what Robinson (1973) describes as a "non-Archimedean valuation".

Now define the function $d : \mathbb{R}(\epsilon) \times \mathbb{R}(\epsilon) \to \mathbb{R}$ so that

$$d(f(\epsilon), g(\epsilon)) := 2^{-v[f(\epsilon)-g(\epsilon)]}$$

for every $f(\epsilon), g(\epsilon) \in \mathbb{R}(\epsilon)$, with the obvious convention that $2^{-\infty} := 0$. Obviously, $d(f(\epsilon), g(\epsilon)) = d(g(\epsilon), f(\epsilon))$ and $d(f(\epsilon), g(\epsilon)) = 0$ if and only if $v[f(\epsilon) - g(\epsilon)] = \infty$, which is true iff $f(\epsilon) = g(\epsilon)$. Finally, it is easy to verify that $d$ is a metric because the triangle inequality is satisfied.

## 6.3 Convergence

Consider any infinite sequence $f^n(\epsilon)$ $(n = 1, 2, \ldots)$ of elements in $\mathbb{R}(\epsilon)$. Evidently $f^n(\epsilon) \to f(\epsilon)$ as $n \to \infty$ iff $d(f^n(\epsilon), f(\epsilon)) \to 0$, which is true iff $v[f^n(\epsilon) - f(\epsilon)] \to \infty$. But after defining $g^n(\epsilon) := f^n(\epsilon) - f(\epsilon)$ for $n = 1, 2, \ldots$, this holds iff $g^n(\epsilon) \to 0$ because $v[g^n(\epsilon)] \to \infty$. Ignoring any zero terms of the sequence $g^n(\epsilon)$ $(n = 1, 2, \ldots)$, this requires that the infinitesimal order $k^n$ of

$$g^n(\epsilon) := \frac{a^n_{k^n}\, \epsilon^{k^n} + \sum_{i=k^n+1}^{\ell^n} a^n_i\, \epsilon^i}{1 + \sum_{j=1}^{m^n} b^n_j\, \epsilon^j}$$

should tend to $+\infty$. But then, for each number $M = 1, 2, \ldots$, there must exist $n(M)$ such that $n > n(M)$ implies $k^n > M$ because $a^n_i = 0$ for all $i \leq M$. That is, all coefficients $a^n_i$ of the numerator must become 0 for $n$ sufficiently large. This is a necessary and sufficient condition for convergence.

An equivalent test of convergence involves looking at a *non-Archimedean* or *field* metric $\rho : \mathbb{R}(\epsilon) \times \mathbb{R}(\epsilon) \to \mathbb{R}(\epsilon)$ defined by $\rho(f(\epsilon), g(\epsilon); \epsilon) := |f(\epsilon) - g(\epsilon)|$. This is not a metric because it is not real-valued over the whole of its domain. Nevertheless, in the domain $\mathbb{R}(\epsilon)$ it is symmetric, has the value 0 iff $f(\epsilon) = g(\epsilon)$, and satisfies the triangle inequality. Moreover, one can say that $f^n(\epsilon) \to f(\epsilon)$ as $n \to \infty$ iff for every $\delta(\epsilon) \in \mathbb{R}_+(\epsilon)$ there exists $n(\delta)$ such that $n > n(\delta)$ implies $\rho(f^n(\epsilon), f(\epsilon); \epsilon) < \delta(\epsilon)$ in the natural ordering of $\mathbb{R}(\epsilon)$. It is then an easy exercise to verify that this gives exactly the same test of convergence as the ordinary metric defined above.

Note one important implication of these definitions: An infinite sequence of real numbers converges if and only if it is eventually equal to a real constant. Thus, the (very fine) topology we have defined on $\mathbb{R}(\epsilon)$ induces the discrete topology on the subspace $\mathbb{R}$, meaning that every subset of $\mathbb{R}$ is open in the subspace topology.

