

RECURRENCE CLASSIFICATION  
FOR A FAMILY OF NON-LINEAR STORAGE MODELS

BY

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*Dedicated to Professor Tomasz Rolski on the occasion of his 70th birthday*

*Abstract.* Necessary and sufficient conditions for positive recurrence of a discrete-time non-linear storage model with power law dynamics are derived. In addition, necessary and sufficient conditions for finiteness of  $p$ -th stationary moments are obtained for this class of models.

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1. INTRODUCTION

It is a pleasure to recognize the extensive contributions of Tomasz Rolski to applied probability, across a wide range of different topic areas. This paper connects to his important work at the intersection of Markov processes (see, for example, [16], [17], and [19]) and stochastic modeling. As will be seen below, the class of models includes, as a special case, the waiting time sequence for the single-server queue, a model to which Professor Rolski has made many important contributions (see, for example, [5], [6], [18], and [14]).

This paper considers a discrete-time storage model in which  $S_n$  represents the content stored in the reservoir at the beginning of the time period  $n$ . Flow conservation implies that

$$S_{n+1} = S_n + Z_{n+1} - O_{n+1},$$

where  $Z_{n+1}$  is the inflow during the period  $(n, n + 1]$ , and  $O_n$  is the outflow during the period  $(n, n + 1]$ . In this paper, we study the non-linear storage model in which  $(Z_n : n \geq 1)$  is a sequence of non-negative independent and identically distributed

random variables, and the outflow  $O_n$  follows the “power law” relation

$$O_n = aS_n^b$$

for  $a$  and  $b$  positive. Under the above assumptions,  $S = (S_n : n \geq 0)$  is a Markov chain taking values in  $\mathbb{R}_+$ .

Such non-linear “power law” dynamics have been widely studied in the hydrology literature, where discrete-time models are used almost exclusively; see, for example, [13] and [4]. There is theoretical justification for using “power law” dynamics for modeling both surface runoff (see [21]) and, as well, for modeling groundwater discharge (see [20]). A diffusion approximation for a network of continuous-time reservoirs, each with “power law” dynamics, was investigated in [9]. The discrete-time storage process with general release rule was studied in [8], in which it was shown that for the “power law” release rule in the case where  $EZ_1 < a$ , there is a continuous dependence of the moments of storage and outflow on  $b$  for small  $b$ . Sufficient conditions for the positive recurrence of autoregressive processes, obtained from our model when  $b = 1$ , are discussed in [1], while necessary and sufficient conditions for transience, null recurrence, and positive recurrence can be found in [10].

Our main results in this paper are:

- (i) necessary and sufficient conditions on the distribution of  $Z_1$  under which  $S$  is a positive recurrent Markov chain (Theorem 2.1);
- (ii) necessary and sufficient conditions on the distribution of  $Z_1$  under which the stationary distribution of  $S$  has finite  $p$ -th moment (Theorem 2.2).

An important tool in our analysis is the use of (stochastic) Lyapunov functions. Our Lyapunov function construction takes advantage of approximations obtained from a related deterministic model which approximates the dynamics of  $S$  when the chain takes on large values. Such an approach to the construction of Lyapunov functions has also been used with great effectiveness in the queueing context. Therein, the so-called “fluid” approximations provide great insight into the construction of Lyapunov functions for the corresponding stochastic queueing network; see, for example, [15]. We view the results of this paper as another excellent illustration of the power of such methods.

The recurrence classification of storage processes has been investigated in continuous time in [11] and [3], among others. Harrison and Resnick [11] proved a duality in recurrence between a storage process with a general release rule and its associated risk process. Brockwell et al. [3] also investigated the recurrence classification of storage systems with general release rule, together with specific application to the case of “power law” models; however, their results differ from ours in the following ways:

- (a) Brockwell et al. [3] work with a continuous-time formulation, whereas we work with a discrete-time formulation.
- (b) Their main result on power law release rules (see p. 429) assumes that the inflow distribution follows a stable law. When the exponent  $b$  is less than one, we

only make hypotheses about moments and make no assumptions at all about the functional form of the distribution.

(c) They have no analog to our recurrence result for  $b$  greater than one. For  $b = 1$ , they have precisely our result (see p. 428).

(d) They do not examine necessary and sufficient conditions for finiteness of moments for the stationary distribution.

This paper is organized as follows. Section 2 states the main results of the paper, while Section 3 provides proofs of the more complicated arguments supporting the results of Section 2.

### 2. THE MAIN RESULTS

Let us start by introducing the notation  $P_x(\cdot) = P(\cdot | S_0 = x)$  and  $E_x(\cdot) = E(\cdot | S_0 = x)$ . For a probability distribution  $\mu$  on  $\mathbb{R}_+$ , let

$$P_\mu(\cdot) = \int_{\mathbb{R}_+} \mu(dx) P_x(\cdot)$$

and

$$E_\mu(\cdot) = \int_{\mathbb{R}_+} \mu(dx) E_x(\cdot).$$

Our first result is a ‘‘solidarity’’ statement that asserts that the long-run behavior of  $S$  is, under suitable conditions, independent of the initial state  $x$ .

