On f-Core Equivalence with General Widespread Externalities

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Abstract: This paper partially extends the f-core equivalence theorem of Hammond, Kaneko and Wooders (1989) for continuum economies with widespread externalities — i.e., those over which each individual has negligible control. Externalities need not result directly from trading activities. Neither free disposal of divisible goods nor monotone preferences are assumed. Instead, a slightly strengthened form of local non-satiation suffices. However, in general it is proved only that any f-core allocation is a compensated Nash–Walrasian equilibrium. Finally, the proof uses an elementary argument which does not rely on Lyapunov's theorem or convexity of the integral of a correspondence w.r.t. a non-atomic measure.

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f-Core Equivalence

1. Introduction

The earlier paper by Hammond, Kaneko and Wooders (1989) (henceforth referred to as HKW) presented a particular model of an economy with a continuum of individual agents and widespread externalities. The externalities were assumed to result directly from trading activities. That paper considered the "f-core" consisting of allocations that could not be blocked or improved by any finite coalition outside a null set. The main result (Theorem 1) concerned the equivalence of the f-core and the set of Walrasian equilibrium allocations.

The main purpose of this note is to extend this equivalence result to a broader class of economic environments, as described in Section 2. This class includes those considered in Kaneko and Wooders (1994) and in Hammond (1995, 1998) — see also the work cited in Section 7 of Khan, Rath and Sun (1997). The widespread externalities need not result directly from trading activities. In addition, rather than fixed endowments, individuals will be assumed to have feasible sets of net trades that may allow for domestic production. Moreover, these feasible sets may not allow free disposal even of divisible goods, and preferences are required only to satisfy a slightly strengthened form of local non-satiation instead of monotonicity.

The cost of these generalizations will be a somewhat weaker result, showing only that any f-core allocation must be a compensated Nash–Walrasian equilibrium, as defined in Section 3. That is, the individuals' choices of net trade vectors must be a compensated Walrasian equilibrium in the sense of Khan and Yamazaki (1981), while their choices of externalities must be a Nash equilibrium. The final transition to a Nash–Walrasian equilibrium is then possible under extra assumptions such as those presented in HKW, or in Yamazaki (1978, 1981), Hammond (1993), or Coles and Hammond (1995).

Section 4 adapts Hildenbrand's (1982) proof of the classical core equivalence theorem, which is based upon Aumann's (1964) original ingenious argument. In fact, unlike the classical theorem or the argument used in HKW, the proof here is more elementary because it does not rely on Lyapunov's theorem concerning the convexity of the range of an $\Re^n$-valued non-atomic measure, even in the case when $n = 1$. Nor does it use the important implication due to Richter (1963) that the integral of a correspondence w.r.t. a non-atomic measure is a convex set (see Hildenbrand (1974), Theorem 3, p. 62).

The final Section 5 contains concluding remarks.

2. A Continuum Economy with Widespread Externalities

Throughout the paper the setting will be an economy with a continuum of agents, as originally formulated by Aumann (1964, 1966). See also Hildenbrand (1974). In fact, using the notation of HKW, let the space of agents be $(A, \mathcal{A}, \mu)$, where A can be regarded as the interval $[0, 1]$ of the real line, with $\mathcal{A}$ as the Borel σ-algebra and μ Lebesgue measure.

Let T be the finite set of traded goods, with corresponding commodity space $\Re^T$. Let E denote the measurable space whose points represent all possible combinations e of externality creating activities. Note that E could be a finite or infinite dimensional linear space, or it could even be a space that is not linear. It is only required to have a σ-algebra of measurable sets.

Write $L(A, E)$ for the set of all measurable functions $\mathbf{e} : A \to E$, whose values will be denoted by e^a for each $a \in A$. Here $\mathbf{e}$ and $\bar{\mathbf{e}}$ will be treated as equivalent if $e^a = \bar{e}^a$ a.e. in A. These functions represent the *widespread externalities.* Then the commodity/externality space will be the Cartesian product $\Re^T \times E \times L(A, E)$.