## 6.4 Completing the Metric Space

Just as the ordinary continuity concept requires completing the space of rationals by going to the real line $\mathbb{R}$ in which Cauchy sequences converge, here I will consider a similar completion $\mathbb{R}^\infty(\epsilon)$ of $\mathbb{R}(\epsilon)$. Now, any sequence $f^n(\epsilon)$ $(n = 1, 2, \ldots)$ of elements in $\mathbb{R}(\epsilon)$ is a *Cauchy sequence* iff for every real $\delta > 0$ there exists $M(\delta)$ such that $d(f^m(\epsilon), f^n(\epsilon)) < \delta$ whenever $m, n > M(\delta)$. Equivalently, the infinitesimal order of $f^m(\epsilon) - f^n(\epsilon)$ must exceed $-\log_2 \delta$ whenever $m, n > M(\delta)$. Finally, using the non-Archimedean metric $\rho$, another equivalent condition for $f^n(\epsilon)$ $(n = 1, 2, \ldots)$ to be a Cauchy sequence is that for every $\delta(\epsilon) \in \mathbb{R}_+(\epsilon)$ there must exist a number $M$ such that $\rho(f^m(\epsilon), f^n(\epsilon); \epsilon) < \delta(\epsilon)$ whenever $m, n > M$.

In $\mathbb{R}(\epsilon)$ there are many Cauchy sequences that do not converge. Indeed, consider any sequence of the form $f^n(\epsilon) = \sum_{k=0}^n a_k \, \epsilon^k$ $(n = 1, 2, \ldots)$ where the infinite sequence $\langle a_k \rangle_{k=1}^\infty$ of real coefficients is non-recurring — for example, the power series $\sum_{r=0}^\infty \epsilon^{r^2}$. An analogy is the non-recurring decimal expansion of any irrational real number such as $\sqrt{2} = 1.4142\ 1356\ 2373\ \ldots$, which has no limit among the set of rational numbers. The obvious limit of the sequence $f^n(\epsilon)$ should be the power series $\sum_{k=0}^\infty a_k \, \epsilon^k$, just as the limit of the decimal expansion $1.4142\ 1356\ 2373\ \ldots$ is $\sqrt{2}$. Yet a non-recurring power series $\sum_{k=0}^\infty a_k \, \epsilon^k$ does not correspond to a rational expression in $\mathbb{R}(\epsilon)$, because multiplying it by any polynomial expression always leaves one with an infinite power series, never a polynomial.

The obvious way to complete $\mathbb{R}(\epsilon)$, therefore, is to allow such "irrational" infinite power series. As before, one wants an ordered algebraic field, so the ratios of such power series must also be accommodated. However, the reciprocal of any power series is itself a power series, but of the form $\sum_{k=k_0}^\infty a_k \, \epsilon^k$ where the leading power $k_0$ could be negative. Accordingly, define $\mathbb{R}^\infty(\epsilon)$ as the set of all such power series.[5] Of course, the reciprocal of any polynomial in $\mathbb{R}(\epsilon)$ is a power series in $\mathbb{R}^\infty(\epsilon)$, so any rational expression in $\mathbb{R}(\epsilon)$ is a power series in $\mathbb{R}^\infty(\epsilon)$. Hence, $\mathbb{R}^\infty(\epsilon)$ does extend $\mathbb{R}(\epsilon)$.

For convenience, write the typical member of $\mathbb{R}^\infty(\epsilon)$ as the doubly infinite power series $\sum_{k=-\infty}^\infty a_k \, \epsilon^k$, where it is understood that there must exist $k_0$ such that $a_k = 0$ for all $k < k_0$. Then both the metric $d$ and the non-Archimedean metric $\rho$ that were defined on $\mathbb{R}(\epsilon)$ can obviously be extended to $\mathbb{R}^\infty(\epsilon)$. The criterion for convergence of any sequence is therefore exactly the same, except that the denominator is clearly irrelevant. In fact, the

