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Deterministic and Stochastic Wireless Network Games: Equilibrium, Dynamics, and Price of Anarchy

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Abstract. Power control over wireless networks has been an active area of research with significant applied impact. A well-motivated line of this research, which has received increasing attention, is applying game-theoretic tools for both gaining insight and design of algorithms. In this paper, we build on the existing work and present a simple game-theoretic formulation of power control on wireless networks that incorporates two novel features. First, we do not impose exogenous power bounds on the feasible transmission power. Second, we allow the channel environment to be stochastic and time varying. Within this model, we first examine the deterministic game under a fixed environment, in which we develop a novel fixed-point theorem of independent interest that operates in general and unbounded partially ordered sets. We then leverage this customized fixed-point theorem to establish various equilibrium-related results: existence, uniqueness, and convergence, followed by a novel Price-of-Anarchy bound characterization. Finally, we study the stochastic behavior of the best response dynamics and establish a number of desirable properties in the presence of a stochastic and time-varying channel.

Keywords: game theory • stochastic stability • power control • fixed point theory • wireless networks

1. Introduction

Communications on wireless networks is a broad field with important applications and with power control being an important problem (Rappaport 2001, Goldsmith 2005). Over the past two decades, distributed power control (by which each transmitter regulates its own power) has emerged to be the predominant power control paradigm for good reasons: centralized coordination is extremely difficult in an often-case large-scale wireless network; further, a single point failure in a central power allocator can have devastating effects.

Consequently, dedicated efforts have been devoted to designing good distributed power control algorithms that achieve certain performance (quality-of-service) guarantees. Much of the work in this space traces back to a simple distributed power control algorithm proposed in Foschini and Miljanic (1993) that is shown to achieve target signal-to-interference-and-noise ratio (SINR) with minimum power, Subsequently, various refinements and extensions have appeared in a series of articles (Mitra 1994, Yates 1996, Ulukus and Yates 1998, Holliday et al. 2003, Zhou et al. 2016b) also with the principal objective of achieving certain SINR thresholds. The landscape of objectives under consideration have since been expanded significantly in wireless communications (as well as in the closely related wireline networks), resulting in more sophisticated models that study, as a highly incomplete list, throughput (El Gamal et al. 2006, Reddy et al. 2008, Seferoglu et al. 2008), fairness (Eryilmaz et al. 2006), delays (Eryilmaz et al. 2008, Altman et al. 2010), and backlog (Gitzenis and Bambos 2002, Reddy et al. 2012, Gopalan et al. 2015).

An important thread of work, which has much to contribute on distributed power control, is a class of utility-based models on the transmitter side that induces strategic interactions, typically in the form of a noncooperative multiplayer game. There have been several such formulations that fall into this thread (Famolari et al. 1999; Alpcan et al. 2002; Saraydar et al. 2002; Xiao et al. 2003; Han and Liu 2005; Srivastava et al. 2005; Alpcan et al. 2006; Meshkati et al. 2007; Zhu and Pavel 2007, 2008; Candogan et al. 2010; Hoang et al. 2015; Zhou et al. 2016a), depending on the specific quantities that the utility model aims to capture as well as the underlying wireless networks. More broadly, this research thread belongs to the well-known and well-motivated paradigm of applying game theory to wireless networks. See Menache and Ozdaglar (2010) and Han et al. (2014) for two seminal monographs that give articulate and comprehensive treatments on this field, including both theory and applications.

At its core, such a utility-based model provides an economic–theoretic explanation for the power-selection process for a transmitter: why it chooses what it chooses or, equivalently, how it should make a choice. Such a game-theoretic model then naturally induces a distributed power control scheme (the well-known best response update): at each iteration, a transmitter chooses the power to maximize its utility. An attractive feature in this resulting distributed power control scheme is that they converge to the unique Nash equilibrium (NE) of the game (shown in Menache and Ozdaglar 2010 and Han et al. 2014), thereby providing a check on self-consistency.

However, in the particular power control games, prior models tend to assume the underlying channel environment is static. That is, the power gain matrix (inherently a function of the underlying wireless network topology) and/or the channel noise are typically assumed to be static over time. Although in certain applications (e.g., slow-varying channels), this provides a good approximation, in certain others, moving wireless links and random channel disturbances can cause either (or both) assumption(s) to fail. Consequently, a natural question arises: what is the behavior of the best response power control scheme under stochastic and time-varying channels? Given the strong merits of the elegant best response power control scheme as articulated by Menache and Ozdaglar (2010) and Han et al. (2014), this question is fully motivated because a positive result on stochastic stability will provide a characterization of its robustness, thereby increasing its applicability. Our message, fortunately, is that this is indeed the case even in the absence of exogenous bounds on feasible transmission power.

1.1. Our Contributions

Our contributions are threefold. First, building on existing models as the ones given in prior work (Famolari et al. 1999, Menache and Ozdaglar 2010, Han et al. 2014, etc.), we present a game-theoretic formulation of power control in wireless communications that incorporates two additional novel features (Section 2). First, we do not impose exogenous power bounds on the feasible transmission power as is commonly done in the abovementioned power control games literature. This not only gives a mathematically interesting perspective, but also provides a good proxy for understanding the regime in which transmission power is large, particularly when one is interested in stability-type results as is our focus here. Second, we also allow the channel environment to be stochastic and time varying and thereby enable a robustness characterization of the results obtained from the simplified case in which the environment is fixed.

Second, we examine the deterministic game under the fixed environment in which power is not exogenously assumed to be bounded. We start by establishing a novel and Tarksi-like fixed point theorem that dispenses with the normal "bounded-lattice" assumption (Granas and Dugundji 2003) prevalent in different variants of Tarski's theorem. This is a crucial step in our case because the unbounded power assumption poses a challenge that cannot be directly addressed using the existing fixed-point theorems. We then establish the existence and uniqueness of the Nash equilibrium by combining our customized fixed-point theorem and the special structures in this particular power control context. The customized fixed-point theorem is stated in a general poset setting and hence is, in our view, interesting in its own right and can potentially be used in other applications that deal with unbounded decision variables. Furthermore, we also place emphasis on convergence to the unique Nash equilibrium. In particular, we show that, under our setting, both synchronous and asynchronous best response dynamics converge to the unique Nash equilibrium. Finally, we provide a case study on the Price-of-Anarchy bounds for the power control game. See Section 3 for details.

Third, we study in depth the stochastic behavior of the best response dynamics under a simple model of stochastic time-varying channel environments, in which we allow both the channel gains (determined by the underlying network link topology) and the noise to change randomly over time. We show that the system admits a unique stationary distribution, and the power iterates under the best response dynamics will converge to this stationary distribution irrespective of the initial conditions. Furthermore, we show that the convergence rate is exponential. These results together provide a complete characterization of the stochastic stability of the best response–based power control scheme. Such a characterization is enabled by certain interesting structural properties present in the model, which we highlight in Section 4 by promoting the geometric insights associated with the analysis. Finally, we establish finite-time high-concentration bounds for the long-run average transmission power under random environments and end the section with a comparative statics discussion that reveals an interesting insight on how noise affects the system performance.

Some initial results are presented in the preliminary conference version (Zhou and Bambos 2015). This paper has significantly expanded upon Zhou and Bambos (2015) in several aspects. First, it provides a streamlined presentation with full proofs. Second, for the fixed environment case, we include the general poset fixed-point theorem as well as detailed characterizations of synchronous and asynchronous best response updates, all of which are not discussed in Zhou and Bambos (2015). Third, for the stochastic environment case, Zhou and Bambos (2015) is limited only to a simulation-based study that has shown the promise of stability in the presence of stochastic and time-varying channels. Further, the randomness considered in Zhou and Bambos (2015) is only limited to the noise: the power gain matrix is still assumed to be fixed and constant over time. Here, we give an extensive and general theoretical treatment that, as a by-product, satisfactorily settles all conjectures raised in Zhou and Bambos (2015) in a forthright manner.

2. Game Model on Wireless Networks

In this section, we consider a game-theoretical formulation of communications over wireless networks in which the environment (to be defined in Section 2.1) is fixed over time. This not only provides a simple starting point, which will be generalized in Section 4 to incorporate a random environment, but it also gives crisp intuition of the properties of the system that prove important for the random environment case.

2.1. Power Control in Wireless Communications

We consider a wireless communications setting as introduced in Foschini and Miljanic (1993) in which there is a network of N communication links, each of which is composed of a transmitter and a corresponding intended receiver. The power vector for transmission is denoted by $P = (P_1, \ldots, P_N)$, where P_i is the power used by (the transmitter of) link i. Throughout the paper, it is assumed that $P \in \mathbf{R}_+^N$.[1] A commonly used (Weeraddana et al. 2012, Tan 2014) measure of link service quality is the SINR. Given a power vector P, link i's SINR $R_i(P)$ is given by

$$R_i(P) = \frac{G_{ii}P_i}{\sum_{j\neq i} G_{ij}P_j + \eta_i}, \qquad (1)$$

where G_{ij} is the channel gain from the transmitter j to receiver i, and η_i is the overall noise, comprising both the intrinsic noise (e.g., thermal noise) and the extraneous noise induced by interferers outside of the system under consideration. Collecting each individual generalized noise into the vector $\boldsymbol{\eta}$ and all the channel gains into the channel gain matrix $\mathbf{G}$ with $G \in \mathbf{R}_+^{N\times N}$ and $G_{ii} \in \mathbf{R}_{++}$, the environment of a wireless network can be compactly represented by the pair $(\mathbf{G}, \boldsymbol{\eta})$. For the rest of this section and Section 3, the environment is assumed to be constant. This assumption is lifted in Section 4.

2.2. Cost Model and Game

Each link acts as a rational agent, aiming to minimize its own cost, which consists of two parts: first, a cost associated with power and, second, a cost associated with the quality of the service received (its SINR), which can be interpreted as the inverse utility derived from the quality of service. Hence, given P, the total cost C_i of link i is given by

$$C_i(P) = r_i(P_i) + f_i\left(\frac{G_{ii}P_i}{\sum_{j\neq i} G_{ij}P_j + \eta_i}\right),$$

where $r_i(\cdot)$ is link i's cost of power and can be any convex and increasing function, and $f_i : R_+ \to R_+$ is the function that maps a given SINR to a cost for link i. Note that, in particular, each link can have a different f_i (and/or v_i); links need not agree on the amount of displeasure (i.e., cost) for the same SINR given. However, because of the nature of the quantity (inverse utility) that f_i models, it is reasonable to make the following structural assumptions (on all f_i's):

Assumption 1.
1. *f_i is strictly decreasing and strictly convex;*
2. $\lim_{x\to+\infty} f_i(x) = 0, \lim_{x\to 0^+} f_i(x) = +\infty$[2];
3. *f_i is continuously differentiable.*

Remark 1. Note that these assumptions imply that f_i' is continuous (and also defined) on $(0, +\infty)$ and satisfies the following conditions: $f_i'(x) < 0, \forall x \in (0, +\infty)$; $\lim_{x\to+\infty} f_i'(x) = 0, \lim_{x\to 0^+} f_i'(x) = -\infty$. Examples of the cost functions $f_i(x)$ include $\frac{a}{x^p}, \forall a, p > 0$, $be^{\frac{a}{x^p}} - 1$, $\forall b, a, p > 0$, $\frac{c}{\log^q(x+1)}, \forall c, q > 0$, $\frac{d}{x\log(x+1)}, \forall d > 0$,[3] and so on or any convex combinations of those functions.

This setup naturally induces a N-player noncooperative game. We proceed with the standard solution concept: Nash equilibrium (Başar and Olsder 1998), which is defined as follows in our current context.

Definition 1. P is a (pure strategy) Nash equilibrium if, for each i, $C_i(P_1, \ldots, P_{i-1}, P_i, \ldots, P_N) \le C_i(P_1, \ldots, P_{i-1}, P_i', \ldots, P_N)$, $\forall P_i' \in \mathbf{R}_+$.

That is, P is a (pure strategy) Nash equilibrium if and only if, when every transmitter uses the power according to P, no transmitter i has any incentive to unilaterally deviate from P_i. In what follows, it is understood that any Nash equilibrium we refer to is a pure-strategy Nash equilibrium.

Under this game-theoretical formulation, several questions naturally arise:

- Does there exist a Nash equilibrium? If so, is it unique?
- What dynamics of update schemes, if any, would converge to the unique Nash equilibrium (if one exists)?

Contrary to Han et al. (2014), Menache and Ozdaglar (2010), and Famolari et al. (1999), which also consider utility models of power control games, here we do not restrict the action space (i.e., power level P) to be bounded. Consequently, one cannot directly invoke the standard existence of Nash equilibrium results as done in prior work. In general, when the action space is a compact set, such existence of NE results typically amount to an application of either Brouwer's fixed-point theorem or Kukatani's fixed point theorem (Fudenberg and Tirole 1991, Granas and Dugundji 2003). In Section 3, we provide a novel fixed-point theorem, which can be viewed as a variant of Tarski's fixed-point theorem, which operates in the absence of the endogenous bound on action space.

2.3. Best Response Function

An alternatively equivalent and helpful view of the Nash equilibrium is via best response functions.

Definition 2. The best response function $g_i: R_+^N \to R_+$ of link i is defined as follows:

$$g_i(P) = \arg\min_{P_i' \in \mathbf{R}_+} C_i(P_1, \ldots, P_{i-1}, P_i', \ldots, P_N).$$

In other words, g_i gives the best (minimum cost) P_i, assuming all the other links choose their power according to the fixed given P. Similarly, the joint best response function $g: R_+^N \to R_+^N$ is defined to be $g(P) = (g_1(P), \ldots, g_N(P))$. Therefore, P is a Nash equilibrium if and only if P is a fixed point of g (i.e., $P = g(P)$).

