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Bounding Stationary Expectations for Queues and Storage Processes Fed by Stationary Input Sequences

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Abstract: In many queueing applications, it is of interest to allow dependence within the input sequence to the model, thereby permitting auto-correlation within the service times or inter-arrival times. This leads naturally to models that are fed by stationary sequences of random variables. When the model has a stationary distribution, conditions guaranteeing finiteness of a given stationary expectation may be needed. It may also be of value to obtain computable bounds on such stationary expectations. This paper develops a Lyapunov-type condition that provides such a computable bound. An implication of the bound is a criterion that guarantees tightness of the associated queueing model. In addition, the paper studies such queueing systems via the perspective of iterated random functions, showing how one can establish existence of stationary distributions via monotonicity or contraction arguments. The results are illustrated via an application to the delay sequence of the single server queue.

Keywords: Bounds on stationary expectations, iterated random functions, queues with stationary inputs.

1. Introduction

Many queues and storage processes can be analyzed as solutions to stochastic recursive equations taking the form

$$X_{n+1} = r(X_n, Z_{n+1}) \tag{1}$$

for some deterministic function r. When $(Z_n : n \geq 1)$ is a sequence of independent and identically distributed (iid) random variables (rv's) independent of X_0, then $(X_n : n \geq 0)$ is a Markov chain with stationary transition probabilities. However, in some applications settings, it is more appropriate to permit the Z_i's to be auto-correlated. In such settings, it is natural to assume instead that $(Z_n : n \geq 1)$ forms a stationary sequence. We then say that the process $(X_n : n \geq 0)$ is a *dynamical system fed by a stationary input sequence.*

Under suitable assumptions, one can establish that

$$\frac{1}{n}\sum_{j=0}^{n-1} P(X_j \in \cdot) \Rightarrow X_\infty \tag{2}$$

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as $n \to \infty$, where $\Rightarrow$ denotes weak convergence. In this paper, we provide a vehicle for bounding the *stationary expectation* $\mathbb{E} f(X_\infty)$ for real-valued f. Such bounds are of intrinsic interest in their own right, and are also of critical value in various computations that require taking expectations of a system in equilibrium. The rigorous invocation of such arguments often requires the knowledge that such equilibrium expectations are finite; see, for example, the argument used by [11] to deduce Theorem 1 there. Such finiteness is, of course, implied by our bound.

Section 2 is devoted to describing and developing our bound. In that section, we also show how the bound can be used to establish tightness for sequences fed by stationary inputs; such tightness results are needed in order to invoke certain theorems that can be found in the literature; see, for example, Theorem A of [2]. Section 2 also discusses the use of the iterated random function viewpoint, as a vehicle for establishing existence of stationary distributions for models fed by stationary input sequences. Section 3 focuses on illustrating the use of the bound in the setting of the delay sequence for a single server queue fed by a stationary sequence of service times and inter-arrival times.

2. The Bound

Suppose that $X = (X_n : n \geq 0)$ is an S-valued sequence satisfying (1) subject to $X_0 = x \in S$, where S is a complete separable metric space. We assume that $(Z_n : -\infty < n < \infty)$ is a stationary sequence for which (2) holds. Put $\mathcal{F}_n = \sigma(Z_j : j \leq n)$.

Theorem 1. *Suppose that $f : S \to \mathbb{R}_+$ is a continuous function. If there exists $g : S \to \mathbb{R}_+$ and $b \in \mathbb{R}_+$ such that*

$$\mathbb{E}[g(r(y, Z_1))|\mathcal{F}_0] \leq g(y) - f(y) + b \tag{3}$$

for $y \in S$, then $\mathbb{E} f(X_\infty) \leq b$.

This result generalizes a similar bound that is known for Markov chains; see [7]. In that context, g is known as a Lyapunov function.

Proof. By stationarity of $(Z_n : -\infty < n \leq \infty)$, (3) implies that

$$\mathbb{E}[g(r(y, Z_{j+1}))|\mathcal{F}_j] \leq g(y) - f(y) + b$$

for $j \geq 0$. Since X_j is $\mathcal{F}_j$-measurable, it follows that

$$\mathbb{E}[g(r(X_j, Z_{j+1}))|\mathcal{F}_j] \leq g(X_j) - f(X_j) + b$$

and hence

$$\mathbb{E}[g(X_{j+1})|\mathcal{F}_j] \leq g(X_j) - f(X_j) + b \tag{4}$$

for $y \geq 0$.