**PROPOSITION 2.1.** *Let  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$  be bounded and uniformly continuous. Then:*

- (i)  $(P^n f : n \geq 0)$  is an equicontinuous family of functions.
- (ii) For  $b \geq 1$  and any two probabilities  $\mu$  and  $\nu$  on  $\mathbb{R}_+$ ,  $E_\mu f(S_n) - E_\nu f(S_n) \rightarrow 0$  as  $n \rightarrow \infty$ .
- (iii) If there exists  $x \geq 0$  such that  $P_x(\liminf_{n \rightarrow \infty} S_n < \infty) = 1$ , then we have  $E_\mu f(S_n) - E_\nu f(S_n) \rightarrow 0$  as  $n \rightarrow \infty$  for any two probabilities  $\mu$  and  $\nu$  on  $\mathbb{R}_+$ .

Part (i) implies that  $S$  is an e-chain, in the terminology of [15]. Note that part (ii) applies even if  $S_n \rightarrow \infty$  a.s. (and is therefore transient), whereas part (iii) can apply even if  $S$  is null recurrent.

Recall that a probability distribution  $\pi$  on  $\mathbb{R}_+$  is said to be *stationary* for  $S$  if  $P_\pi(S_1 \in \cdot) = P_\pi(S_0 \in \cdot)$ . Our next result states that if  $S$  has a stationary distribution  $\pi$ , then that stationary distribution is necessarily unique, and  $S_n$  converges weakly to  $\pi$ , regardless of the initial distribution.

**PROPOSITION 2.2.** *Suppose that  $S$  has a stationary distribution  $\pi$ . Let  $\mu$  be a probability on  $\mathbb{R}_+$  and let  $x \in \mathbb{R}_+$ . Then,*

- (i)  $P_x(\liminf_{n \rightarrow \infty} S_n < \infty) = 1$ ;
- (ii)  $P_\mu(S_n \in \cdot) \Rightarrow P_\pi(S_0 \in \cdot)$  as  $n \rightarrow \infty$ ;
- (iii)  $\pi$  is necessarily the unique stationary distribution of  $S$ .

Propositions 2.1 and 2.2 make clear that  $S$  is well behaved, in terms of its long-run characteristics, once one establishes existence of a stationary distribution. Thus, we now turn to studying the question of when  $S$  has a stationary distribution (which must then necessarily be unique, by Proposition 2.2).

**PROPOSITION 2.3.** *For  $c > 0$ , let  $\tau(c) = \inf\{n \geq 1 : S_n \leq c\}$ . The Markov chain  $S$  has a stationary distribution  $\pi$  if and only if there exists a  $c < \infty$  such that  $E_c\tau(c) < \infty$ .*

Proposition 2.3 shows that the key to the analysis of  $S$  is the study of the finiteness of  $E_c\tau(c)$ . Since

$$(2.1) \quad E_c\tau(c) = 1 + \int_{(c, \infty)} P_c(S_1 \in dx) E_x\tau(c),$$

we need to analyze the behavior of  $E_x\tau(c)$  for large values of  $x$ .

We can now exploit an idea that has proved very effective in the analysis of queueing networks. Specifically, we attempt to find a deterministic dynamical system that approximates the behavior of  $S_n$  when  $S_n$  is large. In the queueing context, such deterministic approximations are called “fluid models”. In our storage theory setting, the deterministic “fluid” approximation to  $(S_n : n \geq 0)$ , conditional on  $S_0 = x$ , will be the deterministic sequence  $(s_n(x) : n \geq 0)$ , for which  $s_0(x) = x$  and

$$s_{n+1}(x) + as_{n+1}(x)^b = s_n(x)$$

for  $n \geq 0$ . For  $c > 0$ , let  $t_c(x) = \inf\{n \geq 0 : s_n(x) \leq c\}$  be the first “hitting time” of  $[0, c]$  for  $(s_n(x) : n \geq 0)$ . Our hope is that  $E_x\tau(c)$  can be approximated by  $t_c(x)$  when  $x$  is large. For  $x \geq 0$ , let

$$h(x) = \begin{cases} x^{1-b}/(a(1-b)) & \text{if } 0 < b < 1, \\ \log(1+x)/\log(1+a) & \text{if } b = 1, \\ \log \log(e+x)/\log(b) & \text{if } b > 1. \end{cases}$$

**PROPOSITION 2.4.** *For each  $c \in (0, \infty)$ ,*

$$\liminf_{x \rightarrow \infty} t_c(x)/h(x) \geq 1.$$

According to Proposition 2.4,  $t_c(x)$  is at least as large as  $h(x)$  for  $x$  large. Because  $(s_n(x) : n \geq 0)$  is obtained from  $(S_n : n \geq 0)$  (conditional on  $S_0 = x$ ) by setting all the inflows to zero, it follows that  $s_n(x) \leq S_n(x)$  for  $n \geq 0$ . Consequently,  $t_c(x) \leq E_x\tau(c)$  for  $x \geq 0$  and  $c > 0$ . In view of (2.1) and Proposition 2.3, we may conclude that the existence of  $\pi$  implies that

$$(2.2) \quad E_ch(S_1) < \infty.$$

Our next result simplifies the moment condition (2.2).