In our current setting, for a given P, there exists a unique minimizer $P_i^\star \in \mathbf{R}_+$ that minimizes $C_i(P_1, \ldots, P_{i-1}, P_i^\star, \ldots, P_N)$. Hence, the best response function g is well-defined, and each link i's best response $P_i^\star = g_i(P)$ can be found via the following first-order optimality condition:

$$-f_i'\left(\frac{G_{ii}P_i^\star}{\sum_{j\neq i} G_{ij}P_j + \eta_i}\right)\left(\frac{G_{ii}}{\sum_{j\neq i} G_{ij}P_j + \eta_i}\right) = r_i'(P_i^*). \quad (2)$$

Note that the best response function g implicitly defined by Equation (2) is a continuous function by the properties of f_i and r_i.

As it turns out, best response functions are not only important in that they serve as a vehicle for establishing results related to NE, but they also have significance serving as a reasonable explanation for the process through which the unique NE is eventually achieved, starting from any initial condition. For this reason, we close this section by studying in some detail the best response function for our current game, which not only provides some structural insight into the system, but also proves useful later in charactering the Nash equilibrium results.

Lemma 1. *Let each $f_i'(y)y$ be increasing in $y \geq 0$.*

1. $\forall P, \hat{P} \in \mathbf{R}_+^N$, $P \leq \hat{P} \Rightarrow g(P) \leq g(\hat{P})$.
2. *For any given P, $\alpha g(P) > g(\alpha P), \forall \alpha > 1$.*
3. *For any $P \in \mathbf{R}_{++}^{\mathbf{N}}$, there exist $\alpha_0 > 0$, such that $\alpha P > g(\alpha P), \forall \alpha \geq \alpha_0$.*

In all three cases, inequality is interpreted as componentwise.

Proof. For (1), assume for contradiction purposes, $g_i(P) > g_i(\hat{P})$ for some i. Denote $g_i(P) = P_i^\star, g_i(\hat{P}) = \hat{P}_i^\star$. By convexity of $r_i(\cdot)$, it follows that $r_i(\hat{P}_i^\star) \leq r_i(P_i^\star)$. Consider Equation (2) that defines g_i, which we rewrite as

$$-f_i'\left(\frac{G_{ii}P_i^\star}{\sum_{j\neq i} G_{ij}P_j + \eta_i}\right)\left(\frac{G_{ii}P_i^\star}{\sum_{j\neq i} G_{ij}P_j + \eta_i}\right)\frac{1}{P_i^\star} = r_i'(P_i^*),$$

$$-f_i'\left(\frac{G_{ii}\hat{P}_i^\star}{\sum_{j\neq i} G_{ij}\hat{P}_j + \eta_i}\right)\left(\frac{G_{ii}\hat{P}_i^\star}{\sum_{j\neq i} G_{ij}\hat{P}_j + \eta_i}\right)\frac{1}{\hat{P}_i^\star} = r_i'(\hat{P}_i^*).$$

Because $P_i^\star > \hat{P}_i^\star$ and $P \leq \hat{P}$, we have $\frac{G_{ii}P_i^\star}{\sum_{j\neq i} G_{ij}P_j + \eta_i} > \frac{G_{ii}\hat{P}_i^\star}{\sum_{j\neq i} G_{ij}\hat{P}_j + \eta_i}$. Further, because $-f_i'(y)y$ is a positive function strictly decreasing in y, we have

$$\begin{aligned} r_i'(P_i^*) &= -f_i'\left(\frac{G_{ii}P_i^\star}{\sum_{j\neq i} G_{ij}P_j + \eta_i}\right)\left(\frac{G_{ii}P_i^\star}{\sum_{j\neq i} G_{ij}P_j + \eta_i}\right)\frac{1}{P_i^\star} \\ &< -f_i'\left(\frac{G_{ii}\hat{P}_i^\star}{\sum_{j\neq i} G_{ij}\hat{P}_j + \eta_i}\right)\left(\frac{G_{ii}\hat{P}_i^\star}{\sum_{j\neq i} G_{ij}\hat{P}_j + \eta_i}\right)\frac{1}{\hat{P}_i^\star} \\ &= r_i'(\hat{P}_i^*), \end{aligned}$$

which contradicts $r_i(\hat{P}_i^\star) \leq r_i(P_i^\star)$. The conclusion therefore follows.

For (2), pick any i and define $\bar{P}_i = g_i(P), \hat{P}_i = g_i(\alpha P)$. First, because $\alpha > 1$, we have $\hat{P}_i \geq \bar{P}_i$ per Statement (1). By convexity of $r_i(\cdot)$, this implies

$$r_i'(\hat{P}_i) \geq r_i'(\bar{P}_i) \quad (3)$$

Next, by the optimality condition, we have

1. $$-f_i'\left(\frac{G_{ii}\bar{P}_i}{\sum_{j\neq i} G_{ij}P_j + \eta_i}\right)\left(\frac{G_{ii}}{\sum_{j\neq i} G_{ij}P_j + \eta_i}\right) = r_i'(\bar{P}_i); \quad (4)$$

2. $$-f_i'\left(\frac{G_{ii}\hat{P}_i}{\alpha\sum_{j\neq i} G_{ij}P_j + \eta_i}\right)\left(\frac{G_{ii}}{\alpha\sum_{j\neq i} G_{ij}P_j + \eta_i}\right) = r_i'(\hat{P}_i). \quad (5)$$

Assume $g_i(\alpha P) \geq \alpha g_i(P)$; then $\hat{P}_i = \beta \bar{P}_i$ for some $\beta \geq \alpha > 1$. Combined with Equation (2), this implies

$$-f_i'\left(\frac{G_{ii}\bar{P}_i}{\frac{\alpha}{\beta}\sum_{j\neq i} G_{ij}P_j + \frac{\eta_i}{\beta}}\right)\left(\frac{G_{ii}}{\alpha\sum_{j\neq i} G_{ij}P_j + \eta_i}\right) = r_i'(\hat{P}_i).$$

Because $\frac{G_{ii}}{\alpha\sum_{j\neq i}G_{ij}P_j+\eta_i} < \frac{G_{ii}}{\sum_{j\neq i}G_{ij}P_j+\eta_i}$ and $-f_i'\left(\frac{G_{ii}\bar{P}_i}{\frac{\alpha}{\beta}\sum_{j\neq i}G_{ij}P_j+\frac{\eta_i}{\beta}}\right) < -f_i'\left(\frac{G_{ii}\bar{P}_i}{\sum_{j\neq i}G_{ij}P_j+\eta_i}\right)$ (because $\frac{G_{ii}\bar{P}_i}{\frac{\alpha}{\beta}\sum_{j\neq i}G_{ij}P_j+\frac{\eta_i}{\beta}} > \frac{G_{ii}\bar{P}_i}{\sum_{j\neq i}G_{ij}P_j+\eta_i}$ and $-f_i'$ is a strictly decreasing function), we have

$$r_i'(\hat{P}_i) = -f_i'\left(\frac{G_{ii}\bar{P}_i}{\frac{\alpha}{\beta}\sum_{j\neq i}G_{ij}P_j + \frac{\eta_i}{\beta}}\right)\left(\frac{G_{ii}}{\alpha\sum_{j\neq i}G_{ij}P_j + \eta_i}\right)$$
$$< -f_i'\left(\frac{G_{ii}\bar{P}_i}{\sum_{j\neq i}G_{ij}P_j + \eta_i}\right)\left(\frac{G_{ii}}{\sum_{j\neq i}G_{ij}P_j + \eta_i}\right) = r_i'(\bar{P}_i),$$

which contradicts Equation (13). Hence, $g_i(\alpha P) < \alpha g_i(P)$. Because this is true for any i, the claim is established.

For (3), we break the proof into two steps. In the first step, we prove the assertion assuming r_i is linear, in which case the cost $C_i(P) = c_iP_i + f_i\left(\frac{G_{ii}P_i}{\sum_{j\neq i}G_{ij}P_j+\eta_i}\right)$ for some $c_i > 0$. In the second step, we establish that the linear cost case $(r_i(P_i) = c_iP_i)$ is the hardest case. In other words, if r_i is any convex increasing function, then $g(\alpha P)$ would only increase more slowly compared with the linear case as α increases, thereby establishing the full statement.

For the first step, note that because the function $-f'(x)$ is strictly decreasing, its inverse, which we denote by h_i, exists (and is defined on $(0, +\infty)$) and is also strictly decreasing. In addition, by the properties of $f'(x)$, it follows that $\lim_{x\to+\infty} h_i(x) = 0, \lim_{x\to 0^+} h_i(x) = \infty$. As a result, we have $\lim_{\alpha\to+\infty} h_i(\alpha x) = 0, \forall x \in (0, +\infty)$, implying that $h_i(\alpha x) = \frac{o(\alpha)}{\alpha}$ as $\alpha \to \infty$ when $h_i(\alpha x)$ is viewed as a function of α for a fixed x.

Fix an arbitrary $P \in \mathbf{R}^N_{++}$. We have, for $\alpha > 1$,

$$g_i(\alpha P) = \frac{\alpha\sum_{j\neq i}G_{ij}P_j + \eta_i}{G_{ii}} h_i\Big(\frac{c_i}{G_{ii}}\Big(\alpha\sum_{j\neq i}G_{ij}P_j + \eta_i\Big)\Big)$$
$$< \frac{\alpha\sum_{j\neq i}G_{ij}P_j + \eta_i}{G_{ii}} h_i\Big(\frac{c_i}{G_{ii}}\alpha\sum_{j\neq i}G_{ij}P_j\Big)$$
$$= \frac{\alpha\sum_{j\neq i}G_{ij}P_j + \eta_i}{G_{ii}} \frac{o(\alpha)}{\alpha} = o(\alpha), \forall i.$$

Hence, for α large enough, $\alpha P_i > g_i(\alpha P_i)$ for every i. Pick such an α and set α_0 to be this α. We then have $\alpha_0 P > g(\alpha_0 P)$. Consequently, for any $\alpha > \alpha_0$, $\alpha P = \frac{\alpha}{\alpha_0}\alpha_0 P > \frac{\alpha}{\alpha_0} g(\alpha_0 P) > g(\frac{\alpha}{\alpha_0}\alpha_0 P) = g(\alpha P)$, where the second inequality follows from (2) of this lemma because $\frac{\alpha}{\alpha_0} > 1$.

For the second step, fix an arbitrary $P \in \mathbf{R}^N_{++}$ and denote $P^\star = g(P)$. By the optimality condition, we have, for each i,

$$-f_i'\left(\frac{G_{ii}P_i^\star}{\sum_{j\neq i}G_{ij}P_j + \eta_i}\right)\left(\frac{G_{ii}}{\sum_{j\neq i}G_{ij}P_j + \eta_i}\right) = r_i'(P_i^\star) \triangleq c_i. \quad (6)$$

Now pick an $\alpha > 1$ and plug αP into Equation (6) in replacement of P. For Equation (6) to continue to hold, P_i^* must increase by the same argument as in Statement (1) of this lemma. However, because r_i is convex, as P_i^* increases, $r_i'(P_i^\star)$ also increases and will exceed c_i. Consequently, because $-f_i'(y)y$ is a positive function strictly decreasing in y, P_i^* needs to increase less for Equation (6) to hold compared with if $r_i(P_i) = c_iP_i$, where c_i takes the particular value $r_i'(P_i^\star)$ as defined in Equation (6). This implies that the value $g(\alpha P)$ under a convex cost r_i will be no larger than the value $g(\alpha P)$ under the corresponding linear cost r_i. The claim, therefore, follows immediately. ■

Remark 2. Because $f_i'(\cdot) < 0$ is an increasing function, $-f_i'(y)$ is positive and decreasing in y. Consequently, the requirement that $-f_i'(y)y$ is decreasing in y means that $-f_i'(y)$ must decrease faster than y increases. If f_i is twice differentiable, this condition is equivalent to $y \geq \frac{-f_i'(y)}{f_i''(y)}, \forall y > 0$. This condition is equivalent to the relative risk aversion assumption given in Menache and Ozdaglar (2010) under the utility (as opposed to cost) formulation. Note that all the examples mentioned in Remark 1 satisfy this assumption. For the rest of this paper, unless noted otherwise, we assume that each f_i satisfies this condition in addition to Assumption 1.

3. Nash Equilibrium Characterizations: Existence, Uniqueness and Convergence

In this section, we address the set of questions raised at the end of the previous section. As mentioned before, a primary challenge in establishing the existence of the Nash equilibrium lies in the absence of the endogenous bounds on the power levels, resulting in both unbounded action space and unbounded cost. We meet this challenge by introducing a customized, Tarski-like fixed-point theorem that operates on unbounded structures as previous fixed-point theorems (Brouwer's, Kukatani's, and Tarski's fixed-point theorems and variants) cannot be directly applied here. We begin by a brief discussion on the existing fixed-point theorems. This discussion serves as a motivation to introduce our own customized fixed-point theorem: we do so in a general setting (i.e., not tied to the domain $\mathbf{R}^n_+$ in the current case) and in a streamlined fashion so as to facilitate the comparison with the classical Tarski's fixed-point theorem. This customized fixed-point theorem, together with the structural properties of the best response function in our setting, then lead to the existence and uniqueness of the Nash equilibrium.