Assume that, for each fixed $\mathbf{e} \in L(A, E)$, every agent $a \in A$ has a *conditionally feasible set* $F^a(\mathbf{e})$ of pairs $(x^a, e^a) \in \Re^T \times E$ and a binary irreflexive *conditional strict preference relation* $\succ^a(\mathbf{e})$ defined on $F^a(\mathbf{e})$. Note that the conditional strict preference relation is not necessarily transitive, and the corresponding weak preference relation is not necessarily complete or transitive.

Of course, individuals typically have preferences that extend to the set of all triples $(x^a, e^a, \mathbf{e})$ satisfying $(x^a, e^a) \in F^a(\mathbf{e})$. These preferences are relevant for any efficiency or other welfare properties. The preferences for widespread externalities $\mathbf{e}$, however, have no effect on the equilibrium or f-core allocations considered in this paper.

The projection $F^a_T(\mathbf{e}) = \text{proj}_{\Re^T}\ F^a(\mathbf{e})$ of $F^a(\mathbf{e})$ onto $\Re^T$ is the *conditionally feasible set of net trade vectors.* Note that endowments of traded goods have been netted out. Nor are such endowments necessarily fixed vectors because agents may be endowed with production possibility sets instead, as in Rader (1964).

In HKW we considered a special case of the model presented here. We assumed that $E = \Re^T$, so externalities were equivalent to net trade vectors. Furthermore, T was partitioned into the two sets D of divisible goods and I of indivisible goods. Also, for each agent $a \in A$, the conditionally feasible set was assumed to be independent of $\mathbf{e}$, and given by

$$F^a = \{ (x^a, e^a) \in \Re^T \times E \mid x^a = e^a \quad \text{and} \quad x^a + \omega^a \in Z_+^I \times \Re_+^D \},$$

where ω^a denotes agent a's fixed endowment vector, and Z_+^I is the set of possible non-negative vectors of indivisible goods, with integer coordinates.

Let $Q \subset \Re$ denote the set of rational numbers. Let Q^T denote the subset of $\Re^T$ whose members are vectors with all their coordinates in Q. The conditional feasible set $F \subset \Re^T \times E$ and strict preference relation $\succ$ on F are said to be *locally non-satiated in rational net trade vectors* if, for any $(x, e) \in F$ and any neighbourhood N of x in $\Re^T$, there exist $x' \in N \cap Q^T$ and $e' \in E$ such that $(x', e') \in F$ with $(x', e') \succ (x, e)$. The same pair F and $\succ$ on F are said to be *continuously non-satiated in rational net trade vectors* if, for every pair $(x, e), (x', e') \in F$ with $(x', e') \succ (x, e)$ or $(x', e') = (x, e)$ and for every neighbourhood N' of x' in $\Re^T$, there exist $x'' \in N \cap Q^T$ and $e'' \in E$ such that $(x'', e'') \in F$ with $(x'', e'') \succ (x, e)$.

Continuous non-satiation in rational net trade vectors, as defined above, trivially implies local non-satiation in rational net trade vectors. Also, the two conditions are logically equivalent when the relation $\succ$ is transitive, or even if it is intransitive but instead satisfies a standard continuity condition on $\Re^T$. Furthermore, both conditions are satisfied whenever preferences for traded goods satisfy any of the usual monotonicity conditions. None of these properties is assumed here, however, so the second stronger assumption is required for the main theorem.

A *feasible allocation* is a measurable mapping $(\mathbf{x}, \mathbf{e}) : A \to \Re^T \times E$ for which $(x^a, e^a) \in F^a(\mathbf{e})$ a.e. in A, and also $\int_A x^a \, d\mu = 0$. Given any such feasible allocation and any agent $a \in A$, define the set

$$P^a(x^a, e^a; \mathbf{e}) := \{ x \in \Re^T \mid \exists e \in E : (x, e) \in F^a(\mathbf{e}) \quad \text{and} \quad (x, e) \succ^a(\mathbf{e}) \; (x^a, e^a) \}$$

of net trade vectors that, when combined with suitable individual externality vectors, are conditionally strictly preferred by a to (x^a, e^a). Following Aumann [1] assume that, given any feasible allocation $(\mathbf{x}, \mathbf{e})$, the set

$$A(t) := \{ a \in A \mid t \in P^a(x^a, e^a; \mathbf{e}) \}$$

is measurable for every fixed $t \in \Re^T$.