Put

$$T_m = \inf\{j \geq 0 : \mathbb{E}[g(X_{j+1})|\mathcal{F}_j] > m\}$$

and note that on $\{T_m > j\}$, $\mathbb{E}[g(X_{j+1})|\mathcal{F}_j] \leq m$ and

$$\begin{aligned} &\mathbb{E}\, g(X_j) I(T_m \geq j) \\ =& \,\mathbb{E}\,\mathbb{E}[g(X_j)|\mathcal{F}_{j-1}] I(T_m \geq j) \\ \leq& \, m P(T_m > j). \end{aligned}$$

Hence,

$$\sum_{j=1}^{T_m} g(X_j) \text{ and } \sum_{j=0}^{T_m-1} \mathbb{E}[g(X_{j+1})|\mathcal{F}_j]$$

are both integrable. Furthermore,

$$\sum_{j=1}^{T_m \wedge n} (g(X_j) - \mathbb{E}[g(X_j)|\mathcal{F}_{j-1}])$$

is a martingale adapted to $(\mathcal{F}_j : j \geq 1)$; where $a \wedge b \triangleq \min(a, b)$.

Consequently,

$$\mathbb{E} \sum_{j=1}^{T_m \wedge n} (g(X_j) - \mathbb{E}[g(X_j)|\mathcal{F}_{j-1}]) = 0$$

for $n \geq 1$. So,

$$0 = -\,\mathbb{E} \sum_{j=0}^{(T_m \wedge n)-1} (\mathbb{E}[g(X_{j+1}|\mathcal{F}_j)] - g(X_j)) + \mathbb{E}\, g(X_{T_m \wedge n}) - \mathbb{E}\, g(X_0).$$

The non-negativity of $g(X_{T_m \wedge n})$ ensures that

$$g(x) \geq -\,\mathbb{E} \sum_{j=0}^{(T_m \wedge n)-1} (\mathbb{E}[g(X_{j+1})|\mathcal{F}_j] - g(X_j)),$$

so that (4) yields the inequality

$$\begin{aligned} b\,\mathbb{E}\, T_m \wedge n + g(x) &\geq \mathbb{E} \sum_{j=0}^{(T_m \wedge n)-1} (b - \mathbb{E}[g(X_{j+1})|\mathcal{F}_j] + g(X_j)) \\ &\geq \mathbb{E} \sum_{j=0}^{(T_m \wedge n)-1} f(X_j). \end{aligned}$$

Sending $m \to \infty$, we find that the Monotone Convergence Theorem implies that

$$bn + g(x) \geq \sum_{j=0}^{n-1} \mathbb{E}\, f(X_j). \tag{5}$$

Since f is continuous, $f(X_n) \Rightarrow f(X_\infty)$. Fatou's lemma therefore guarantees that

$$\mathbb{E} f(X_\infty) \leq \liminf_{n\to\infty} \mathbb{E} f(X_n). \tag{6}$$

Dividing both sides of (5) by n, sending $n \to \infty$, and applying (6), we obtain the bound.

A key assumption in Theorem 1 is the requirement (2). We now elaborate further on this weak convergence result, using the iterated random function perspective. In particular, we define $\varphi_n : S \to S$ via

$$\varphi_n(y) = r(y, Z_n)$$

for $y \in S$. We can then view X_n as $X_n(x)$, where $X_0(y) = y$ and

$$\begin{aligned} X_n(y) &= \varphi_n(X_{n-1}(y)) \\ &= (\varphi_n \circ \cdots \circ \varphi_1)(y) \end{aligned}$$

for $y \in S$. The stationarity of the Z_i's implies, of course, that $(\varphi_n : -\infty < n < \infty)$ is a stationary sequence of random mappings. While the iterated random function viewpoint has become a widely used tool in the Markov chain setting (see, for example, [4], [12], [5], [6]), it has been exploited less frequently when analyzing dynamical systems fed by stationary inputs. Notable exceptions are the papers by [1, 2], which focus on the application of tightness and compactness arguments to provide sufficient conditions for (2).