After establishing the existence and uniqueness of the Nash equilibrium, we then proceed to establish that the best response dynamics are guaranteed to converge to the unique Nash equilibrium in our setting—something that is not at all common in a general multiplayer noncooperative game. We establish convergence in both the traditional, synchronous best response update and the less stringent, as well as more practical, asynchronous best response update.

3.1. Fixed-Point Theorems

A vast amount of literature exists on fixed-point theorems. The results can usually be classified based on the assumptions of the underlying map or domain: the completeness of the domain (Banach theorem and its variants), the compactness and convexity of the domain (Brouwer fixed-point theorem, Kakutani fixed-point theorem[4] and variants), the order–theoretical structure of the domain (Tarski's theorems and variants), just to name a few. For a detailed exposition, see Granas and Dugundji (2003).

The ones that concern us here are those that operate on partially ordered sets (hereafter referred to as posets) with the prominent ones being the Knaster–Tarski theorem and Tarski–Kantorovitch theorem. They are relevant because the domain $\mathbf{R}_+^n$ of the function g here is a poset. However, these order–theoretical fixed-point theorems (Davey and Priestley 2002, Granas and Dugundji 2003) roughly share one feature in common: the poset of interest must be bounded—something clearly missing in the current setting. As an illustration example and for ease of comparison, we first state (without proof) the classical Tarski–Kantorovitch fixed-point theorem (Granas and Dugundji 2003).

Definition 3. Let $(P, \leq)$ be a partially ordered set and $G: P \to P$ be a map.

1. A subset $S \subset P$ is called a chain if S is totally ordered under $\leq$.
2. G is called a poset-mapping continuous map if, for each countable chain $\{c_i\}$ having a supremum, $G(\sup\{c_i\}) = \sup\{G\{c_i\}\}$.

Remark 3. If the underlying poset P is also a metric space (such as $\mathbf{R}_+^n$) and is, hence, endowed with the normal continuity concept, a continuous map there needs to be distinguished from a poset-mapping continuous map: the former does not imply the latter. Furthermore, as indicated in Granas and Dugundji (2003), a poset-mapping continuous map is necessarily monotonic: $x \leq y \Rightarrow G(x) \leq G(y)$, where $x, y \in P$. This follows easily by noting that if $x \leq y$, then $y = \sup\{x, y\}$; hence, $G(y) = \sup\{G(x), G(y)\}$, implying that $G(x) \leq G(y)$.

We are now ready to state the Tarski–Kantorovitch fixed-point theorem.

Theorem 1. *Let $(P, \leq)$ be a partially ordered set and $G: P \to P$ be a poset-mapping continuous map. Assume the following two conditions hold:*

1. $\exists p_1 \in P, p_1 \leq G(p_1)$;
2. *every countable chain in $\{x \in P \mid x \geq p_1\}$ has a supremum.*

Then G has a fixed point.

Note that the unboundedness of the domain P ($\mathbf{R}^n$ here) implies the second condition does not hold. As a result, Tarski's theorem cannot be directly invoked here. As mentioned earlier, a few other such fixed-point theorems on posets or lattices also suffer from the same problem.

Consequently, in the next subsection, we give our customized theorem (also in a general setting) that works around this unboundedness issue.

3.2. Our Variant of Fixed-Point Theorem

Theorem 2. *Let $(P, \leq)$ be a partially ordered set and $G: P \to P$ be a poset-mapping continuous map. Assume the following three conditions hold:*

1. $\exists p_1 \in P, p_1 \leq G(p_1)$;
2. *every bounded countable chain in $\{x \in P \mid x \geq p_1\}$ has a supremum;*
3. $\exists p_2 \in P, p_2 > p_1, p_2 \geq G(p_2)$.

Then G has a fixed point.

Remark 4. Two things worth mentioning here. First, Theorem 2 is not necessarily a generalization of Tarski's theorem as there can be partially ordered sets that do satisfy Condition (3) in Theorem 2 but not Condition (2) in Theorem 1. From another viewpoint, although the second condition in Theorem 2 is less stringent, it does have a more stringent requirement: Condition (3). Consequently, Theorem 2 can be viewed as a variant of Tarski's theorem that serves complementary purposes. Second, to give some intuition about the theorem, there is still a notion of "boundness" built into Theorem 2: p_2 serves as an effective "upper bound" that prevents the fixed-point iterations from going to infinity.

Proof. Consider the sequence $\{G^n(p_1)\}_{n\in\mathbf{N}}$, where G^n is recursively defined as follows: $G^0(p_1) = p_1, G^n(p_1) = G(G^{n-1}(p_1)), \forall n > 0$. Because the map G is continuous, it is also monotonic (Remark 3). Hence, we have $G^{n-1}(p_1) \leq G^n(p_1), \forall n \in \mathbf{N}$. Therefore, $\{G^n(p_1)\}_{n\in\mathbf{N}}$ forms a chain (that is increasing).

Moreover, $\exists p_2 > p_1, p_2 \in P$ implies that $G^n(p_2) \geq G^n(p_1)$, and $p_2 \geq G(p_2)$ implies that $p_2 \geq G^n(p_2), \forall n \in \mathbf{N}$. It then follows that $p_2 \geq G^n(p_1), \forall n \in \mathbf{N}$, leading to that the chain $\{G^n(p_1)\}_{n\in\mathbf{N}}$ is bounded; hence, it has a supremum: call it p^* (i.e., $\sup\{\{G^n(p_1)\}_{n\in\mathbf{N}}\} = p^*$).

By the definition of the poset-mapping continuity of G, we have

$$G(p^*) = G(\sup\{G^n(p_1)\}_{n\in\mathbf{N}}) = \sup\{G(\{G^n(p_1)\}_{n\in\mathbf{N}})\} = \sup\{\{G^n(p_1)\}_{n\in\mathbf{N}}\} = p^*.$$

Hence, p^* is a fixed point of G. ∎

Remark 5. Note that one could as well apply the same iteration to p_2, using p_1 as a lower bound to establish that another Nash Equilibrium exists as the limit of

$G^n(p_2)$. In general, these two could well be two different Nash equilibria. In the current context, we shall see later that these two Nash equilibria are indeed the same one because of the uniqueness of the Nash equilibrium established by Theorem 3.

3.3. Existence and Uniqueness of the Nash Equilibrium

By combining the customized fixed-point theorem in Theorem 2 and the properties of the best response function given in Lemma 1, we are now ready to characterize the existence and uniqueness of the Nash equilibrium.

Theorem 3. *There exists a unique Nash equilibrium.*

Proof. For existence, by Lemma 1, g is continuous and monotonic. This leads to that g is a poset-mapping continuous map. To see this, take any sequence $\{p^i\} \subset \mathbf{R}_+^N$:

- By monotonicity of g, $g(\sup\{p^i\}) \geq g(p^i)$, $\forall i$, thereby implying that $g(\sup\{p^i\}) \geq \sup\{g(p^i)\}$.
- By continuity of g, $g(\sup\{p^i\}) \leq \sup\{g(p^i)\}$.

Together, these two statements imply that $g(\sup\{p^i\}) = \sup\{g(p^i)\}$; hence, g is poset-mapping continuous.

Fix any $P \in \mathbf{R}_{++}^N$. By Lemma 1, we can find α sufficiently large with $\alpha P > g(\alpha P)$. Setting $p_1 = 0$, $p_2 = \alpha P$, we have $p_1 \leq g(p_1)$, $p_2 \geq g(p_2)$, $p_1 < p_2$. By the topological structure of $\mathbf{R}_+^n$, we have that every bounded (nonempty) subset has a least upper bound (and, hence, of course, implying every bounded countable chain has a supremum). Applying Theorem 2 establishes the existence of the Nash equilibrium.

For the uniqueness of Nash equilibrium, suppose we have two different Nash equilibria $x, y \in \mathbf{R}_{++}^n$ (note that any Nash equilibrium must reside in $\mathbf{R}_{++}^n$). Let i be the index such that $\frac{y_i}{x_i} = \max_j\{\frac{y_j}{x_j}\}$. Set $\beta = \frac{y_i}{x_i}$. Without loss of generality, assume $\beta > 1$.

Scale x by β so that $\beta x \geq y$ and $\beta x_i = y_i$. We have

$$g_i(\beta x) < \beta g_i(x) = \beta x_i. \quad (7)$$

On the other hand, we also have

$$g_i(\beta x) >= g_i(y) = y_i. \quad (8)$$

This indicates that $y_i < \beta x_i$, contradicting the fact that $\beta x_i = y_i$. Hence, $x = y$. ∎

We end this subsection with a discussion on the uniqueness of Nash equilibrium. First, unlike establishing the existence of a Nash equilibrium, which can be done in sufficient generality (as in our current version of a fixed-point theorem or any of the existing fixed-point theorems), the uniqueness result rarely holds in a general game, and in the case that it does hold, one typically needs to work with the special properties of the specific class of games at hand in order to establish it. In the current setting, to establish the uniqueness result, we have exploited both the monotonicity property and the scalability property of the best response function. This is a rather standard argument and consistent with the existing literature (Alpcan et al. 2002, Menache and Ozdaglar 2010, Han et al. 2014) that do impose bounds on the feasible action set. We do note that there is a slight difference in the argument used here in the unbounded action set case versus that used in the bounded action set case. Specifically, if one imposes bounds on maximum power, then g's domain is some bounded feasible set. In this case, when we multiply β by x, because $\beta > 1$, βx may well lie outside of the bounded feasible power set. Consequently, a priori, one cannot directly apply g to βx. If there is no maximum power constraint, then βx will always be feasible (no matter how large β is), and our uniqueness result relies on this operation to reach a contradiction. In any case, it should be pointed out that establishing a Nash equilibrium's uniqueness often needs to be done on a case-by-case basis (typically) via ad hoc methods that fully exploit the problem structure.

3.4. Synchronous Best Response Dynamics: Convergence to the Unique NE

The best response function g also suggests a natural way for the N links to arrive at the unique Nash equilibrium: at each iteration, each link selects the best power that minimizes its total cost, assuming all the other links will choose their respective best powers from the previous iteration. Hence, each link at each iteration proceeds according to the following update scheme (Algorithm 1).

Algorithm 1. (Synchronous Best Response Dynamics for Power Update)

1: Each link i arbitrarily chooses an initial power $P_i^0 \in \mathbf{R}_+$
2: **for** iteration $k = 0, 1, 2, 3, \ldots$ **do**
3: **for** $i = 1, \ldots, N$ **do**
4: $P_i^{k+1} = g_i(P^k)$
5: **end for**
6: **end for**

Note that each link i will not be able to access the powers (P_j^k's, $j \neq i$) used by other links in the previous iteration; however, it need not because the total interference and noise $\sum_{j\neq i} G_{ij} P_j^k + \eta_i$ that each link (more precisely, each link's receiver) can detect is enough for computing the current best power P_i^{k+1}. In particular, each receiver i can sense the SINR from the previous iteration and send it back to the corresponding transmitter, which can then infer the quantity $\frac{G_{ii}}{\sum_{j\neq i} G_{ij} P_j^k + \eta_i}$, which is sufficient for best response updates. Second, for a general f_i, the function values g_i can be computed efficiently by doing a binary search because of the monotonicity of f_i'. In certain cases, analytical solutions can also be obtained (e.g., $f_i(x) = \frac{1}{x}$).

Next, we show that, under this update scheme, P^k (the k-th power vector iterate) will converge to the unique Nash equilibrium irrespective of the initial power vector P^0.

Lemma 2. *Under Algorithm 1, P^k (the k-th power vector iterate) converges to the unique Nash equilibrium irrespective of the initial power vector P^0.*

Proof. Note that the best response update in Algorithm 1 can be compactly written as $P^k = g^k(P^0)$, where $g^k(\cdot)$ denotes k repeated applications of the best response update. By Lemma 1, for any initial power vector P^0, there exists a sufficiently large $\alpha > 0$, such that

$$0 \le P^0 \le \alpha P^0, \tag{9}$$

with $0 < g(0), \alpha P^0 > g(\alpha P^0)$. Consequently, by applying g repeatedly to the three sides of (9), it follows from Remark 5 that both $g^k(0)$ and $g^k(\alpha P^0)$ converge as $k \to \infty$. The corresponding limits are fixed points of g and are, therefore, Nash equilibria. However, because there is a unique Nash equilibrium by Theorem 3, they must coincide, and it then follows that $g^k(P^0)$, always satisfying $g^k(0) < g^k(P^0) < g^k(\alpha P^0)$, $\forall k$, converges to the same unique Nash equilibrium. ∎

We end this subsection with a brief discussion on convergence speed. First, we note that when a fixed-point iteration is applied in a general poset as defined in Definition 3, it is not meaningful to speak of convergence speed because, at that level of generality, there is no notion of distance to measure the progress toward a fixed point. Consequently, we directly discuss convergence speed in the current power control setting, in which the underlying poset domain has an additional metric space structure: in $\mathbf{R}^N$, any finite dimensional norm $\|\cdot\|$ can serve as a distance metric to measure progress toward the Nash equilibrium (the fixed point in the current setting).