3. Nash–Walrasian Equilibrium Allocations and the f-Core

Say that the finite coalition $S \subset A$ can *improve* the feasible allocation $(\hat{\mathbf{x}}, \hat{\mathbf{e}})$ if, for all $a \in S$, there exists an alternative $x^a \in P^a(\hat{x}^a, \hat{e}^a; \hat{\mathbf{e}})$ such that $\sum_{a \in S} x^a = 0$. Note that any finite coalition has measure zero, so its choice of externalities leaves $\hat{\mathbf{e}}$ essentially unaffected.

The feasible allocation $(\hat{\mathbf{x}}, \hat{\mathbf{e}})$ is said to be in the *f-core* iff there is a full set of agents $A^1 \subset A$ (with $\mu(A^1) = 1$) such that no finite coalition $S \subset A^1$ can improve $(\hat{\mathbf{x}}, \hat{\mathbf{e}})$. That is, after excluding a null set of agents if necessary, it must be impossible to form any improving or blocking finite coalition S.

It seems evident that finite coalitions lack the power to improve any allocation in the f-core. But do they also really have the power to improve any allocation outside the f-core? Instead of answering this question directly, I offer the following apparently stronger definition of "discernible improvements," which leads to an apparently larger set of "weak" f-core allocations.

Specifically, say that the finite family of m non-null measurable sets $S_k \in \mathcal{A}$ ($k = 1, 2, \ldots, m$) (with $\mu(S_k) > 0$, and not necessarily disjoint), can *discernibly improve* the feasible allocation $(\hat{\mathbf{x}}, \hat{\mathbf{e}})$ if there exist corresponding natural numbers n_k such that every finite coalition made up of exactly n_k agents from each set S_k in the finite family, but no other agents, can improve this allocation. The feasible allocation $(\hat{\mathbf{x}}, \hat{\mathbf{e}})$ is said to be in the *weak f-core* iff no finite family of measurable sets can discernibly improve the allocation. Evidently, if some finite family S_k ($k = 1, 2, \ldots, m$) can discernibly improve an allocation, then any full set of agents A^1 must include improving finite coalitions with an appropriate number of members selected from each of the non-null sets $A^1 \cap S_k$. So the f-core is a subset of the weak f-core.

Because neither free disposal of traded goods nor monotonicity of preferences has been assumed, any equilibrium price vector $p \in \Re^T$ could have some negative components. Accordingly, say that any feasible allocation $(\hat{\mathbf{x}}, \hat{\mathbf{e}})$ and price vector $p \neq 0$ together constitute a *Nash–Walrasian equilibrium* if, a.e. in A, both $p\,\hat{x}^a = 0$ and also any $x^a \in P^a(\hat{x}^a, \hat{e}^a; \hat{\mathbf{e}})$ satisfies $p\,x^a > 0$. This is a Walrasian equilibrium insofar as almost all agents choose private good net trade vectors in order to maximize preferences subject to their budget constraints. It is a Nash equilibrium insofar as almost all agents choose externalities and private goods simultaneously in order to maximize preferences given the widespread externalities chosen by almost everybody else. Because of the assumption that there is a continuum of agents, these externalities are described by $\hat{\mathbf{e}}$ up to an irrelevant null set of agents. Moreover, no individual agent has the power to change $\hat{\mathbf{e}}$. Finally, note that in Nash–Walrasian equilibrium only traded goods in the set T are rationed by price; each agent is free to choose any combination e of externality creating activities in the space E at will, subject only to individual feasibility.

Finally, say that the feasible allocation $(\hat{\mathbf{x}}, \hat{\mathbf{e}})$ and price vector $p \neq 0$ together constitute a *compensated Nash–Walrasian equilibrium* if, a.e. in A, both $p\,\hat{x}^a = 0$ and also any x^a in the closure of $P^a(\hat{x}^a, \hat{e}^a; \hat{\mathbf{e}})$ satisfies $p\,x^a \geq 0$. This is an obvious adaptation of the standard concept of compensated Walrasian equilibrium (or quasi-equilibrium).