We illustrate the iterated random function perspective with the discussion of a couple of sufficient conditions for (2). The first is an elaboration of the famous argument due to [10], expressed in the language of iterated random functions. Suppose that $S = \mathbb{R}_+$ and that φ_i is a non-decreasing continuous function for $i \in \mathbb{Z}$. Put $Y_0(y) = y$ and

$$Y_n(y) = (\varphi_{-1} \circ \cdots \circ \varphi_{-n})(y)$$

for $n \geq 1$. Then, $Y_n(y) = Y_{n-1}(\varphi_{-n}(y))$. Since $Y_{n-1}(\cdot)$ is non-decreasing and $\varphi_{-n}(0) \geq 0$, clearly $Y_n(0) \geq Y_{n-1}(0)$. It follows that there exists Y_∞ such that $Y_n(0) \nearrow Y_\infty$ as $n \to 0$. Because

$$Y_n(0) \stackrel{\mathcal{D}}{=} X_n(0)$$

where $\stackrel{\mathcal{D}}{=}$ denotes equality in distribution, it follows that if Y_∞ is finite-valued, then

$$X_n(0) \Rightarrow Y_\infty$$

as $n \to \infty$. Furthermore, because

$$\tilde{Y}_n(0) = (\varphi_{-2} \circ \cdots \circ \varphi_{-n-1})(0) \stackrel{\mathcal{D}}{=} Y_n(0)$$

and $Y_n(0) = \varphi_{-1}(\tilde{Y}_{n-1}(0))$, it follows from the continuity of φ_{-1} that

$$Y_\infty \stackrel{\mathcal{D}}{=} \varphi_{-1}(\tilde{Y}_\infty), \tag{7}$$

where $\tilde{Y}_\infty = \lim_{n\to\infty} \tilde{Y}_n(0) \overset{\mathcal{D}}{=} Y_\infty$. The distributional identity (7) is the stationary process analog of the steady-state equations that arise in the setting of Markov chain models. It is in this sense that the distribution of Y_∞ is a stationary distribution of the dynamical system (1), and that we refer to $\mathbb{E}\, f(Y_\infty)$ as a *stationary expectation*.

It remains to provide a sufficient condition for the finiteness of Y_∞. We say that the function $f : S \to \mathbb{R}_+$ is *coercive* if the sub-level set $K_m = \{x \in S : f(x) \le m\}$ is a compact subset of S for each $m \ge 1$. Recall that a sequence $(P_n : n \ge 1)$ of probabilities on S is *tight* if for each $\epsilon > 0$, there exists a compact subset K for which $P_n(K) \ge 1-\epsilon$ for $n \ge 1$.

Proposition 1. *Suppose that S is a complete separable metric space and that X satisfies* (3) *with $f : S \to \mathbb{R}_+$ coercive, $b \in \mathbb{R}_+$, and $g : S \to \mathbb{R}_+$. If*

$$P_n(\cdot) = \frac{1}{n}\sum_{j=0}^{n-1} P(X_j \in \cdot),$$

then $(P_n : n \ge 1)$ is tight.

Proof. The proof of Theorem 1 establishes that when $X_0 = x \in S$,

$$\frac{1}{n}\sum_{j=0}^{n-1} \mathbb{E}\, f(X_j) = \int_S f(y)P_n(dy) \le b + \frac{g(x)}{n}.$$

Hence, Markov's inequality implies that

$$P_n(K_m^c) \le \int_{K_m^c} \frac{f(y)}{m} P_n(dy) \le \frac{1}{m}\int_S f(y)P_n(dy) \le \frac{b}{m} + \frac{g(x)}{mn},$$

so that by choosing m sufficiently large, $P_n(K_m^c) < \epsilon$, thereby proving tightness.

Because $Y(0) \overset{\mathcal{D}}{=} X_n(0)$, it is evident that

$$P_n(\cdot) = \frac{1}{n}\sum_{j=0}^{n-1} P(Y_j(0) \in \cdot),$$

and hence Proposition 1 establishes that the almost sure (and weak) limit Y_∞ is finite-valued in the presence of a coercive f.

We conclude this section with a second style of argument that can be used to establish the validity of (2). This approach, based on an assumption of contractiveness in r, has found many applications in the Markov chain setting where the Z_i's (and hence the φ_i's) are iid.