Second, we point out that all the existing results in the literature on convergence speed can be inherited and utilized directly in the current setting. This is because our version of a fixed-point theorem (as introduced in Theorem 2) is a theorem concerning existence of a fixed point with which we have lifted the boundedness assumption. However, the current results on convergence speed are oblivious to whether the domain is bounded or not. (Another way to see that those existing convergence speed results will carry through in the current setting is that because we have shown convergence in Lemma 2, the sequence generated from any initial point is automatically bounded.) As an example, one would get a geometric convergence rate if the best response function (the fixed point map in the current setting) is contractive: $g(\cdot)$ satisfies $\|g(P) - g(\tilde{P})\| \le \rho \|P - \tilde{P}\|$ for some $\rho < 1$. Another well-known condition that gives a geometric convergence rate is that the first-order derivative around the fixed point is well behaved. More formally, let $\mathbf{J}(x)$ be the Jacobian matrix of first partial derivatives of the best response function g at a point $x \in \mathbf{R}^N$:

$$\mathbf{J}_{ij}(x) = \frac{\partial g_i(x)}{\partial x_j}.$$

If we already know that a unique fixed point exists (in this case it's the Nash equilibrium) and that convergence to the unique fixed point is guaranteed, then the geometric convergence rate holds (Kirk and Sims 2001) if $\|\mathbf{J}(P^{Nash})\|_M \le \rho$ for some $\rho < 1$, where $\|\cdot\|_M$ is the matrix norm induced by the vector norm $\|\cdot\|$ (that is used to measure distance). Note that this condition only requires good properties around the fixed point; however, it is important to emphasize that this result presupposes the convergence to the fixed point. Consequently, our existence, uniqueness, and convergence results established previously allow us to take advantage of all the existing convergence rate results.[5]

3.5. Asynchronous Best Response Dynamics: Convergence to the Unique NE

An alternative and less demanding update scheme is the asynchronous best response update (Algorithm 2) in which, in each iteration, not every link is necessarily updating its power. In comparison with the synchronous update scheme, this is a more practical and easily implementable scheme because, in a decentralized update setting, it may not be easy to enforce a fixed time step by which everyone updates simultaneously.

Algorithm 2. (Asynchronous Best Response Dynamics for Power Update)

1: Each link i arbitrarily chooses an initial power $P_i^0 \in \mathbf{R}_+$
2: **for** iteration $k = 0, 1, 2, 3, \ldots$ **do**
3: Let $\mathcal{N}^k \subset \{1, 2, \ldots, N\}$ be a (possibly empty) set of updating links at k
4: **for** each $i \in \mathcal{N}^k$ **do**
5: $P_i^{k+1} = g_i(P^k)$
6: **end for**
7: **end for**

Next, we show that, so long as each link updates its power infinitely often, the iterate will converge to the unique Nash equilibrium. To that end, we first introduce some notation and characterize an important property that is used for the asynchronous best response update.

Definition 4. Let $\mathcal{N} \subset \{1, 2, \ldots, N\}$ and $\{\mathcal{N}^k\}_{k=1}^{\infty}$ be a given sequence of sets with $\mathcal{N}^k \subset \{1, 2, \ldots, N\}$, $\forall k$.

- Define the partial best response function $g^{\mathcal{N}}: \mathbf{R}_+^N \to \mathbf{R}_+^N$ to be

$$g_i^{\mathcal{N}}(P) := \begin{cases} g_i(P), & i \in \mathcal{N} \\ P_i, & 1 \notin \mathcal{N}. \end{cases} \tag{10}$$

Define the corresponding composition $g^{\{\mathcal{N}^l\}_{l=1}^k}$ of partial updates as $g^{\{\mathcal{N}^l\}_{l=1}^k}(P)$.

- Define the set of complete cycle times $\{t_j\}_{j=0}^{\infty}$ to be

$$t_j := \begin{cases} 0, & j = 0 \\ \inf\{k \mid k > t_{j-1}, \forall 1 \le i \le N, \exists k, i \in \mathcal{N}^k\}, & j \ge 1. \end{cases} \tag{11}$$

Intuitively, $g^{\mathcal{N}}$ is the one-step asynchronous best response update, and t_j is the first iteration by when every link has updated its power at least j times.

Theorem 4. *Under Algorithm 2, if each link i is updated infinitely often, that is, for each i, $|\{k \mid i \in \mathcal{N}^k\}| = +\infty$, then P^k (the k-th power vector iterate) converges to the unique Nash equilibrium irrespective of the initial power vector P^0.*

Proof. First note that, by definition, monotonicity still holds under partial best response updates: If $P \ge \tilde{P}$, then $g^{\mathcal{N}}(P) \ge g^{\mathcal{N}}(\tilde{P}), \forall \mathcal{N}$.

Next, let $\{\mathcal{N}^l\}_{l=1}^{\infty}$ be the sequence of updating sets given in Algorithm 2. We show that if $P \le g(P)$, then $g^n(P) \le g^{\{\mathcal{N}^l\}_{l=1}^{t_n}}(P), \forall n \ge 1$, where $g^n(\cdot)$, as before, is n compositions of g with itself. We proceed by an induction argument.

We first establish the base case: $g(P) \le g^{\{\mathcal{N}^l\}_{l=1}^{t_1}}(P)$. To see this, first note that, by monotonicity, we have

$$P \le g^{\{\mathcal{N}^l\}_{l=1}^{1}}(P) \le g^{\{\mathcal{N}^l\}_{l=1}^{2}}(P) \le \dots \le g^{\{\mathcal{N}^l\}_{l=1}^{t_1}}(P). \tag{12}$$

For each $i \in \{1, 2, \dots, N\}$, let $\mathcal{N}^{k_i}$ be the first set that contains i. Per the definition of t_1, we have $k_i \le t_1$. Furthermore, because $i \in \mathcal{N}^{k_i}$, we have

$$g_i^{\{\mathcal{N}^l\}_{l=1}^{k_i}}(P) = g_i^{\mathcal{N}^{k_i}}\left(g^{\{\mathcal{N}^l\}_{l=1}^{k_i-1}}(P)\right) \ge g_i^{\mathcal{N}^{k_i}}(P) = g_i(P),$$

where we have the implicit convention that $g^{\mathcal{N}_{l=1}^{k_i-1}}(P) = P$ if $k_i - 1 = 0$, and the inequality follows from (12). Finally, invoking (12) again, we have that, for each i, $g_i^{\{\mathcal{N}^l\}_{l=1}^{t_1}}(P) \ge g_i^{\{\mathcal{N}^l\}_{l=1}^{k_i}}(P) \ge g_i(P)$, thereby leading to the conclusion.

The rest of the induction follows easily because, by assuming $g^n(P) \le g^{\{\mathcal{N}^l\}_{l=1}^{t_n}}(P), \forall n \ge 1$, we have

$$\begin{aligned} g^{n+1}(P) &= g(g^n(P)) \le g(g^{\{\mathcal{N}^l\}_{l=1}^{t_n}}(P)) \le g^{\{\mathcal{N}^l\}_{l=t_n+1}^{t_{n+1}}}\left(g^{\{\mathcal{N}^l\}_{l=1}^{t_n}}(P)\right) \\ &= g^{\{\mathcal{N}^l\}_{l=1}^{t_{n+1}}}(P). \end{aligned}$$

By a similar argument, it follows that if $P \ge g(P)$, then $g^n(P) \ge g^{\{\mathcal{N}^l\}_{l=1}^{t_n}}(P), \forall n \ge 1$.

Finally, as in the proof to Lemma 2, pick $\alpha > 0$ such that $0 \le P^0 \le \alpha P^0$ with $0 < g(0), \alpha P^0 > g(\alpha P^0)$. Because $g^{\{\mathcal{N}^l\}_{l=1}^{n}}(0)$ is an increasing sequence with $g^{\{\mathcal{N}^l\}_{l=1}^{n}}(0) \le g^n(0)$, it has a limit with $\lim_{n \to +\infty} g^{\{\mathcal{N}^l\}_{l=1}^{n}}(0) \le p^*$, where $p^* = \lim_{n \to +\infty} g^n(0)$ is the unique Nash equilibrium. On the other hand, because $g^n(0) \le g^{\{\mathcal{N}^l\}_{l=1}^{t_n}}(0), \forall$, we have

$$\begin{aligned} \lim_{n \to +\infty} g^{\{\mathcal{N}^l\}_{l=1}^{n}}(0) &= \lim_{n \to +\infty} g^{\{\mathcal{N}^l\}_{l=1}^{t_n}}(0) \\ &\ge \lim_{n \to +\infty} g^n(0) = p^*, \end{aligned}$$

and it follows that $\lim_{n \to +\infty} g^{\{\mathcal{N}^l\}_{l=1}^{n}}(0) = p^*$. By a similar argument, we have $\lim_{n \to +\infty} g^{\{\mathcal{N}^l\}_{l=1}^{n}}(\alpha P) = p^*$. This establishes that $\lim_{n \to +\infty} g^{\{\mathcal{N}^l\}_{l=1}^{n}}(P^0) = p^*$. ■

However, if such an update scheme is used as a distributed power control scheme in practice, then therein lies a fundamental weakness of the modeling assumption: why would the environments (η and $\mathbf{G}$) be fixed over time? The answer is, of course, it wouldn't: neither the power gain matrix nor the generalized noise will. Therefore, it is of immediate interest to consider the stochastic environment case.

3.6. Price of Anarchy in Power Control Games

An interesting problem for noncooperative games is to compare the performance of a Nash equilibrium solution with that of the social optimal solution. The standard metric in comparing equilibrium performances to social optimal performance is known as the Price of Anarchy (PoA), a notion first developed in the theoretical computer science literature (Koutsoupias and Papadimitriou 1999). Specifically, PoA is defined as the ratio between the cost achieved by the worst-performing Nash equilibrium and the cost achieved by the social optimal. The social optimal is the centralized optimal solution for an aggregate social objective, typically formed as a sum of all the individual costs. The smaller the PoA (note that PoA is at least one by definition), the better a Nash equilibrium fares in comparison with the social optimal (even in the worst case).

PoA is mostly thoroughly analyzed in the class of routing games (Roughgarden 2005, 2007; Papadimitriou and Valiant 2010), in which the special structures of routing games are exploited to obtain constant bounds on PoA on the class of routing games independent of the problem parameters. Broadly speaking, there are two classes of routing games (each corresponding to a different special structure). The first class is the atomic routing games, in which each agent has a non-negligible impact on the entire routing network (hence, the word "atomic") and has only a finite set of actions (corresponding to the routes to take). In this (finite game) setting, a variational inequality approach is then employed (see Roughgarden 2007) to obtain PoA bounds. For instance, when all the players' cost functions are affine, PoA is upper bounded by 2.618. The second class is the nonatomic routing games, in which each agent has a negligible impact on the routing network and can, therefore, be abstracted as a single point in a continuum of agents. In other words, nonatomic routing is akin to population games. In this second class, assumptions are typically imposed such that resulting games admit convex potentials based on which bounds on PoA can be derived.

However, in general, unless special structures (as in the routing games) exist, there may not exist any PoA bounds. In particular, the power control games studied here are neither finite games (every player has a continuum space of actions; further, the space of actions is also unbounded) nor potential games (let alone convex potential games). In fact, as we see next, for a general power control game defined in this paper, there cannot exist a constant bound on PoA.

Consider the two-link case in which each $f_i(x) = \frac{1}{x}$, $i = 1, 2$ and $r_1(x) = cx, r_2(x) = x$ with $0 < c < 1$. Fix $\eta_1 = 0$, η_2=0 and gain matrix to be $\begin{bmatrix} G & G' \\ G' & G \end{bmatrix}$, where $G > G'$. In this case, we have $C_1(P) = \frac{G'P_2}{GP_1} + cP_1$ and $C_2(P) = \frac{G'P_1}{GP_2} + P_2$.

It is straightforward to verify that the unique Nash equilibrium $P^{Nash} = (P_1^{Nash}, P_2^{Nash})$ satisfies

$$\frac{G'P_2^{Nash}}{G(P_1^{Nash})^2} = c, \frac{G'P_1^{Nash}}{G(P_2^{Nash})^2} = 1.$$

This then leads to that $\frac{(P_2^{Nash})^3}{(P_1^{Nash})^3} = c$ and $P_2^{Nash} = \frac{G'}{Gc^{\frac{1}{3}}}$.

Let P^* be the optimal solution to the social objective $C(P) = C_1(P) + C_2(P_2)$. It follows that

$$C(P^*) \le C((1,1)) = 2\frac{G'}{G} + c + 1 < 2\left(\frac{G'}{G} + 1\right).$$

On the other hand, evaluating the social objective at P^{Nash} results in

$$C(P^{Nash}) > C_2(P^{Nash}) = 2\frac{G'}{Gc^{\frac{1}{3}}}.$$

Consequently, the PoA is $\frac{C(P^{Nash})}{C(P^*)} > \frac{c^{-\frac{1}{3}}}{1 + \frac{G}{G'}} > \frac{c^{-\frac{1}{3}}}{2\frac{G}{G'}}$. Consequently, we can pick a sequence of triples (G, G', c) such that $\frac{c^{-\frac{1}{3}}}{2\frac{G}{G'}} \to \infty$, in which case the PoA will diverge to infinity and cannot be bounded by a universal constant.

This discussion establishes that there cannot be a bound on PoA for the general case. We now proceed to characterize a PoA bound in the special case of the fully homogeneous wireless network under the same cost functions ($f_i(x) = \frac{1}{x}$ and $r_i(x) = cx$) as in the preceding discussion (in which the negative result is given). The fully homogeneous case is one in which every player shares the same characteristics as all the other players. We believe this is an interesting yet illuminating simple special case because, as it turns out, perhaps surprisingly, in this case, the PoA is at most two, independent of the number of players N; wireless environment parameters G, G', η; and the cost per unit power c. To the best of our knowledge, no prior PoA bound on power control games has been identified even in this special case. We formalize the result in the following theorem.