Suppose that each agent $a \in A$ has a closed convex conditional feasible set $F^a_T(\hat{\mathbf{e}})$ and a conditional strict preference relation $\succ^a(\hat{\mathbf{e}})$ whose graph is open relative to $F^a(\hat{\mathbf{e}}) \times F^a(\hat{\mathbf{e}})$. Suppose too that preferences are continuously non-satiated in rational vectors, and so locally non-satiated *a fortiori*. Then a well known sufficient condition for the compensated Nash–Walrasian equilibrium $(\hat{\mathbf{x}}, \hat{\mathbf{e}}, p)$ to be a (full) Nash–Walrasian equilibrium is that almost every agent $a \in A$ has a *cheaper point* $\underline{x}^a \in F^a_T(\hat{\mathbf{e}})$ with $p\,\underline{x}^a < 0$. Sufficient conditions for individuals to have such cheaper points are discussed in Hammond (1993). So are somewhat weaker hypotheses than convexity of $F^a_T(\hat{\mathbf{e}})$ under which, in an economy without externalities, a compensated Walrasian equilibrium is still a Walrasian equilibrium. Where these hypotheses are satisfied, there is exact equivalence between the f-core and the set of Nash–Walrasian (or Walrasian) equilibrium allocations.

4. f-Core Equivalence

A standard rather easy proof shows that any Nash–Walrasian equilibrium allocation is in the f-core. Conversely:

Weak Equivalence Theorem. *Suppose that preferences are continuously non-satiated w.r.t. rational vectors. For any allocation* $(\hat{\mathbf{x}}, \hat{\mathbf{e}})$ *in the weak f-core, there exists a price vector* $p \neq 0$ *such that* $(\hat{\mathbf{x}}, \hat{\mathbf{e}}, p)$ *is a compensated Nash–Walrasian equilibrium.*

Proof: Let $(\hat{\mathbf{x}}, \hat{\mathbf{e}})$ be *any* feasible allocation, not necessarily in the weak f-core. Then, following the beginning of Aumann's (1964, p. 45) and Hildenbrand's (1982, pp. 843–4) proof of core equivalence, for every $t \in Q^T$, define the set

$$\hat{A}(t) := \{\, a \in A \mid t \in P^a(\hat{x}^a, \hat{e}^a; \hat{\mathbf{e}}) \,\}$$

By our earlier assumption, the set $\hat{A}(t)$ must be measurable. Then let

$$A' := A \setminus \left[\bigcup \{\, \hat{A}(t) \mid t \in Q^T,\ \mu(\hat{A}(t)) = 0 \,\} \right].$$

Because Q^T is countable, the set A' is also measurable and satisfies $\mu(A') = \mu(A) = 1$. Finally, define the convex hull

$$K := \mathrm{co} \left\{ \bigcup_{a \in A'} [Q^T \cap P^a(\hat{x}^a, \hat{e}^a; \hat{\mathbf{e}})] \right\}$$

Suppose that $0 \in K$. Then there exist m agents $a_k \in A'$ with conditionally preferred rational net trade vectors $t_k \in Q^T \cap P^{a_k}(\hat{x}^{a_k}, \hat{e}^{a_k}; \hat{\mathbf{e}})$ and positive convex weights $r_k \in \Re_+$ $(k = 1, \ldots, m)$, such that $\sum_{k=1}^m r_k = 1$ and $\sum_{k=1}^m r_k \, t_k = 0$. These equations combine to give a single matrix equation of the form $\mathbf{r}'\mathbf{T} = \mathbf{e}_0$, where $\mathbf{T}$ is the $(1 + \#T) \times m$ matrix whose kth column has 1 in the first row followed by the vector t_k, whereas $\mathbf{e}_0$ is the unit vector $(1, 0, \ldots, 0) \in \Re \times \Re^T$. This matrix equation can be solved by elementary row operations. Because each vector $t_k \in Q^T$ has rational co-ordinates, the result will be a row vector $\mathbf{r}'$ of non-negative convex weights which are *rational* numbers $r_k \in Q_+$. So there exist natural numbers n_k $(k = 1, \ldots, m)$ such that $\sum_{k=1}^m n_k \, t_k = 0$.