Our contraction assumption takes the form

$$\mathbb{E}[d(r(y, Z_1), r(y_2, Z_1))|\mathcal{F}_0] \le \gamma d(y_1, y_2) \tag{8}$$

for $y_1, y_2 \in S$, where $\gamma < 1$ and $d : S \times S \to \mathbb{R}_+$ is the metric on S. The inequality (8) is, of course, equivalent to requiring that

$$\mathbb{E}[d(\varphi_n(y_1), \varphi_n(y_2))|\mathcal{F}_{n-1}] \leq \gamma d(y_1, y_2) \tag{9}$$

for $n \in \mathbb{Z}$. Relation (9) implies that

$$\begin{aligned} &\mathbb{E}\, d((\varphi_n \circ \cdots \varphi_1)(y), (\varphi_n \circ \cdots \varphi_2)(y)) \\ =\ &\mathbb{E}\,\mathbb{E}[d((\varphi_n \circ \cdots \varphi_1)(y), (\varphi_n \circ \cdots \varphi_2)(y))|\mathcal{F}_{n-1}] \\ \leq\ &\gamma\, \mathbb{E}\, d((\varphi_{n-1} \circ \cdots \varphi_1)(y), (\varphi_{n-1} \circ \cdots \varphi_2)(y)), \end{aligned}$$

from which we find that

$$\mathbb{E}\, d((\varphi_n \circ \cdots \varphi_1)(y), (\varphi_n \circ \cdots \varphi_2)(y)) \leq \gamma^{n-1}\, \mathbb{E}\, d(\varphi_1(y), y). \tag{10}$$

As in our Loynes-style argument, we put

$$Y_n(y) = (\varphi_{-1} \circ \cdots \varphi_{-n})(y).$$

Since $((\varphi_n \circ \cdots \varphi_1)(y), (\varphi_n \circ \cdots \varphi_2)(y)) \overset{\mathcal{D}}{=} (Y_n(y), Y_{n-1}(y))$, (10) shows that

$$\mathbb{E}\, d(Y_n(y), Y_{n-1}(y)) \leq \gamma^{n-1}\, \mathbb{E}\, d(Y_1(y), y).$$

Hence, if $\mathbb{E}\, d(Y_1(y), y) < \infty$,

$$\mathbb{E} \sum_{n=1}^{\infty} d(Y_n(y), Y_{n-1}(y)) \leq (1-\gamma)^{-1}\, \mathbb{E}\, d(Y_1(y), y) < \infty,$$

so that

$$\sum_{n=1}^{\infty} d(Y_n(y), Y_{n-1}(y)) < \infty \quad a.s. \tag{11}$$

This inequality (11) implies that $(Y_n(y) : n \geq 0)$ is a.s. Cauchy in the metric d. Since S is assumed to be complete, it follows that there exists $Y_\infty(y) \in S$ such that $Y_n(y) \to Y_\infty(y)$ a.s. in the metric d. Since $X_n = (\varphi_n \circ \cdots \varphi_1)(x)$, $X_n \overset{\mathcal{D}}{=} Y_n(x)$ and we may conclude that

$$X_n \Rightarrow Y_\infty(x)$$

as $n \to \infty$, proving (2) under (8).

Remark 1. *The contraction assumption ensures that* $\mathbb{E}\, d(Y_n(y_1), Y_n(y_2)) \to 0$ *as* $n \to \infty$, *thereby guaranteeing that* $Y_\infty(y)$ *is independent of* $y \in S$.

Remark 2. *If the* φ_i*'s are continuous, then the same argument as used in the monotone case proves that the equation* (7) *holds, establishing that* (1) *possesses a stationary distribution.*

3. A Queueing Example

In this section, we illustrate our bound by applying it to the well-studied delay sequence for the single-server queue. Specifically, we consider a queue with an infinite capacity waiting room that operates according to a first in/first out (FIFO) queue discipline. In this setting, it is well known that the delay X_n associated with the n'th customer to arrive satisfies a recursion of the form (1), where $S = \mathbb{R}_+$, $r(x, z) = [x + z]^+$ (with $[w]^+ \triangleq \max(w, 0)$ for $w \in \mathbb{R}$), and

$$Z_n = V_{n-1} - \chi_n,$$

with V_n representing the service time requirement for customer n, and χ_n corresponding to the inter-arrival time separating the arrivals of customers $n - 1$ and n; see [8, 9, 3] for an illustration of this recursion. We note that r satisfies the monotonicity and continuity assumptions that were discussed in Section 2 in the context of our Loynes-style argument.