[5]In fact $\mathbb{R}^\infty(\epsilon)$ is a proper subset of the space $\mathcal{L}$ described by Levi-Civita (1892/3) and by Laugwitz (1968), whose members are power series of the form $\sum_{k=k_0}^\infty a_k \, \epsilon^{\nu_k}$ where $\langle \nu_k \rangle_{k=1}^\infty$ is any increasing sequence of real numbers that tends to $\infty$ as $k \to \infty$.

infinite sequence $f^n(\epsilon)$ $(n = 1, 2, \ldots)$ of power series $\sum_{k=-\infty}^{\infty} a_k^n \, \epsilon^k$ in $\mathbb{R}^\infty(\epsilon)$ will converge to the limit $f(\epsilon) = \sum_{k=-\infty}^{\infty} a_k \, \epsilon^k$ in $\mathbb{R}^\infty(\epsilon)$ iff for every $k$ there exists $n_k$ such that $a_k^n = a_k$ for all $n \geq n_k$.

Finally, for any sequence $\sum_{k=-\infty}^{\infty} a_k^n \, \epsilon^k$ $(n = 1, 2, \ldots)$ of power series in $\mathbb{R}^\infty(\epsilon)$ to be a Cauchy sequence, it must be true that: (i) there exist $k_0, n_0 \in \mathbb{N}$ such that $a_k^n = 0$ whenver $k < k_0$ and $n > n_0$; (ii) for every $k \geq k_0$ there exist both $n_k \in \mathbb{N}$ and $a_k \in \mathbb{R}$ such that $a_k^n = a_k$ for all $n > n_k$. But then the sequence obviously converges to $f(\epsilon) := \sum_{k=-\infty}^{\infty} a_k \, \epsilon^k$, which is equal to $\sum_{k=k_0}^{\infty} a_k \, \epsilon^k$ because $a_k$ must be 0 for all $k < k_0$. So the Cauchy sequence has a limit in $\mathbb{R}^\infty(\epsilon)$. This confirms that $\mathbb{R}^\infty(\epsilon)$ is indeed a complete metric space.

Two Cauchy sequences $\langle \sum_{k=-\infty}^{\infty} a_k^n \, \epsilon^k \rangle_{n=1}^{\infty}$ and $\langle \sum_{k=-\infty}^{\infty} b_k^n \, \epsilon^k \rangle_{n=1}^{\infty}$ are said to be "limit equivalent" in $\mathbb{R}^\infty(\epsilon)$ whenever $\sum_{k=-\infty}^{\infty} (a_k^n - b_k^n) \, \epsilon^k$ converges to 0 in $\mathbb{R}^\infty(\epsilon)$ as $n \to \infty$. Because the quotient space of limit equivalence classes of Cauchy sequences in $\mathbb{R}(\epsilon)$ is easily seen to be an ordered field which is isomorphic to $\mathbb{R}^\infty(\epsilon)$, it follows that $\mathbb{R}^\infty(\epsilon)$ is effectively the smallest complete metric space containing $\mathbb{R}(\epsilon)$, just as $\mathbb{R}$ is the smallest complete metric space containing the ordered field of rationals.

# 7 Lexicographic Expected Utility

## 7.1 Continuity

In the special case when the non-Archimedean ordered field $\mathbb{F}$ is the complete metric space $\mathbb{R}^\infty(\epsilon)$, the unsatisfactory non-Archimedean continuity (NAC) axiom of Section 4.2 will emerge as an implication of two weaker continuity axioms. Of these, the first is a non-Archimedean version of the familiar continuity axiom for preferences which requires that the sets of weakly preferred and weakly dispreferred mixtures of any two lotteries both be closed.

(C*) *Continuity.* For all lotteries $\lambda^S, \mu^S, \nu^S \in \Delta(Y^S; \mathbb{R}^\infty(\epsilon))$ satisfying $\lambda^S \succ^S \mu^S$ and $\mu^S \succ^S \nu^S$, the two sets

$$\begin{aligned} A &:= \{ \alpha(\epsilon) \in (0,1)_{\mathbb{R}^\infty(\epsilon)} \mid \alpha\lambda^S + (1-\alpha)\nu^S \succsim^S \mu^S \} \\ B &:= \{ \alpha(\epsilon) \in (0,1)_{\mathbb{R}^\infty(\epsilon)} \mid \alpha\lambda^S + (1-\alpha)\nu^S \precsim^S \mu^S \} \end{aligned}$$

are both closed.