Theorem 5. *Consider a fully homogeneous power control game in which every player shares the same cost functions $f_i(x) = \frac{1}{x}, r_i(x) = cx$ and the wireless network is fully symmetric: $G_{ii} = G, G_{ij} = G_{ji} = G', \forall i, j,\ \eta_i = \eta,\ \forall i$. Then the PoA is at most two: $\forall N \ge 1, \forall c, G > 0,\ \forall \eta,\ G' \ge 0$,*

$$\frac{C(P^{Nash})}{C(P^*)} \le 2.$$

Proof. For ease of exposition, we break the proof into the following three steps:

1. Characterizing the social optimal solution.

The social cost function is given by

$$C(P) = \sum_{i=1}^{N}\left\{\frac{\sum_{j\ne i} G'P_j + \eta}{GP_i} + cP_i\right\}$$

We first show that there must be a unique minimum to this total cost function with all components equal: the unique minimum has the form $P^* = (p, p, \ldots, p)$.

To see this, assume on the contrary $P^* = (P_1, \ldots, P_N)$ in which not all components are equal. Then, by the symmetry of the cost function, $C(P)$ is invariant under permutation. This implies that, in particular, all the N cyclic permutations of P $((P_1, P_2, P_3, \ldots, P_N), (P_2, P_3, \ldots, P_N, P_1), (P_3, P_4, \ldots, P_1, P_2), \ldots, (P_N, P_1, P_2, \ldots, P_{N-1}))$ are all minima of the function $C(P)$.

Now we evaluate the function $C(P)$ at the point $\tilde{P}$ that is the average of those N cyclical permutations: $\tilde{P} = \left(\frac{\sum_{i=1}^N P_i}{N}, \frac{\sum_{i=1}^N P_i}{N}, \ldots, \frac{\sum_{i=1}^N P_i}{N}\right)$ and obtain

$$\begin{aligned} C(\tilde{P}) &= N\left\{\frac{\sum_{j\ne i} G'\frac{\sum_{i=1}^N P_i}{N} + \eta}{G\frac{\sum_{i=1}^N P_i}{N}} + c\frac{\sum_{i=1}^N P_i}{N}\right\} \\ &= N(N-1)\frac{G'}{G} + \frac{\eta}{G}\frac{N^2}{\sum_{i=1}^N P_i} + c\sum_{i=1}^N P_i. \end{aligned} \tag{13}$$

Next, note that

$$\begin{aligned} C(P) &= \frac{G'}{G}\sum_{i=1}^N \sum_{j\ne i} \frac{P_j}{P_i} + \frac{\eta}{G}\sum_{i=1}^N \frac{1}{P_i} + c\sum_{i=1}^N P_i \\ &= 0.5\frac{G'}{G}\sum_{i=1}^N \sum_{j\ne i}\left(\frac{P_j}{P_i} + \frac{P_i}{P_j}\right) + \frac{\eta}{G}\sum_{i=1}^N \frac{1}{P_i} + c\sum_{i=1}^N P_i \end{aligned} \tag{14}$$

$$> N(N-1)\frac{G'}{G} + \frac{\eta}{G}\sum_{i=1}^{N}\frac{1}{P_i} + c\sum_{i=1}^{N}P_i$$

$$= N(N-1)\frac{G'}{G} + \frac{\eta}{G}N\frac{1}{\frac{N}{\sum_{i=1}^{N}\frac{1}{P_i}}} + c\sum_{i=1}^{N}P_i \tag{15}$$

$$> N(N-1)\frac{G'}{G} + \frac{\eta}{G}\frac{N^2}{\sum_{i=1}^{N}P_i} + c\sum_{i=1}^{N}P_i = C(\tilde{P}), \tag{16}$$

where the first strict inequality follows because there must be at least one pair (i,j) such that $\frac{P_j}{P_i} + \frac{P_i}{P_j} > 2$ because not all P_i's are the same and where the second strict inequality follows from the classical arithmetic-mean–harmonic-mean inequality (again strict inequality holds because not all P_i's are equal). However, this yields an immediate contradiction because $C(P) > C(\tilde{P})$ but P is a minimum. Consequently, there is a unique minimum $P^* = (p, p, \ldots, p)$.

Plug this p into C and take the derivative and set it to zero; it then follows, after some algebra, that $p = \sqrt{\frac{\eta}{cG}}$. Consequently, $P^* = \left(\sqrt{\frac{\eta}{cG}}, \sqrt{\frac{\eta}{cG}}, \ldots, \sqrt{\frac{\eta}{cG}}\right)$.

2. Characterizing the Nash equilibrium solution.

By the characterization of a Nash equilibrium, it must satisfy $g(P^{Nash}) = P^{Nash}$, where $g(\cdot)$ is the best response function. This means that for each i, we have

$$\frac{\sum_{j\neq i}G'P_j + \eta}{GP_i^2} = c.$$

Per Theorem 3, there exists a unique Nash equilibrium. Therefore, by symmetry, the unique Nash equilibrium must satisfy

$$\frac{\sum_{j\neq i}G'p^{Nash} + \eta}{G(p^{Nash})^2} = c,$$

thereby implying (after some algebra) that $P^{Nash} = (p^{Nash}, p^{Nash}, \ldots, p^{Nash})$, where

$$p^{Nash} = \frac{(N-1)G' + \sqrt{((N-1)G')^2 + 4Gc\eta}}{2Gc}.$$

3. Bounding the performance loss between the social optimal and the NE.

We now analyze PoA, which is defined as $\frac{C(P^{Nash})}{C(P^*)}$:

$$\frac{C(P^{Nash})}{C(P^*)} = \frac{\sum_{i=1}^{N}\left\{\frac{\sum_{j\neq i}G'p^{Nash}+\eta}{Gp^{Nash}} + cp^{Nash}\right\}}{\sum_{i=1}^{N}\left\{\frac{\sum_{j\neq i}G'p^*+\eta}{Gp^*} + cp^*\right\}}$$

$$= \frac{(N-1)\frac{G'}{G} + \frac{\eta}{Gp^{Nash}} + cp^{Nash}}{(N-1)\frac{G'}{G} + \frac{\eta}{Gp^*} + cp^*}$$

$$= \frac{(N-1)\frac{G'}{G} + \frac{\eta}{Gp^{Nash}} + cp^{Nash}}{(N-1)\frac{G'}{G} + 2\sqrt{\frac{\eta C}{G}}} \tag{17}$$

$$= \frac{(N-1)\frac{G'}{G} + \frac{2c\eta}{(N-1)G' + \sqrt{((N-1)G')^2 + 4Gc\eta}} + \frac{(N-1)G' + \sqrt{((N-1)G')^2 + 4Gc\eta}}{2G}}{(N-1)\frac{G'}{G} + 2\sqrt{\frac{\eta C}{G}}} \tag{18}$$

$$\le \frac{(N-1)\frac{G'}{G} + \frac{2c\eta}{\sqrt{4Gc\eta}} + \frac{(N-1)G' + \sqrt{((N-1)G')^2} + \sqrt{4Gc\eta}}{2G}}{(N-1)\frac{G'}{G} + 2\sqrt{\frac{\eta C}{G}}}$$

$$= \frac{2(N-1)\frac{G'}{G} + 2\sqrt{\frac{c\eta}{G}}}{(N-1)\frac{G'}{G} + 2\sqrt{\frac{\eta C}{G}}} \le 2, \tag{19}$$

where, for the second-to-last inequality, we have used the fact that $\sqrt{x+y} \le \sqrt{x} + \sqrt{y}, \forall, x, y \ge 0$. ■

4. Stochastic Stability of Best Response Updates Under Random Environments

Motivated by the preceding discussion, we now consider a wireless communications setting in which the environment (both the gain matrix and the generalized noise) is random across time steps. Our primary focus is on studying the behavior of the best response update under such random environments. We start with a worst-case stability characterization of the best response dynamics under varying environments.

4.1. Worst-Case Stability Characterization

Throughout the section, we use $\boldsymbol{\theta}_i = (\{G_{ij}\}_{j=1}^{N}, \eta_i)$ to denote the environment for link i, namely, all the interferences and noise on receiver i. Collecting the θ_i's into a single matrix, $\theta = (\mathbf{G}, \boldsymbol{\eta}) \in \mathbf{R}_+^{N\times(N+1)}$ then denotes the joint environment of all the links. We denote by $\theta^k = (\mathbf{G}^k, \boldsymbol{\eta}^k) \in \mathbf{R}_+^{N\times(N+1)}$ the joint environment (which now varies across iterations) in the k-th iteration when following the distributed power control update in Algorithm 1.

Our first observation is that there is a natural partial ordering on the set of all environments. For each individual link i, all else being equal, it is intuitive that the environment is more "friendly" to it if G_{ii} is larger and all other G_{ij}'s and the noise η_i are smaller because a more "friendly" such environment allows link i to transmit using less power while still achieving the same SINR. As characterized by the following definition, under this ordering, we can think of a smaller θ as representing a more "friendly" joint environment.

Definition 5. Let $\theta_i = (\{G_{ij}\}_{j=1}^N), \eta_i), \tilde{\theta}_i = (\{\tilde{G}_{ij}\}_{j=1}^N), \tilde{\eta}_i) \in \mathbf{R}_+^{N+1}$. The partial ordering $\preceq$ on $\mathbf{R}_+^{N+1}$ is defined as

$$\theta_i \preceq \tilde{\theta}_i \text{ if and only if } G_{ii} \geq \tilde{G}_{ii}, G_{ij} \leq \tilde{G}_{ij}, \forall j \neq i, \eta_i \leq \tilde{\eta}_i.$$

Furthermore, let $\theta, \tilde{\theta} \in \mathbf{R}_+^{N(N+1)}$ be joint environments. Denote by $\theta \preceq \tilde{\theta}$ if and only if $\theta_i \preceq \tilde{\theta}_i, \forall i$.

Now that the environment itself is a variable, the best response function should be viewed as a bivariate function $g(P, \theta)$ in which the dependence of the best response function on the joint environment is made explicit. It is intuitive that when the joint environment becomes less friendly (under the abovementioned ordering), every link's best response becomes more aggressive. It turns out that this is indeed the case as stated by the following lemma, which follows from a similar argument as in the proof of Statement 1 in Lemma 1. Note that Lemma 1 and Lemma 3 together establish that the best response is bimonotonic (i.e., monotonic in each of the two variables when holding the other fixed), a crucial property at play for establishing the various stability results to come.

Lemma 3. *For any fixed* $P \in \mathbf{R}_+^N, \theta \preceq \tilde{\theta} \Rightarrow g(P, \theta) \leq g(P, \tilde{\theta})$.

Because of the nature and physical limits of the interference and noise, we assume that the gain matrix and the generalized noise are bounded. Specifically, we assume that $\forall k, 0 < \underline{G}_{ii} \leq G_{ii}^k \leq \overline{G}_{ii}, 0 \leq G_{ij}^k \leq \overline{G}_{ij}, i \neq j, 0 < \underline{\eta}_i \leq \eta_i^k \leq \overline{\eta}_i$. Per the ordering in Definition 5, this directly translates to the following bounded environment assumption:

$$\underline{\theta}_i \preceq \theta_i^k \preceq \overline{\theta}_i, \forall k, i, \tag{20}$$

where $\underline{\theta}_i = (0, \ldots, 0, \overline{G}_{ii}, 0, \ldots, 0, \underline{\eta}_i), \overline{\theta}_i = (\overline{G}_{i1}, \ldots, \overline{G}_{ij-1}, \underline{G}_{ii}, \overline{G}_{ij+1}, \ldots, \overline{G}_{iN}, \overline{\eta}_i)$; equivalently, $\underline{\theta} \preceq \theta^k \preceq \overline{\theta}, \forall k$. We denote by $\mathcal{U}$ the set of all such environments: $\mathcal{U} = \{\theta \mid \underline{\theta} \preceq \theta \preceq \overline{\theta}\}$.

The first question that arises concerns the worst-case stability of the best response update. The following lemma indicates that a bounded environment results in a bounded power iterate, something to be contrasted with the classical Foschini–Miljanic power update as given in Foschini and Miljanic (1993), in which the power iterate can go to infinity even if the environment is finite. The following theorem makes this statement precise.

Theorem 6. *For a given constant* $\epsilon \in \mathbf{R}_+^{N\times(N+1)}$, *let* $P^e(\epsilon)$ *be the equilibrium power vector under Algorithm 1 when the joint environment is constant:* $\theta^k = \epsilon, \forall k$. *Then, under the bounded environment assumption* $\theta^k \in \mathcal{U}, \forall k$ *and the update in Algorithm 1, for any initial power vector* $P^0 \in \mathbf{R}_+^N$, *we have*

$$\limsup_{k\to\infty} P^k \leq P^e(\overline{\theta}), \ \liminf_{k\to\infty} P^k \geq P^e(\underline{\theta}),$$

where P_i^k *is given in Algorithm 1. Furthermore, if* $P^e(\underline{\theta}) \leq P \leq P^e(\overline{\theta})$, *then* $P^e(\underline{\theta}) \leq g(P, \theta) \leq P^e(\overline{\theta})$ *for any* θ *satisfying* $\underline{\theta} \preceq \overline{\theta}$.

Proof. Pick an arbitrary initial vector $P^0 \in \mathbf{R}_+^N$. By Lemma 1, for any given $Q > 0$, we can always pick an $\alpha > 0$ large enough such that $\alpha Q \geq g(\alpha Q), \alpha Q \geq P^0 \geq 0$. Set $\overline{p}^0 = \alpha Q, \underline{P}^0 = 0$.