Put $g(x) = x^2$ and observe that

$$g(r(x, z)) = ([x + z]^+)^2 \leq (x + z)^2,$$

so

$$\mathbb{E}[g(r(y, Z_1))|\mathcal{F}_0] - g(y) = 2y\,\mathbb{E}[Z_1|\mathcal{F}_0] + \mathbb{E}[Z_1^2|\mathcal{F}_0].$$

Hence, if we assume that

$$\mathbb{E}[Z_1|\mathcal{F}_0] \leq -\delta < 0 \quad a.s. \tag{12}$$

and

$$\mathbb{E}[Z_1^2|\mathcal{F}_0] \leq b < \infty \quad a.s., \tag{13}$$

it follows that we may choose $f(x) = \delta x$ in Theorem 1. We conclude that in the presence of assumptions (12) and (13), Theorem 1 and Proposition 1 imply the existence of a rv X_∞ satisfying (7), and also the bound

$$\mathbb{E}\,X_\infty \leq \frac{b}{\delta}.$$

One means of inducing auto-correlation in the Z_i's is to assume that $(Z_i : -\infty < i < \infty)$ can be represented in terms of a *moving average*. In particular, suppose that

$$Z_i = \sum_{j=0}^{\infty} a_j \epsilon_{i-j} \tag{14}$$

where $(\epsilon_k : -\infty < k < \infty)$ is an iid sequence with finite second moment, and

$$\sum_{j=0}^{\infty} |a_j| < \infty.$$

Then,

$$\mathbb{E}[Z_1|\mathcal{F}_0] = a_0\,\mathbb{E}\,\epsilon_1 + \sum_{j=1}^{\infty} a_j \epsilon_{i-j}$$

and

$$\mathrm{Var}[Z_1|\mathcal{F}_0] = a_0^2\,\mathrm{Var}\,\epsilon_1$$

A sufficient condition for (12) to hold is to then assume that the ϵ_i's are bounded rv's (say, bounded in absolute value by 1), with

$$a_0\,\mathbb{E}\,\epsilon_1 + \sum_{j=1}^{\infty} |a_j| < 0. \tag{15}$$

We note that condition (15) is a strong requirement on the Z_i's.

In the spirit of weakening this assumption, we consider the following generalization of condition (3), namely

$$\mathbb{E}[g(X_m(y))|\mathcal{F}_0] \le g(y) - f(y) + b \tag{16}$$

for $y \in S$. Condition (16) involves consideration of the conditional expectation over $m \ge 1$ steps of X, rather than the single step postulated in Theorem 1. The identical argument as used in Theorem 1 establishes that

$$\frac{1}{n}\sum_{j=0}^{n-1} \mathbb{E}\,f(X_{jm}(y)) \le b + \frac{g(y)}{n}. \tag{17}$$

Note that $X_{i+jm}(y) = X_{jm}(X_i(y))$, so that (17) implies that

$$\frac{1}{n}\sum_{j=0}^{n-1} \mathbb{E}\,f(X_{i+jm}(y)) \le b + \frac{\mathbb{E}\,g(X_i(x))}{n}$$

for $0 \le i < m$, from which it is evident that

$$\frac{1}{nm}\sum_{j=0}^{nm-1} \mathbb{E}\,f(X_j(x)) \le b + \sum_{i=0}^{m-1} \frac{\mathbb{E}\,g(X_i(x))}{nm}. \tag{18}$$

Suppose that (16) holds and

$$\sum_{i=0}^{m-1} \mathbb{E}\,g(X_i(x)) < \infty. \tag{19}$$

Under these assumptions, the same argument as used for $m = 1$ shows that

$$P_n(\cdot) = \frac{1}{n}\sum_{i=0}^{n-1} P(X_i(x) \in \cdot) \Rightarrow P(X_\infty \in \cdot).$$

We may further conclude that

$$\mathbb{E}\, f(X_\infty) \le b.$$

Returning to our delay sequence example, put

$$S_n = Z_1 + \cdots + Z_n.$$

It is well known that

$$X_m(y) = (y + S_m)I(S_m > -y) + \max_{0\le i\le m}(S_m - S_i)I(S_m \le -y); \tag{20}$$

see Chapter X of [3]. Consequently,

$$\begin{aligned}
&g(X_m(y)) - g(y)\\
&\le (y+S_m)^2 I(S_m > -y) - y^2 + \max_{0\le i\le m}(S_m - S_i)^2\\
&\le (y+S_m)^2 - y^2 + \max_{0\le i\le m}(S_m - S_i)^2\\
&= 2yS_m + S_m^2 + \max_{0\le i\le m}(S_m - S_i)^2.
\end{aligned} \tag{21}$$