### 7.2 Extended Continuity

Before presenting the second continuity axiom, let me first re-state axiom (NAC) for the case when probabilities take values in $\mathbb{R}^\infty(\epsilon)$.

(NAC) *Non-Archimedean Continuity.* Given any three lotteries $\lambda^S, \mu^S, \nu^S \in \Delta(Y^S; \mathbb{R}^\infty(\epsilon))$ satisfying $\lambda^S \succ^S \mu^S$ and $\mu^S \succ^S \nu^S$, there exists $\alpha(\epsilon) = \sum_{k=0}^\infty \alpha_k \, \epsilon^k \in (0,1)_{\mathbf{R}^\infty(\epsilon)}$ such that $\alpha(\epsilon)\lambda^S + [1-\alpha(\epsilon)]\nu^S \sim^S \mu^S$.

Then it is evident that axiom (NAC) can only be satisfied if the following logically weaker axiom is as well:

(XC) *Extended Continuity.* Suppose that $\alpha_m(\epsilon) := \sum_{k=0}^m a_k \, \epsilon^k \in (0,1)_{\mathbf{R}^\infty(\epsilon)}$ has the property that whenever $\delta_m^+(\epsilon) = \alpha_m(\epsilon) + \delta' \, \epsilon^m$ and $\delta_m^-(\epsilon) = \alpha_m(\epsilon) - \delta'' \, \epsilon^m$ for some real $\delta', \delta'' > 0$, then

$$\delta_m^+(\epsilon)\lambda^S + [1-\delta_m^+(\epsilon)]\nu^S \succ^S \mu^S \succ^S \delta_m^-(\epsilon)\lambda^S + [1-\delta_m^-(\epsilon)]\nu^S \tag{14}$$

In this case, there must exist $a'_{m+1}, a''_{m+1} \in \mathbb{R}$ such that, if $\gamma'_{m+1}(\epsilon) = \alpha_m(\epsilon) + a'_{m+1} \, \epsilon^{m+1}$ and $\gamma''_{m+1}(\epsilon) = \alpha_m(\epsilon) + a''_{m+1} \, \epsilon^{m+1}$, then

$$\gamma'_{m+1}(\epsilon)\lambda^S + [1-\gamma'_{m+1}(\epsilon)]\nu^S \succ^S \mu^S \succ^S \gamma''_{m+1}(\epsilon)\lambda^S + [1-\gamma''_{m+1}(\epsilon)]\nu^S$$

Clearly, axiom (XC) is specific to the field $\mathbb{R}^\infty(\epsilon)$. When $\alpha_m(\epsilon)$ satisfies (14), any positive probability perturbation of order $\epsilon^m$ is strictly preferred, and any negative perturbation of the same order is strictly dispreferred. Then axiom (XC) requires there to be some perturbations of order $\epsilon^{m+1}$ that are strictly preferred, and others that are strictly dispreferred. Note the similarity here with the requirements of the following non-Archimedean version of a standard weakening of axiom (C*):

(C) *Continuity.* For all $\lambda^S, \mu^S, \nu^S \in \Delta(Y^S; \mathbb{R}^\infty(\epsilon))$ satisfying $\lambda^S \succ^S \mu^S$ and $\mu^S \succ^S \nu^S$, there exist $\alpha'(\epsilon), \alpha''(\epsilon) \in (0,1)_{\mathbf{R}^\infty(\epsilon)}$ such that

$$\alpha'(\epsilon)\lambda^S + [1-\alpha'(\epsilon)]\nu^S \succ^S \mu^S \succ^S \alpha''(\epsilon)\lambda^S + [1-\alpha''(\epsilon)]\nu^S$$

### 7.3 Non-Archimedean Continuity

It will now be shown that axioms (C*) and (XC) can replace the questionable (NAC) among the sufficient conditions for the SEU hypothesis.