Let $P^k = h(P^0, \theta^1, \ldots, \theta^k)$ denote the k-iterate in Algorithm 1 with the initial power vector being P^0 and the realizations of the joint environments being $\theta^1, \ldots, \theta^k$. Set $\overline{P}^k = h(\overline{p}^0, \overline{\theta}, \overline{\theta}, \ldots, \overline{\theta}), \underline{P}^k = h(\underline{P}^0, \underline{\theta}, \underline{\theta}, \ldots, \underline{\theta})$. Because $\overline{P}^0 \geq P_i^0 \geq \underline{P}^0$, by monotonicity (in power) of the best response function (Lemma 1), we have

$$h(\underline{P}^0, \theta^1, \ldots, \theta^k) \leq h(P^0, \theta^1, \ldots, \theta^k) \leq h(\overline{P}^0, \theta^1, \ldots, \theta^k).$$

By monotonicity (in environment) of the best response function, we have

$$h(\underline{P}^0, \underline{\theta}, \underline{\theta}, \ldots, \underline{\theta}) \leq h(\underline{P}^0, \theta^1, \ldots, \theta^k),$$
$$h(\overline{P}^0, \theta^1, \ldots, \theta^k) \leq h(\overline{P}^0, \overline{\theta}, \overline{\theta}, \ldots, \overline{\theta}).$$

This leads to $\underline{P}^k \leq P^k \leq \overline{P}^k, \forall k$.

Furthermore, $\overline{P}^k$ is a decreasing sequence with $\lim_{k\to\infty} \overline{P}^k = P^e(\overline{\theta})$, and $\underline{P}^k$ is an increasing sequence with $\lim_{k\to\infty} \underline{P}^k = P^e(\underline{\theta})$. Therefore, we have

$$\overline{P}^k \geq P^j, \forall j \geq k, \tag{21}$$
$$\underline{P}^k \leq P^j, \forall j \geq k. \tag{22}$$

Define $U^k = \sup_{j\geq k} P^j, L^k = \inf_{j\geq k} P^j$. Inequalities (21) and (22) imply that $U^k \leq \overline{P}^k, L^k \leq \underline{P}^k$. Therefore,

$$\limsup_{k\to\infty} P^k \leq P^e(\overline{\theta}), \ \liminf_{k\to\infty} P^k \geq P^e(\underline{\theta}).$$

The second part of the theorem also follows from the bimonotonicity of the best response function: if $P^e(\underline{\theta}) \leq P \leq P^e(\overline{\theta})$, then for any θ with $\underline{\theta} \preceq \overline{\theta}$, we have $P^e(\underline{\theta}) = g(P^e(\underline{\theta}), \underline{\theta}) \leq g(P, \theta) \leq g(P^e(\overline{\theta}), \overline{\theta}) = P^e(\overline{\theta})$. ∎

4.2. Stochastic Stability: Overview and Intuition

Theorem 6 gives a worst-case characterization of the behavior of the best response update under a random environment. The next step is to characterize the probabilistic behavior of the power vector iterate P^k in Algorithm 1 when the environments $\Theta^k{}_{k=0}^{\infty}$ now follow some stochastic process. In what follows, we assume the environments are independent and identically distributed across time (they can have correlation across links). In particular, for simplicity and without loss of generality, we assume Θ^k are **iid** with a continuous density function $f_\Theta(\theta)$ supported on $\mathcal{U}$. Consequently, the power iterates $\{P^k\}_{k=0}^{\infty}$ in Algorithm 1 then form a general state space Markov chain. We then investigate the probabilistic behavior by answering the following questions:

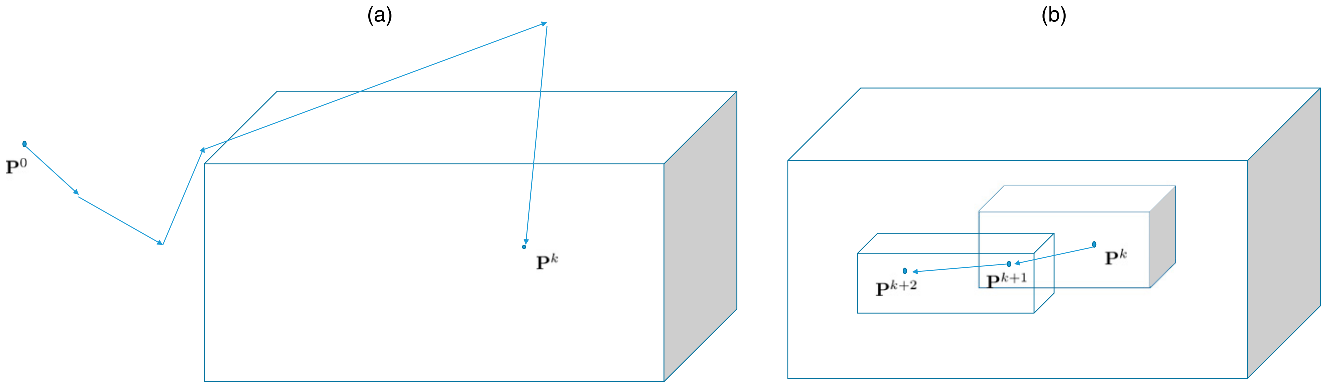
- Does there exist a stationary distribution for the Markov chain?
- If there indeed exists a stationary distribution, is it unique?
- Will P^k converge to that unique stationary distribution (should a unique one exist) irrespective of the initial condition?
- If convergence is ensured, what is the convergence rate?

The news is quite positive: the first three questions all have yes answers; for the last question, the Markov chain enjoys an exponential convergence rate provided the initial power is bounded. Note that one aspect of why instability could be an issue despite the fact that a unique Nash equilibrium exists in the deterministic network environment case is that once randomness enters the network, there are effectively many Nash equilibria, one for each network environment. Consequently, it is a priori not clear at all how best response dynamics would behave when jumping from the process of "chasing" one Nash equilibrium to another. As we characterize with rigor later, the end process is very well behaved, and stochastic stability (in a very strong form) is present.

We first give some intuition on why the Markov chain has a unique stationary distribution and will converge to it. First, Theorem 6 indicates that, irrespective of the initial condition, after finitely many iterations, the power vector is confined in a N-dimensional hyper-rectangle $\mathcal{H} = \prod_{i=1}^{N}[P_i^e(\underline{\theta}), P_i^e(\overline{\theta})]$, which is the set of all equilibrium power vectors, each corresponding to a particular realization of environment held constant throughout all iterations. Hence, without loss of generality, we can assume the process starts in $\mathcal{H}$. Note that, again by Theorem 6, once the power vector enters $\mathcal{H}$, it will remain in it in subsequent iterations. Second, because the best-response function is bimonotonic, there is a one-to-one correspondence between a disturbance vector $\theta \in \mathcal{U}$ and the corresponding equilibrium $P_i^e(\theta)$ (i.e., the equilibrium generated if the disturbance is fixed to be θ across time steps). Third, another consequence of this bimonotonicity is that if the initial condition is somewhere in $\mathcal{H}$, then, after one iteration, the possible locations of the current state is in another smaller hyper-rectangle that contains the starting point and that is contained in $\mathcal{H}$. From there, each possible location of the current state (in the smaller hyper-rectangle) will, in the next iteration, produce another hyper-rectangle (representing the set of all possible locations in the next iteration) also contained in $\mathcal{H}$. This process will continue on until the possible locations of the current state permeate the entire hyper-rectangle $\mathcal{H}$ and mixing will happen.

Figure 1 provides a pictorial illustration of the geometry embodied in the previous discussion. First, irrespective of what the initial power vector is, in a finite number of iterations, the power iterate (P^k) will enter a hyper-rectangle in Figure 1. In fact, P^k is trapped in this hyper-rectangle: the crucial property of this hyper-rectangle is that, starting anywhere inside, the power iterate will never exit it. Second, once the power iterate enters the hyper-rectangle (P^k in Figure 1(b)), in the next iteration, there will be an area within the hyper-rectangle that the next power iterate can reach

Figure 1. (Color online) Illustration of Power Update Dynamics



Notes. (a) Regardless of the initial condition, the power iterate will enter a hyper-rectangle and will thereafter be trapped in it. (b) Propagation of the current power iterate within the hyper-rectangle. The smaller hyper-rectangle around P^k represents the possible region that it can reach with positive probability in one iteration. Similarly for P^{k+1}.

with positive probability: this area is the smaller hyper-rectangle around P^k. In other words, the next power iterate P^{k+1} will enter any measurable subset of this smaller hyper-rectangle with positive probability and will not enter anywhere outside of it. Similarly, suppose after one iteration, the power iterate is at P^{k+1} as in Figure 1(b). Then, again, the smaller hyper-rectangle around P^{k+1} represents the possible region that it will reach with positive probability in one iteration. The same can be said for P^{k+2}, and it goes on. One can then intuitively sense that such hyper-rectangles, representing possible locations of the power iterate, will eventually "permeate" the larger hyper-rectangle in which the power iterate is trapped.

4.3. Stationary Distribution: Existence, Uniqueness, and Convergence

To formally answer those questions, we now employ theory from discrete–time general state space Markov chains because $\{P^k\}_{k=0}^{\infty}$ forms a $\mathbf{R}_+^N$-valued, time-homogeneous Markov chain. To this end, we first introduce some of the key concepts and terminologies.[6]

Definition 6. Let $(S, \mathcal{S})$ be a measurable space[7] and $A \subset S$ be an element in $\mathcal{S}$. Let $\{X^n\}_{n=0}^{\infty}$ be a $(S, \mathcal{S})$-valued Markov chain with transition kernel $\mathbf{K}(s; A)$.[8]

1. A σ-finite measure π on $(S, \mathcal{S})$ is called an invariant measure if $\pi(A) = \int_S \pi(ds)K(s, A), \forall A \in \mathcal{S}$. An invariant measure π that is also a probability measure is called a stationary probability measure.
2. $\{X^n\}_{n=0}^{\infty}$ is called ϕ-irreducible if there exists a nontrivial measure ϕ on $(S, \mathcal{S})$ such that

$$\phi(A) > 0 \Rightarrow \mathbf{P}_s(\tau_A < \infty) > 0, \forall s \in S, \tag{23}$$

where $\tau_A = \min\{n \geq 1 \mid X^n \in A\}$ is the first return time[9] and $\mathbf{P}_s(\tau_A < \infty)$ denotes the probability of the first return time being finite given that the process starts at s (i.e., $X^0 = s$).
3. A set A is called Harris recurrent if $\mathbf{P}_s(\sum_{n=1}^{\infty} \mathbf{1}_{\{X^n \in A\}} = \infty) = 1, \forall s \in S$.
4. $\{X^n\}_{n=0}^{\infty}$ is called a Harris recurrent chain if it is ϕ-irreducible and $\phi(A) > 0 \Rightarrow A$ is Harris recurrent, $\forall A \in \mathcal{S}$.
5. A set A is called a v_m-small set if there exists a positive integer m, a nontrivial measure v_m on $(S, \mathcal{S})$ such that $\forall s \in A, \forall B \in \mathcal{S}, K^m(s, B) \geq v_m(B)$, where K^m is the m-step transition kernel. For brevity, we also call A a small set.
6. Let $\{X^n\}_{n=0}^{\infty}$ be ϕ-irreducible. $\{X^n\}_{n=0}^{\infty}$ is strongly aperiodic if there exists a v_1-small set A with $v_1(A) > 0$.
7. $\{X^n\}_{n=0}^{\infty}$ is called a positive Harris chain if it is Harris recurrent and there exists a small set A such that $\sup_{a \in A} \mathbf{E}_a[\tau_A] < \infty$.[10]

A general Markov chain $\{X^n\}_{n=0}^{\infty}$ need not have an invariant measure; even if it does, it can still fail to have a stationary probability measure. For any practical stochastic system, having a unique stationary probability measure and convergence to that stationary probability measure is crucial because they characterize the stochastic stability of that system. Definition 6 gives the essential ingredients for guaranteeing the desired features (existence, uniqueness, convergence) as formalized by the following theorem in Meyn and Tweedie (2009).

Theorem 7. *Let $\{X^n\}_{n=0}^{\infty}$ be a time-homogeneous Markov chain on $(S, \mathcal{S})$ that is positive Harris.*

- *$\{X^n\}_{n=0}^{\infty}$ has a unique stationary probability measure $\pi(\cdot)$.*
- *If $\{X^n\}_{n=0}^{\infty}$ is strongly aperiodic, then the chain converges to the stationary probability measure in total variation distance: $\forall s \in S, \lim_{n\to\infty} \|\mathbf{K}^n(s, \cdot) - \pi(\cdot)\|_{\mathbf{TV}} = 0$, where the n-step transition kernel gives the probability measure after n steps when starting at s.*

In our current setting, $\{P^k\}_{k=0}^{\infty}$ has state space $\mathbf{R}_+^N$ (with the standard Borel σ-algebra). As we show next, those sufficient conditions are satisfied, which we record in the following lemma.

Lemma 4. *The Markov chain $\{P^k\}_{k=0}^{\infty}$ has the following properties:*

1. *It is ϕ-irreducible for some (nontrivial) ϕ.*
2. *It is Harris recurrent.*
3. *It is a positive Harris chain.*
4. *It is strongly aperiodic.*

Proof.