Suppose that

$$\mathbb{E}[S_m|\mathcal{F}_0] \le -d < 0 \tag{22}$$

and

$$\mathbb{E}[S_m^2 + \max_{0\le i\le m}(S_m - S_i)^2|\mathcal{F}_0] \le \beta < \infty. \tag{23}$$

If $f(x) = dx$, then (22) implies that condition (16) is in force. Also, it is evident that (23) yields the inequality

$$\mathbb{E}\, S_j^2 = \mathbb{E}(S_m - S_{m-j})^2 \le \mathbb{E}\max_{0\le i\le m}(S_m - S_i)^2 \le \beta < \infty$$

for $0 \le j \le m$. Hence, (20) shows that

$$\begin{aligned}
\mathbb{E}\, g(X_i(x)) &\le \mathbb{E}(x+S_i)^2 + \mathbb{E}\max_{0\le j\le i}(S_i - S_j)^2\\
&\le \mathbb{E}(x+S_i)^2 + \sum_{j=0}^{i}\mathbb{E}(S_i - S_j)^2\\
&\le x^2 + 2x\,\mathbb{E}\, S_i + \mathbb{E}\, S_i^2 + \sum_{j=0}^{i}\mathbb{E}\, S_{i-j}^2\\
&< \infty
\end{aligned}$$

for $0 \leq i < m$, proving the validity of (19). Hence, we may conclude that (22) and (23) guarantee that

$$P_n(\cdot) = \frac{1}{n}\sum_{i=0}^{n-1} P(X_i(x) \in \cdot) \Rightarrow P(X_\infty \in \cdot)$$

as $n \to \infty$, where $\mathbb{E}\, X_\infty$ satisfies the bound

$$\mathbb{E}\, X_\infty \leq \frac{\beta}{d}.$$

Use of the m-step conditions (22) and (23) weaken the requirements demanded of the moving average defined in (14). In particular, (22) requires that when the ϵ_i's are again bounded by 1, the condition analogous to (15) now takes the form

$$\sum_{i=0}^{m-1} a_i\, \mathbb{E}\, \epsilon_1 + \sum_{j \geq m} |a_j| < 0. \tag{24}$$

Condition (24) significantly weakens the requirements on the coefficients of the moving average, thereby generalizing the dependency structures present within the Z_i's for which bounds can be computed.

References

[1] Anantharam, V., & Konstantopoulos, T. (1997). Stationary solutions of stochastic recursions describing discrete event systems. *Stochastic Processes and Their Applications*, 68(2), 181–194.

[2] Anantharam, V., & Konstantopoulos, T. (1999). A correction and some additional remarks on: stationary solutions of stochastic recursions describing discrete event systems. *Stochastic Processes and Their Applications*, 80, 271–278.

[3] Asmussen, S. (2003). *Applied Probability and Queues*, Volume 2. Springer.

[4] Diaconis, P., & Freedman, D. (1999). Iterated random functions. *SIAM review*, 41(1), 45–76.

[5] Duflo, M. (2013). *Random Iterative Models*, Volume 34. Springer Science & Business Media.

[6] Glynn, P. W., & Rhee, C.-H. (2014). Exact estimation for Markov chain equilibrium expectations. *Journal of Applied Probability*, 51(A), 377–389.

[7] Glynn, P. W., & Zeevi, A. (2008). Bounding stationary expectations of Markov processes. *Markov Processes and Related Topics: a Festschrift for Thomas G. Kurtz*, 4, 195–214.

[8] Kendall, D. G. (1951). Some problems in the theory of queues. *Journal of the Royal Statistical Society: Series B (Methodological)*, 13(2), 151–173.

[9] Lindley, D. V. (1952). The theory of queues with a single server. *Mathematical Proceedings of the Cambridge Philosophical Society*, Volume 48, 277–289. Cambridge University Press.

[10] Loynes, R. M. (1962). The stability of a queue with non-independent inter-arrival and service times. *Mathematical Proceedings of the Cambridge Philosophical Society*, Volume 58, 497–520. Cambridge University Press.

[11] Marshall, K. T. (1968). Some inequalities in queuing. *Operations Research*, 16(3), 651–668.

[12] Stenflo, Ö. (2012). A survey of average contractive iterated function systems. *Journal of Difference Equations and Applications*, 18(8), 1355–1380.