**Lemma 9** *On the space $\Delta(Y^S; \mathbb{R}^\infty(\epsilon))$ of probability distributions taking values in the complete metric space $\mathbb{R}^\infty(\epsilon)$, the four axioms (O), (I*), (C*) and (XC) together imply (NAC).*

**Proof:** Suppose that $\lambda^S, \mu^S, \nu^S \in \Delta(Y^S; \mathbb{R}^\infty(\epsilon))$ satisfy $\lambda^S \succ^S \mu^S$ and $\mu^S \succ^S \nu^S$. The following proof involves a recursive construction. This begins at stage $m = 0$ with $\alpha_{-1}(\epsilon) := 0$ and the two disjoint subsets

$$\begin{aligned} A_0 &:= \{\, \alpha \in (0,1)_{\mathbf{R}} \mid \alpha\lambda^S + (1-\alpha)\nu^S \succ^S \ \mu^S \,\} \\ B_0 &:= \{\, \alpha \in (0,1)_{\mathbf{R}} \mid \alpha\lambda^S + (1-\alpha)\nu^S \prec^S \ \mu^S \,\} \end{aligned}$$

of the real interval $(0,1)_{\mathbf{R}}$. By Lemma 1 of Section 4.4, it follows from axioms (O) and (I*) that there must exist a unique $a_0 \in (0,1)_{\mathbf{R}}$ such that $a_0 = \inf A_0 = \sup B_0$.

Suppose that recursion steps $k = 0, 1, \ldots, m$ ($m \geq 0$) have already been completed, and that they yield the coefficients $a_0, a_1, \ldots, a_m$ of the polynomial expression $\alpha_m(\epsilon) := \sum_{k=0}^m a_k\,\epsilon^k \in (0,1)_{\mathbf{R}^\infty(\epsilon)}$, together with the associated sets

$$\begin{aligned} A_m &:= \{\, \alpha \in \mathbb{R} \mid [\alpha_{m-1}(\epsilon) + \alpha\,\epsilon^m]\lambda^S + [1 - \alpha_{m-1}(\epsilon) - \alpha\,\epsilon^m]\nu^S \succ^S \ \mu^S \,\} \\ B_m &:= \{\, \alpha \in \mathbb{R} \mid [\alpha_{m-1}(\epsilon) + \alpha\,\epsilon^m]\lambda^S + [1 - \alpha_{m-1}(\epsilon) - \alpha\,\epsilon^m]\nu^S \prec^S \ \mu^S \,\} \end{aligned}$$

which satisfy $a'_m > a_m > a''_m$ whenever $a'_m \in A_m$ and $a''_m \in B_m$.

At the recursion step $m+1$ ($m \geq 0$), axiom (XC) implies that the next two sets $A_{m+1}$ and $B_{m+1}$ of the construction are both non-empty. Applying Lemma 1 once again, it follows that there must exist a unique $a_{m+1} \in \mathbb{R}$ such that $a_{m+1} = \inf A_{m+1} = \sup B_{m+1}$. Let $\alpha_{m+1}(\epsilon) := \sum_{k=0}^{m+1} a_k\,\epsilon^k \in (0,1)_{\mathbf{R}^\infty(\epsilon)}$. Then $a'_{m+1} > a_{m+1} > a''_{m+1}$ whenever $a'_{m+1} \in A_{m+1}$ and $a''_{m+1} \in B_{m+1}$. This completes recursion step $m+1$.