(1) By Theorem 6, without loss of generality, we can assume the initial state P^0 is in the hyper-rectangle $\mathcal{H} = \prod_{i=1}^{N} [P_i^e(\underline{\theta}), P_i^e(\overline{\theta})]$. Take any $P \in \mathcal{H}$; by the bimonotonicity of the best response function, there exists a unique $\theta \in \mathcal{U}$ such that $g(P, \theta) = P$. Further, $P = P^e(\theta)(= \lim_{k\to\infty} g^k(P^0, \theta), \forall P^0)$. Because the best response function is continuous (in both P and θ), it follows that, for any r-neighborhood of P, denoted by $\mathcal{N}_P(r) = \{\tilde{P} \in \mathcal{H} \mid \|P - \tilde{P}\|_1 < r\}$, there exists an $\gamma > 0$ small enough and a positive integer T large enough (γ, T can depend on P^0) such that if $\forall k \leq T, \|\theta - \theta^k\|_1 < \gamma$, then $P^T \in \mathcal{N}_P(r)$. Because the density function $f_\Theta(\theta)$ is continuous and supported on $\mathcal{U}$ (hence, $f_{\min} \triangleq \min_{\theta \in \mathcal{U}} f_\Theta(\theta)$ exists and is positive), it follows that with probability at least $(f_{\min}\gamma^N)^T, P^T \in \mathcal{N}_P(r)$.

We now take the point $P^e(\underline{\theta})$ (i.e., the minimum equilibrium power vector) and consider the neighborhood $\mathcal{N}_{P^e(\underline{\theta})}(r)$. For each starting point $P^0 \in \mathcal{H}$, in one iteration, P^1 can reach anywhere in the hyper-rectangle:

$$\mathcal{H}_{P^0} \triangleq \{P \mid g(P^0, \underline{\theta}) \leq P \leq g_i(P^0, \overline{\theta})\} \subset \mathcal{H}.$$

Consequently, if $A \in \mathcal{S}, A \subset \mathcal{H}_{P^0}, \lambda(A) > 0$, we have that $K(P^0, A) \geq f_{\min}\lambda(A) > 0$, where $\lambda(\cdot)$ is the Lebesgue measure.

Next, by continuity of the best response function g, pick $\hat{r}$ small enough such that

$$\mathcal{I} \triangleq \bigcap_{P^0 \in \mathcal{N}_{P^e(\underline{\theta})}(\hat{r})} \mathcal{H}_{P^0} \neq \varnothing.$$

Note that $\mathcal{I}$ is itself a hyper-rectangle with a positive Lebesgue measure. This construction ensures that starting from any point in $\mathcal{N}_{P^e(\underline{\theta})}(\hat{r})$, the Markov chain will, in the next step, with positive probability, reach any measurable subset of $\mathcal{I}$ that has a positive Lebesgue measure. Therefore, starting at any $P^0 \in \mathcal{H}$, the Markov chain will enter $\mathcal{I}$ as well as any of its measurable subsets with a positive Lebesgue measure with positive probability in $T+1$ steps.

Finally, take the (nontrivial) measure ϕ to be $\phi(A) = \frac{\lambda(A \cap \mathcal{I})}{\lambda(\mathcal{I})}, \forall A \in \mathcal{S}$. This construction ensures that the Markov chain is ϕ-irreducible.

(2) Fix any $A \in \mathcal{S}$ with $\phi(A) > 0$. From the previous discussion, we know that starting at $P^0 = P^e(\overline{\theta})$, there exist γ, T such that[11] $P^{T+1} \in A$ with probability at least $(f_{\min}\gamma^N)^T f_{\min}\lambda(A)$. Equivalently, by setting $\epsilon = (f_{\min}\gamma^N)^T f_{\min}$, we have $K^{T+1}(P^e(\overline{\theta}), A) \geq \epsilon\lambda(A)$.

Moreover, by the bimonotonicity of the best response function g, starting at any other point $P \in \mathcal{H}$ only gives easier access to $\mathcal{N}_{P^e(\underline{\theta})}(\hat{r})$: following the same realizations of the environment $\theta^0, \ldots, \theta^k$, $P^{k+1}_{P^0=P} \leq P_{P^0=P^e(\overline{\theta})})^{k+1}$, $\forall P \in \mathcal{H}$, thereby leading to that

$$P_{P^0=P^e(\overline{\theta})})^{k+1} \in \mathcal{N}_{P^e(\underline{\theta})}(\hat{r}) \Rightarrow P^{k+1}_{P^0=P} \in \mathcal{N}_{P^e(\underline{\theta})}(\hat{r}).$$

Consequently, $\mathbf{K}^{T+1}(P^0, A) \geq \epsilon\lambda(A), \forall P^0 \in \mathcal{H}$. Thus, with probability at least $\epsilon\lambda(A) > 0$, A is visited in $T+1$ steps irrespective of the starting point. By the Borel Cantelli Lemma (Billingsley 1986), A is, therefore, visited infinitely often with probability one. Harris recurrence hence follows.

(3) Again set $\epsilon = (f_{\min}\gamma^N)^T f_{\min}$. Take $m = T+1$ and $v_{T+1}(B) = \epsilon\lambda(B \cap \mathcal{I}), \forall B \in \mathcal{S}$ and note that $\mathcal{I}$ is a v_{T+1}-small set. This follows because, for any $P^0 \in \mathcal{I}$ (in fact any $P^0 \in \mathcal{H}$), $\forall B \in \mathcal{S}$, $\mathbf{K}^{T+1}(P^0,B) \geq \mathbf{K}^{T+1}(P^0, B \cap \mathcal{I}) \geq \epsilon\lambda(B \cap \mathcal{I}) = v_{T+1}(B)$.

We can directly compute an upper bound on the expected return time:

$$\begin{aligned}\sup_{P^0 \in \mathcal{I}} \mathbf{E}_{P^0}[\tau_{\mathcal{I}}] &\leq \sup_{P^0 \in \mathcal{H}} \mathbf{E}_{P^0}[\tau_{\mathcal{I}}] \\ &\leq \sum_{i=1}^{\infty} \epsilon\lambda(\mathcal{I})(1 - \epsilon\lambda(\mathcal{I}))^{i-1}\, i(T+1) \\ &= \frac{T+1}{\epsilon\lambda(\mathcal{I})} < \infty.\end{aligned}$$

(4) Again, by continuity of the best response functions, following the construction in (1), pick an $r > 0$ small enough such that

$$C \triangleq \mathcal{N}_{P^e(\underline{\theta})}(r) \cap \bigcap_{P^0 \in \mathcal{N}_{P^e(\underline{\theta})}(\hat{r})} \mathcal{H}_{P^0} \neq \varnothing.$$

We, therefore, have $\mathbf{K}(c, C) > 0, \forall c \in C$. Take $m = 1$ and take the measure $v_1(\cdot)$ to be $v_1(A) = f_{min}\lambda(A \cap C)$, $\forall A \in \mathcal{S}$, where again λ is the Lebesgue measure. It follows that C is a v_1-small set because $\forall B \in \mathcal{S}, \forall P^0 \in \mathcal{H}, \mathbf{K}(P^0, B) \geq \mathbf{K}(P^0, B \cap C) \geq f_{\min}\lambda(B \cap C) = v_1(B)$. Further $v_1(C) > 0$; hence, the conclusion follows. ■

Lemma 4 together with Theorem 7 establishes the following result.

Theorem 8. *There exists a unique stationary probability measure $\pi(\cdot)$ for $\{P^k\}_{k=0}^{\infty}$. Moreover, $\forall p^0 \in \mathbf{R}^N_+$, $\lim_{n\to\infty} \|\mathbf{P}^n_{p^0}(\cdot) - \pi(\cdot)\|_{TV} = 0$, where $\mathbf{P}^n_{p^0}(\cdot)$ denotes the probability measure of the state at time n, starting at p^0.*

4.4. Stationary Distribution: Convergence Rate

Now that Theorem 8 establishes the existence and uniqueness of and convergence to the stationary probability measure, we next turn to studying the convergence rate. As mentioned before, here the Markov chain converges to the unique stationary distribution exponentially fast (i.e., at a geometric rate). As it turns out, the chain is uniformly ergodic. We first define the notion of a petite set, a generalization of small sets, that shall provide an equivalent characterization of the geometric convergence rate.

Definition 7. Consider the same measurable space setup as in Definition 6. Let $d(\cdot)$ be a probability distribution on $\mathbf{Z}_+$, the set of all nonnegative integers.

1. The sampled transition kernel $\mathbf{K}_d(s, A)$ is defined to be $\mathbf{K}_d(s, A) = \sum_{m=0}^{\infty} \mathbf{K}^m(s, A) d(m)$.
2. A set A is called v_d-petite if $\mathbf{K}_d(s, A) \geq v_d(A)$ for some nontrivial measure v_d on $(S, \mathcal{S})$.

The following theorem from Meyn and Tweedie (2009) relates the geometric convergence rate of a Markov chain to a condition on petite sets.

Theorem 9. *Let $\{X^n\}_{n=0}^{\infty}$ be a time-homogeneous Markov chain on $(S, \mathcal{S})$. The following two conditions are equivalent:*

1. *There exist $r > 1, R < \infty$ such that for all $s \in S$,*

$$\|\mathcal{K}^m(s, \cdot) - \pi(\cdot)\|_{TV} \leq Rr^{-m}.$$

2. *The chain is aperiodic, and there exists a petite set A satisfying*

$$\sup_{s \in S} \mathbf{E}_s[\tau_A] < \infty.$$

We are now ready to characterize the convergence rate in the Markov chain formed by the best response dynamics.

Theorem 10. *Let the initial power P^0 be bounded: $0 \le P^0 \le \overline{P}$. The Markov chain $\{P^k\}_{k=0}^{\infty}$ converges to its unique stationary probability measure $\pi(\cdot)$ at a uniform geometric rate. There exist $r > 1, R < \infty$ such that for all $P^0 \in \prod_{i=1}^N [0, \overline{P}_i]$*

$$\|\mathbf{K}^m(P^0, \cdot) - \pi(\cdot)\|_{TV} \le Rr^{-m}.$$

Proof. First note that any v_m-small set A is a v_{δ_m}-petite set, in which δ_m is the probability distribution that puts all probability mass on m and zero on everything else. Per the proof of (3) in Lemma 4, $\mathcal{I}$ is a v_{T+1}-small set and, hence, a $v_{\delta_{T+1}}$-petite set. By Theorem 9, it suffices to show that $\sup_{P^0 \in \prod_{i=1}^N [0,\overline{P}_i]} \mathbf{E}_{P^0}[\tau_{\mathcal{I}}] < \infty$.

To show that, first observe that by Theorem 6, if $P^0 = \overline{P}$, then there exists an $T_{\overline{P}}$ such that $P^{T_{\overline{P}}} \in \mathcal{H}$. Similarly, if $P^0 = 0$, then there exists an T_0 such that $P^{T_0} \in \mathcal{H}$. Take $T^{\max} = \max\{T_{\overline{P}}, T_0\}$. By the monotonicity of the best response function, $\forall P^0 \in \prod_{i=1}^N [0, \overline{P}_i]$, $P^{T^{\max}} \in \mathcal{H}$.

Finally, the conclusion follows because

$$\sup_{P^0 \in \prod_{i=1}^N [0,\overline{P}_i]} \mathbf{E}_{P^0}[\tau_{\mathcal{I}}] \le \sup_{P^0 \in \prod_{i=1}^N [0,\overline{P}_i]} \mathbf{E}_{P^0}[\tau_{\mathcal{H}}] + \sup_{P^0 \in \mathcal{H}} \mathbf{E}_{P^0}[\tau_{\mathcal{I}}] \le T^{\max} + \frac{T+1}{\epsilon\lambda(\mathcal{I})} < \infty,$$

where the last inequality follows from $\sup_{P^0 \in \mathcal{H}} \mathbf{E}_{P^0}[\tau_{\mathcal{I}}] \le \frac{T+1}{\epsilon\lambda(\mathcal{I})} < \infty$ as shown in Lemma 4. ∎

4.5. High-Concentration Bounds on Average Power

So far we have focused on the behavior of the last power iterate P^k. From an engineering standpoint, it is also natural and interesting to understand the behavior of the long-run average power $\frac{1}{T}\sum_{k=1}^T P^k$ used under the best response dynamics. Per our previous characterization, it follows that the average power has the desired feature of asymptotically stabilizing to a constant (almost surely): Theorem 8 and the Birkhoff Ergodic theorem imply the following result.

Corollary 1. *Let $\pi(\cdot)$ be the unique stationary probability measure. Then $\{P^k\}_{k=0}^{\infty}$. $\frac{1}{T}\sum_{k=1}^T P^k$ converges almost surely to $\int_{R_+^N} P\pi(dP)$ as $T \to \infty$.*

Corollary 1 gives an asymptotic characterization of the average power. It would also be desirable to obtain (component-wise) high-concentration bounds of the form $\mathrm{P}(|\frac{1}{n}\sum_{k=1}^n P_i^k - \mathbf{E}[\frac{1}{T}\sum_{k=1}^T P_i^k]| \ge \epsilon) \le \delta_i(T, \epsilon)$ for each i, where $\delta_i(T,\epsilon)$ is decreasing quickly to zero as T increases. Such high-concentration results establish sharp concentration of the random vector $\frac{1}{T}\sum_{k=1}^T P^k$ around its mean. Toward this end, we utilize recent results in Paulin (2012) on concentration inequalities on Markov chains based on spectral methods. We first introduce the necessary terminology.