Because $a_k \in A'$, there is no $t \in Q^T$ such that $a_k \in \hat{A}(t)$ and $\mu(\hat{A}(t)) = 0$. Because $t_k \in Q^T \cap P^{a_k}(\hat{x}^{a_k}, \hat{e}^{a_k}; \hat{\mathbf{e}})$ and so $a_k \in \hat{A}(t_k)$, it follows that $\mu(\hat{A}(t_k)) > 0$ $(k = 1, \ldots, m)$.

Now define S as any finite coalition consisting of n_k members from each of the m sets $\hat{A}(t_k)$ ($k = 1$ to m). Also, for $k = 1, \ldots, m$, define $x^a := t_k$ for each $a \in \hat{A}(t_k) \cap S$. Then $\sum_{a \in S} x^a = \sum_{k=1}^m n_k t_k = 0$. Furthermore, for all $a \in S$, one has $x^a \in P^a(\hat{x}^a, \hat{e}^a; \hat{\mathbf{e}})$. This shows that the finite family $\hat{A}(t_k)$ ($k = 1$ to m) can discernibly improve the allocation $(\hat{\mathbf{x}}, \hat{\mathbf{e}})$. Therefore $(\hat{\mathbf{x}}, \hat{\mathbf{e}})$ is not in the weak f-core.

On the other hand, if $(\hat{\mathbf{x}}, \hat{\mathbf{e}})$ does belong to the weak f-core, the preceding three paragraphs have shown that $0 \notin K$. Because K is convex, the separating hyperplane theorem then implies that there must exist $p \neq 0$ for which $p\,t \geq 0$ for all $t \in K$.

Consider any $a \in A'$ and any x in the closure of $P^a(\hat{x}^a, \hat{e}^a; \hat{\mathbf{e}})$. Then continuous non-satiation in rational net trade vectors implies that there must exist an infinite sequence $(x_m, e_m) \in F^a(\hat{\mathbf{e}}) \cap (Q^T \times E)$ such that $(x_m, e_m) \succ^a(\hat{\mathbf{e}})\ (\hat{x}^a, \hat{e}^a)$ $(m = 1, 2, \ldots)$ and $x_m \to x$ as $m \to \infty$. Therefore $x_m \in Q^T \cap P^a(\hat{x}^a, \hat{e}^a; \hat{\mathbf{e}}) \subset K$ for all m, implying that $p\,x_m \geq 0$ and so $p\,x \geq 0$ in the limit.

Next, given any $a \in A'$, the assumption of local non-satiation in rational net trade vectors implies the existence of an infinite sequence $(x'_m, e'_m) \in F^a(\hat{\mathbf{e}}) \cap (Q^T \times E)$ such that $(x'_m, e'_m) \succ^a(\hat{\mathbf{e}})\ (\hat{x}^a, \hat{e}^a)$ $(m = 1, 2, \ldots)$ and also $x'_m \to \hat{x}^a$ as $m \to \infty$. It follows that $x'_m \in Q^T \cap P^a(\hat{x}^a, \hat{e}^a; \hat{\mathbf{e}}) \subset K$ $(m = 1, 2, \ldots)$, hence $p\,x'_m \geq 0$ and so $p\,\hat{x}^a \geq 0$ in the limit. But feasibility implies that $\int_A \hat{x}^a \, d\mu = 0$, so $\int_A p\,\hat{x}^a \, d\mu = 0$. Because $\mu(A') = 1$ and $p\,\hat{x}^a \geq 0$ for all $a \in A'$, it follows that $p\,\hat{x}^a = 0$ a.e. in A', and so a.e. in A.

The results of the last two paragraphs confirm that $(\hat{\mathbf{x}}, \hat{\mathbf{e}}, p)$ is a compensated Nash–Walrasian equilibrium. ∎

5. Concluding Remarks

As discussed in Section 3, the main result of Section 4 is not in itself a full equivalence theorem for the f-core. Like the usual equivalence theorem for the core, however, there will be full equivalence whenever a compensated Nash–Walrasian equilibrium is also an ordinary Nash–Walrasian equilibrium.

One limitation of the paper is the absence of any existence result such as those discussed by Khan, Rath and Sun (1997) or, if mixed strategies are allowed, by Balder (1996), as well as in earlier work cited in those articles. It remains to be seen how far such existence results extend to economies with private traded goods and widespread externalities.

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