Consider next the two sequences $\alpha_m^+(\epsilon) := \alpha_m(\epsilon) + \epsilon^m$ and $\alpha_m^-(\epsilon) := \alpha_m(\epsilon) - \epsilon^m$ ($m = 1, 2, \ldots$). Then the above construction implies that

$$\alpha_m^+(\epsilon)\lambda^S + [1 - \alpha_m^+(\epsilon)]\nu^S \succ^S \mu^S \succ^S \alpha_m^-(\epsilon)\lambda^S + [1 - \alpha_m^-(\epsilon)]\nu^S$$

for $m = 1, 2, \ldots$. In addition, because $\mathbb{R}^\infty(\epsilon)$ is complete, as $m \to \infty$ both sequences $\alpha_m^+(\epsilon)$ and $\alpha_m^-(\epsilon)$ converge to the same infinite power series $\alpha(\epsilon) := \sum_{k=0}^\infty a_k\,\epsilon^k \in (0,1)_{\mathbf{R}^\infty(\epsilon)}$. Using axiom (C*) and the definitions of the two sets $A$ and $B$ it contains, it follows that $\alpha(\epsilon) \in A \cap B$. Evidently, it is this $\alpha(\epsilon)$ which satisfies $\alpha(\epsilon)\lambda^S + [1-\alpha(\epsilon)]\nu^S \sim^S \mu^S$ and so makes (NAC) true. ∎

## 7.4 Main Theorem

Finally, the following theorem shows that we have provided sufficient conditions for the non-Archimedean version of the SEU hypothesis to hold, with a *real-valued* utility function:

**Theorem 10** *Suppose that the seven axioms (O), (I*), (C*), (RO), (SI), (SIC) and (XC) are all satisfied on* $\Delta(Y^S; \mathbb{R}^\infty(\epsilon))$*. Then, unless there is universal indifference, there exist unique strictly positive non-Archimedean subjective probabilities* $p(\cdot;\epsilon) \in \Delta(S; \mathbb{R}^\infty_+(\epsilon))$ *and a unique cardinal equivalence class of real-valued NMUFs* $v : Y \to \mathbb{R}$ *such that, for each (non-empty) event* $E \subset S$*, the corresponding contingent preference ordering* $\succsim^E$ *on* $\Delta(Y^E; \mathbb{R}^\infty(\epsilon))$ *is represented by the* $\mathbb{R}^\infty(\epsilon)$*-valued subjective expected utility expression*

$$U^E(\lambda^E) \equiv \sum\nolimits_{s\in E} p(s;\epsilon) \sum\nolimits_{y\in Y} \mu_s(y)\, v(y) \tag{15}$$

*in the sense that* $\lambda^E \succsim^E \mu^E$ *if and only if* $U^E(\lambda^E) \geq U^E(\mu^E)$ *in the natural ordering of* $\mathbb{R}^\infty(\epsilon)$*.*

**Proof:** The result follows trivially from applying Theorem 8 to the field $\mathbb{F} = \mathbb{R}^\infty(\epsilon)$ and then using Lemma 9 to replace the particular axiom (NAC) with the two axioms (C*) and (XC). ∎

### 7.5 Lexicographic Expected Utility

For every $s \in S$, the subjective probability $p(s;\epsilon) \in \mathbb{R}^\infty_+(\epsilon)$ can be expressed as the power series $\sum_{k=0}^\infty p_k(s)\,\epsilon^k$. Thus, the SEU expression (15) can be re-written as the power series $U^E(\lambda^E) \equiv \sum_{k=0}^\infty u_k^E(\lambda^E)\,\epsilon^k$ whose coefficients are given by

$$u_k^E(\lambda^E) := \sum\nolimits_{s\in E} p_k(s) \sum\nolimits_{y\in Y} \mu_s(y)\, v(y) \quad (k = 0, 1, 2, \ldots)$$

But then $\lambda^E \succsim^E \mu^E$ if and only if the associated infinite hierarchies of coefficients $\langle u_k^E(\lambda^E)\rangle_{k=0}^\infty$ and $\langle u_k^E(\mu^E)\rangle_{k=0}^\infty$ satisfy $\langle u_k^E(\lambda^E)\rangle_{k=0}^\infty \geq_L \langle u_k^E(\mu^E)\rangle_{k=0}^\infty$ w.r.t. the lexicographic total ordering $>_L$ on $\mathbb{R}^\infty$. In this sense, the preference ordering $\succsim^E$ has a lexicographic expected utility representation.