Definition 8. Let $\{X^k\}_{k=0}^{\infty}$ be a time-homogeneous Markov chain on a Polish[12] state space S with transition kernel $\mathbf{K}(s, A)$ and stationary probability measure $\pi(\cdot)$. Then the mixing time of the chain $t_{mix}(\epsilon)$ is defined as follows:

$$t_{mix}(\epsilon) = \min\left\{t \,\middle|\, \sup_{s \in S} \|\mathbf{K}^t(s, \cdot), \pi(\cdot)\|_{\mathbf{TV}} \le \epsilon\right\}. \quad (24)$$

A constant that will be used later to obtain convenient bounds is $t_{mix} = t_{mix}(1/4)$.

Next we state a general concentration theorem from Paulin (2012).

Theorem 11. *Let $X = (X^1, \ldots, X^T)$ be a time-homogeneous Markov chain, taking values in a Polish state space $\Lambda = (\Lambda_1 \times \cdots \times \Lambda_T)$. Let $f : \Lambda \to \mathbf{R}$ be a measurable function. If there exists a $c \in \mathbf{R}_+^T$ such that*

$$f(x) - f(y) \le \sum_{k=1}^T c_k \mathbf{1}_{\{x^k \ne y^k\}}, \forall x, y \in \Lambda, \quad (25)$$

then $\forall \epsilon > 0$,

$$\mathrm{P}(|f(X) - \mathbf{E}[f(X)]| \ge \epsilon) \le 2\exp\left(\frac{-2\epsilon^2}{9\|c\|_2^2 t_{mix}}\right). \quad (26)$$

By properly specializing Theorem 11 to our current setting, we obtain the following result. Note that $t_{mix}(= t_{mix}(1/4))$ is a finite constant by Theorem 8. Therefore, the convergence rate is exponentially fast in the number of time steps.

Theorem 12. *For each i,*

$$\mathrm{P}\left(\left|\frac{1}{T}\sum_{k=1}^T P_i^k - \mathbf{E}\left[\frac{1}{T}\sum_{k=1}^T P_i^k\right]\right| \ge \epsilon\right) \le 2\exp\left(\frac{-2\epsilon^2 T}{9(P_i^e(\overline{\theta}) - P_i^e(\underline{\theta}))^2 t_{mix}}\right). \quad (27)$$

Proof. Without loss of generality, we can take for each $j = 1, \ldots, T, \Lambda_j = \prod_{i=1}^N [P_i^e(\underline{\theta}), P_i^e(\overline{\theta})]$ because, after finitely many steps, the state space will be confined into the hyper-rectangle $\mathcal{H}$. The space Λ is then clearly a Polish space.

For each i, choose $f_i(x) = \left(\frac{1}{T}\sum_{k=1}^T x^k\right)_i$, where $x^k \in \Lambda_k$, $\forall k = 1, \ldots, T$ and $(\cdot)_i$ extracts the i-th component from a vector; f_i is continuous and, hence, measurable. Take $c_k = \frac{P_i^e(\bar{\theta}) - P_i^e(\underline{\theta})}{T}$, and we have $\forall x, y \in \lambda$:

$$f_i(x) - f_i(y) = \frac{1}{T}\sum_{k=1}^{T}(x^k - y^k)_i = \frac{1}{T}\sum_{k=1}^{T}(x^k - y^k)_i \mathbf{1}_{\{x_i^k \neq y_i^k\}} \leq \frac{1}{T}\sum_{k=1}^{T}(P_i^e(\bar{\theta}) - P_i^e(\underline{\theta}))\mathbf{1}_{\{x_i^k \neq y_i^k\}} = \sum_{k=1}^{T} c_k \mathbf{1}_{\{x_i^k \neq y_i^k\}}.$$

Plugging $\|c\|_2^2 = \frac{(P_i^e(\bar{\theta}) - P_i^e(\underline{\theta}))^2}{T}$ into Equation (26) gives the desired bound. ■

4.6. Comparative Statics: A Discussion

We conclude the section with a discussion on how the network parameters would affect the final equilibrium. We break the discussion into two parts. The first part concerns deterministic network environments, and we use the properties of the best response function and the fixed-point iteration to give a comparative statics result on how changes in the gain matrix $\mathbf{G}$ and noise η would move the final equilibrium point. The second part concerns stochastic network environments, and we discuss whether noise is detrimental or beneficial to the overall system performance under best response–based power control. We start on the deterministic channel case. This result is essentially a simple synthesis of all the property results proved earlier.

Theorem 13. *Let the channel environment be denoted by* $(\mathbf{G}, \eta)$ *and the corresponding unique Nash equilibrium be denoted as* P^{Nash}. *Then*

1. *Pick any set of* G_{ij}*'s with* $i \neq j$ *and increase them to* $\tilde{G}_{ij}$. *Let* $\tilde{P}^{Nash}$ *be the resulting Nash equilibrium. Then* $\tilde{P}^{Nash} \geq P^{Nash}$, *where the inequality is component-wise.*
2. *Pick any set of* G_{ii}*'s and increase them to* $\tilde{G}_{ii}$. *Let* $\tilde{P}^{Nash}$ *be the resulting Nash equilibrium. Then* $\tilde{P}^{Nash} \leq P^{Nash}$, *where the inequality is component-wise.*

Proof. Let θ denote the original channel environment, and let $\tilde{\theta}$ denote the final channel environment. In the first case, because $\tilde{G}_{ij} \geq G_{ij}, \forall i \neq j$, it follows that $\theta \preceq \tilde{\theta}$. Consequently, per Lemma 3, $g(P, \theta) \leq g(P, \tilde{\theta}), \forall P \in \mathbf{R}_{++}^N$. Therefore, pick an arbitrary $P^0 \in \mathbf{R}_{++}^N$ and apply $g(\cdot, \theta)$ and $g(\cdot, \tilde{\theta})$ repeatedly results: $g^n(P^0, \theta) \leq g^n(P^0, \tilde{\theta})$ because $g(P, \theta)$ is monotonic in P per Lemma 1, where g^n means g composed with itself n times. Because $\lim_{n\to\infty} g^n(P^0, \theta) = P^{Nash}, \lim_{n\to\infty} g^n(P^0, \tilde{\theta}) = \tilde{P}^{Nash}$ per Theorem 2, the conclusion, therefore, follows.

The proof to the second case is identical (with the inequality sign reversed). ■

We next provide a discussion on how noise affects the system performance. In general, one would think that noise would be detrimental to the system performance. However, we shall present a simple yet illuminating special case to demonstrate that, although the presence of noise can make the system performance worse, that is not always the case.

Consider a two-link wireless network in which $\mathbf{G} = \begin{bmatrix} G & G' \\ 0 & 1 \end{bmatrix}$, $\eta = [\eta_1, 1]$, $f_1(x) = f_2(x) = \frac{1}{x}$, and $r_1(x) = c_1 x, r_2(x) = c_2 x$, where c_1, c_2 are fixed constants throughout the discussion. Consequently, we have

$$C_1(P_1, P_2) = \frac{G'P_2 + \eta_1}{GP_1} + c_1 P_1,$$
$$C_2(P_1, P_2) = \frac{1}{P_2} + c_2 P_2.$$

Therefore, by the optimality conditions for Nash equilibrium, we have $P_1^{Nash} = \sqrt{\frac{G'P_2 + \eta_1}{Gc_1}}, P_2^{Nash} = \sqrt{\frac{1}{c_2}}$. Because our main inquiry is whether noise is beneficial or not, a fair comparison would be to compare and contrast the following two:

1. Overall system performance under the best response power control scheme when the environment is fixed at $\bar{\mathbf{G}} = \begin{bmatrix} \bar{G} & \bar{G}' \\ 0 & 1 \end{bmatrix}$, $\bar{\eta} = [\bar{\eta}_1, 1]$.
2. Overall system performance under the best response power control scheme when the environment is random but with mean equal to $\bar{\mathbf{G}}, \bar{\eta}$.

In the first case, because the environment is deterministic, the equilibrium power is also a fixed constant. The overall system cost at equilibrium under best response power control is $C(P_1^{Nash}, P_2^{Nash}) = C_1(P_1^{Nash}, P_2^{Nash}) + C_2(P_1^{Nash}, P_2^{Nash}) = 2\sqrt{\frac{c_1\left(\frac{\bar{G}'}{\sqrt{c_2}} + \bar{\eta}_1\right)}{\bar{G}}} + 2\sqrt{c_2}$. In the second case, the overall system cost takes the same expression except now the environment quantities G, G', η_1 are random.

Consequently, denoting by C_e the overall system cost in equilibrium for the first case and by $\tilde{C}_e$ the *expected* overall system cost in equilibrium for the second case, we have

$$C_e - \tilde{C}_e = 2\sqrt{\frac{c_1\left(\frac{\bar{G}'}{\sqrt{c_2}} + \bar{\eta}_1\right)}{\bar{G}}} - 2\mathbf{E}\left[\sqrt{\frac{c_1\left(\frac{G'}{\sqrt{c_2}} + \eta_1\right)}{G}}\right],$$

where $\mathbf{E}[G] = \bar{G}, \mathbf{E}[G'] = \bar{G}'$. We consider two cases:

1. If the interference parameters are noisy (i.e., either G' or η_1 or both are random), then the system is *better off* compared with the deterministic environment case. This is because, by Jensen's inequality, because $\sqrt{x}$ is a strictly concave function, $\mathbf{E}\left[\sqrt{\frac{c_1\left(\frac{G'}{\sqrt{c_2}} + \eta_1\right)}{G}}\right] < \sqrt{\frac{c_1\left(\frac{\bar{G}'}{\sqrt{c_2}} + \bar{\eta}_1\right)}{\bar{G}}}$, thereby implying that $\tilde{C}_e < C_e$.
2. If link 1's own channel gain is noisy (i.e., G is random), then the system is *worse off* compared with the deterministic environment case. This is because, by

Jensen's inequality, because $\frac{1}{\sqrt{x}}$ is a strictly convex function, $\mathbf{E}[\sqrt{\frac{c_1(\frac{G'}{\sqrt{c_2}}+\eta_1)}{G}}] >< \sqrt{\frac{c_1(\frac{\bar{G}'}{\sqrt{c_2}}+\bar{\eta}_1)}{\bar{G}}}$, thereby implying that $C_e < \tilde{C}_e$.

We end the section with a brief discussion on the seemingly counterintuitive phenomenon that the noise can possibly increase the system performance. The main insight that can be taken away from the specific calculations done in the special case discussed previously is that noise has the effect of reducing the effective strength of the parameter in the signal. Consequently, if this strength reduction happens on parameters that are beneficial to communications (such as each link's own channel gain G_{ii}), then this would decrease the overall system performance because then the link resides in a more competitive environment (caused by the randomness). However, if this strength reduction happens on parameters that are harmful to communications in the first place (such as interference gains or thermal noise), then the overall system is lifted to a more benign environment in which less effort is required for communications, thereby driving down the overall system cost. Of course, when everything becomes random, things become much more complicated (even in the special case considered earlier): ascertaining whether the system cost becomes larger or smaller in such cases must be done on a case-by-case basis because one needs to work with specific distributions to discern which "strength reduction" is more severe and which is less to reach a conclusion about the overall effect. Nevertheless, through this case study, we hope to communicate the message that noise can be beneficial.

5. Conclusion

In view of the results in the prior sections, we believe this simple game theoretic model has two elements to contribute in the current literature. First, it provides a theoretical understanding of the "large-power" regime, in which the typical bounded power assumption is lifted. Second, and more importantly from an engineering perspective, our results provide a complete characterization of the best response update in the presence of random channel environments, thereby increasing its applicability. Additionally, we have derived a fixed-point theorem that operates in unbounded poset structures. Such a fixed-point theorem can potentially find other operations research–related applications with which one deals with unbounded decision variables.

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Endnotes

[1] In this paper, $\mathbf{R}_+$ denotes nonnegative reals and $\mathbf{R}_{++}$ denotes positive reals.

[2] We assume $fi(0) = +\infty$ to continuously extend the domain of f_i to nonnegative reals.

[3] This can be easily verified by noting that $f_i'(x) = -d\frac{\frac{1}{x}+\frac{1}{(x+1)\log(x+1)}}{x\log(x+1)}$, which is negative and strictly increasing in x.

[4] Note that the Brouwer fixed-point theorem can be viewed as a special case of the Kakutani fixed-point theorem as the former is a function but the latter is a correspondence, which maps a point to a set.

[5] It is easy to see that, without any further condition on g, such as the two just given, there can be no convergence rate guarantee in general.

[6] The definitions here follow mostly from Meyn and Tweedie (2009).

[7] $\mathcal{S}$ is a σ-algebra of subsets of S.

[8] Intuitively, $\mathbf{K}(s;A)$ gives the probability of the next state being in the set A with the current state at s. Formally, a transition kernel $\mathbf{K}: S \times \mathcal{S} \to [0,1]$ satisfies the following two properties:

- For each $s \in S, A \to \mathbf{K}(s,A)$ is a probability measure on $(S,\mathcal{S})$.
- For each $A \in \mathcal{S}, s \to \mathbf{K}(s,A)$ is a measurable function on $(S,\mathcal{S})$.

In our case, the transition kernel is time-invariant. Similarly, we can define the $m-$step transition kernel $\mathbf{K}^m(s,A)$.

[9] Note the distinction between the first return time ($n \geq 1$) here and the first visit time ($n \geq 0$).

[10] Intuitively, it means the return time to A is uniformly bounded when you start in A.

[11] Here we are suppressing the dependence of γ, T on the initial point $P^e(\overline{\theta})$. It is understood that, from this point onward until the end of the proof, γ, T will always refer to the initial point being $P^e(\overline{\theta})$.

[12] A Polish space is a space homeomorphic to complete separable metric space. In particular, a complete separable metric space is Polish.

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