## References


Anderson, R.M. (1991) "Non-Standard Analysis with Applications to Economics," in W. Hildenbrand and H. Sonnenschein (eds.) *Handbook of Mathematical Economics, Vol. IV* (Amsterdam: North-Holland), ch. 39, pp. 2145–2208.

Anscombe, F.J. and R.J. Aumann (1963) "A Definition of Subjective Probability," *Annals of Mathematical Statistics*, **34**, 199–205.

Arrow, K.J. (1953, 1964) "Le rôle des valeurs boursières pour la répartition la meilleure des risques," in *Econométrie* (Paris: Centre National de la Recherche Scientifique), pp. 41–48; translation of English original, "The Role of Securities in the Optimal Allocation of Risk-bearing," later published in *Review of Economic Studies*, **31**: 91–96.

Barberà, S., P.J. Hammond, and C. Seidl (eds.) (1998) *Handbook of Utility Theory* (Dordrecht: Kluwer Academic Publishers) (in press).

Blume, L., A. Brandenburger and E. Dekel (1991a) "Lexicographic Probabilities and Choice Under Uncertainty" *Econometrica*, **59**: 61–79.

Blume, L., A. Brandenburger and E. Dekel (1991b) "Lexicographic Probabilities and Equilibrium Refinements" *Econometrica*, **59**: 81–98.

Chernoff, H. (1954) "Rational Selection of Decision Functions" *Econometrica*, **22**: 422–443.

Chipman, J.S. (1960) "Foundations of Utility" *Econometrica*, **28**: 193–224.

Chipman, J.S. (1971a ) "On the Lexicographic Representation of Preference Orderings" in Chipman *et al.*, ch. 14, pp. 276–288.

Chipman, J.S. (1971b) "Non-Archimedean Behavior under Risk: An Elementary Analysis — With Application to the Theory of Assets" in Chipman *et al.*, ch. 15, pp. 289–318.

Chipman, J.S., L. Hurwicz, M.K. Richter and H. Sonnenschein (eds.) (1971) *Preferences, Utility and Demand: A Minnesota Symposium* (New York: Harcourt Brace Jovanovic).

Császár, Á. (1955) "Sur la structure des espaces de probabilité conditionelle" *Acta Mathematica Academiae Scientiarum Hungaricae*, **6**: 337–361.

Debreu, G. (1959) *Theory of Value: An Axiomatic Analysis of Economic Equilibrium* (New York: John Wiley).

Feynman, R.P. (1987) "Negative Probability" in Hiley, B.J. and F.D. Peat (eds.) *Quantum Implications: Essays in Honour of David Bohm* (London: Routledge & Kegan Paul), ch. 13, pp. 235–248.

Fishburn, P.C. (1970) *Utility Theory for Decision Making* (New York: John Wiley).

Hammond, P.J. (1988) "Consequentialist Foundations for Expected Utility," *Theory and Decision*, **25**: 25–78.

Hammond, P.J. (1994) "Elementary Non-Archimedean Representations of Probability for Decision Theory and Games," in Humphreys, P. (ed.) *Patrick Suppes: Scientific Philosopher, Vol. I: Probability and Probabilistic Causality* (Dordrecht: Kluwer Academic Publishers), ch. 2, pp. 25–59.

Hammond, P.J. (1998a) "Objective Expected Utility: A Consequentialist Perspective" to appear in Barberà *et al.*, ch. 2.

Hammond, P.J. (1998b) "Subjective Expected Utility" to appear in Barberà *et al.*, ch. 3.

Hammond, P.J. (1998c) "Consequentialism, Non-Archimedean Probabilities, and Lexicographic Expected Utility," to appear in C. Bicchieri, R. Jeffrey and B. Skyrms (eds.) *The Logic of Strategy* (Oxford University Press).

Hausner, M. (1954) "Multidimensional Utilities" in R.M. Thrall, C.H. Coombs and R.L. Davis (eds.) *Decision Processes* (New York: John Wiley), ch. 12, pp. 167–180.

Karni, E., and D. Schmeidler (1991) "Utility Theory with Uncertainty," in W. Hildenbrand and H. Sonnenschein (eds.) *Handbook of Mathematical Economics, Vol. IV* (Amsterdam: North-Holland), ch. 33, pp. 1763–1831.

Kolmogorov, A.N. (1933, 1956) *Grundbegriffe der Wahrscheinlichkeitsrechnung*; translated as *Foundations of the Theory of Probability* (Berlin: Springer; and New York: Chelsea).

Kreps, D. and R. Wilson (1982) "Sequential Equilibrium," *Econometrica*, **50**, 863–894.

LaValle, I.H. and P.C. Fishburn (1996) "On the Varieties of Matrix Probabilities in Nonarchimedean Decision Theory" *Journal of Mathematical Economics*, **25**: 33–54.

Laugwitz, D. (1968) "Eine nichtarchimedische Erweiterung angeordneter Körper" *Mathematische Nachrichten*, **37**: 225–236.

Levi-Civita, T. (1892/3) "Sugli infiniti ed infinitesimali attuali: quali elementi analitici" *Atti Istituto Veneto di scienze, lettere, ed arti*, **7**: 1765–1815; reprinted in *Opere Matematiche di Tullio Levi-Civita: Memorie e Note: Volume primo (1893–1900)*, (Nicola Zanichelli, Bologna, 1954), pp. 1–39.

Lightstone, A.H. and A. Robinson (1975) *Nonarchimedean Fields and Asymptotic Expansions* (Amsterdam: North-Holland).

McLennan, A. (1989a) "The Space of Conditional Systems is a Ball" *International Journal of Game Theory*, **18**: 125–139.

McLennan, A. (1989b) "Consistent Conditional Systems in Noncooperative Game Theory" *International Journal of Game Theory*, **18**: 141–174.

Myerson, R.B. (1978) "Refinements of the Nash Equilibrium Concept," *International Journal of Game Theory*, **7**, 73–80.

Myerson, R.B. (1986) "Multistage Games with Communication," *Econometrica*, **54**: 323–358.

Raiffa, H. (1961) "Risk, Ambiguity, and the Savage Axioms: Comment" *Quarterly Journal of Economics*, **75**, 690–694.

Rényi, A. (1955) "On a New Axiomatic Theory of Probability," *Acta Mathematica Academiae Scientiarum Hungaricae*, **6**, 285–335.

Rényi, A. (1956) "On Conditional Probability Spaces Generated by a Dimensionally Ordered Set of Measures," *Theory of Probability and its Applications*, **1**, 61–71; reprinted in *Selected Papers of Alfréd Rényi, I: 1948–1956* (Budapest: Akadémia Kiadó, 1976), Paper 120, pp. 547–557.

Richter, M.K. (1971) "Rational Choice" in Chipman *et al.*, ch. 2, pp. 29–58.

Robinson, A. (1973) "Function Theory on Some Nonarchimedean Fields" *American Mathematical Monthly: Papers in the Foundations of Mathematics* **80**: S87–S109.

Royden, H.L. (1968) *Real Analysis* (2nd edn.) (New York: Macmillan).

Savage, L.J. (1954, 1972) *The Foundations of Statistics* (New York: John Wiley; and New York: Dover Publications).

Selten, R. (1965) "Spieltheoretische Behandlung eines Oligopolmodells mit Nachfrageträgheit" *Zeitschrift für die gesamte Staatswissenschaft*, **121**: 301–324 and 667–689.

Selten, R. (1973) "A Simple Model of Imperfect Competition, where 4 Are Few and 6 Are Many" *International Journal of Game Theory*, **2**: 141–201.

Selten, R. (1975) "Re-examination of the Perfectness Concept for Equilibrium Points of Extensive Games" *International Journal of Game Theory*, **4**: 25–55.

Skala, H.J. (1975) *Non-Archimedean Utility Theory* (Dordrecht: D. Reidel).