PROPOSITION 2.5.  $E_c h(S_1) < \infty$  if and only if  $Eh(Z_1) < \infty$ .

The following proposition shows that our deterministic dynamical system  $(s_n(x) : n \geq 0)$  does in fact provide good approximations for  $E_x \tau(c)$ .

PROPOSITION 2.6. Suppose that  $Eh(Z_1) < \infty$ . Then, there exists  $c < \infty$  such that

$$\limsup_{x \rightarrow \infty} E_x \tau(c)/h(x) \leq 1,$$

so that

$$E_x \tau(c) \sim h(x)$$

as  $x \rightarrow \infty$ , and

$$t_c(x) \sim h(x)$$

as  $x \rightarrow \infty$ , where we write  $a(x) \sim b(x)$  as  $x \rightarrow \infty$ , whenever  $a(x)/b(x) \rightarrow 1$  as  $x \rightarrow \infty$ .

Our main theorem on recurrence classification for  $S$  follows easily from Propositions 2.1–2.6.

THEOREM 2.1. Suppose  $\mu$  is a probability distribution on  $\mathbb{R}_+$ . Then:

(i) If  $0 < b < 1$ , then  $S$  has a stationary distribution  $\pi$  if and only if

$$EZ_1^{1-b} < \infty.$$

(ii) If  $b = 1$ , then  $S$  has a stationary distribution  $\pi$  if and only if

$$E \log(1 + Z_1) < \infty.$$

(iii) If  $b > 1$ , then  $S$  has a stationary distribution  $\pi$  if and only if

$$E \log \log(e + Z_1) < \infty.$$

If  $S$  has a stationary distribution  $\pi$ , then  $\pi$  is necessarily the unique stationary distribution of  $S$  and

$$P_\mu(S_n \in \cdot) \Rightarrow P_\pi(S_0 \in \cdot) \quad \text{as } n \rightarrow \infty.$$

We can also derive necessary and sufficient conditions for the finiteness of stationary moments. This is our second principal result.

THEOREM 2.2. Suppose  $S$  has a stationary distribution  $\pi$ . Let  $p > 0$ . Then:

(i) If  $0 < b < 1$ , then  $E_\pi S_0^p < \infty$  if and only if  $EZ_1^{p+1-b} < \infty$ .

(ii) If  $b = 1$ , then  $E_\pi S_0^p < \infty$  if and only if  $EZ_1^p < \infty$ .

(iii) If  $b > 1$ , then  $E_\pi S_0^p < \infty$  if and only if  $EZ_1^{p/b} < \infty$ .

It turns out that as  $b \downarrow 0$ ,  $S$  has dynamics that converge to those of the waiting time sequence of the single-server queue with deterministic interarrival times equal to  $a$ . To prove this, we only need to show that the finite-dimensional distributions of the process  $S$  converge to the waiting time sequence of the relevant single-server queue. Let  $\phi_b^a$  be the inverse of  $y \mapsto y + ay^b$ . Then  $\phi_b^a \rightarrow \phi^a$  as  $b$  tends to zero, where  $\phi^a(x) = (x - a)^+$ . To see this, first note that  $\phi_b^a(1 + a) = 1$ . Since  $\phi_b^a$  is monotonically increasing,  $\phi_b^a(x) > 1$  for all  $x > 1 + a$ , but this means that  $\phi_b^a(x) \uparrow \phi^a(x)$ . For  $a < x < 1 + a$ ,  $\phi_b^a(x) < 1$ , and so  $\phi_b^a(x) \downarrow \phi^a(x) > 0$ . So for  $x > a$ ,  $\phi^a(x) = x - a$ . The monotonicity and positivity of  $\phi_b^a$  imply that of  $\phi^a$ , and so  $\phi^a(a) = 0$ , because for any  $\varepsilon > 0$ , we have  $0 \leq \phi^a(a) \leq \phi^a(a + \varepsilon) = \varepsilon$ . Thus,  $\phi^a(a) = 0$  and  $\phi^a(x) = 0$  for all  $0 \leq x \leq a$ . This shows that for all  $n \geq 0$ ,  $S \Rightarrow W$ , where  $W_{n+1} = (W_n + Z_{n+1} - a)^+$ ,  $n \geq 0$ .

The celebrated results of [12] show that  $E Z_1^{p+1} < \infty$  is necessary and sufficient for finiteness of the  $p$ -th moment of the waiting time sequence. Thus, one “extra” moment is needed. Theorem 2.2 shows clearly the continuous transition from the autoregressive/linear dynamics setting ( $b = 1$ ) in which no extra moments on  $Z_1$  are needed to the classical queueing context, in which one extra moment of  $Z_1$  is required. It is also worth noting that, subject to  $b$  being greater than one, the specific value of  $b$  has no impact on recurrence (see Theorem 2.1), whereas the specific value of  $b$  has a clear impact on the moment structure of  $S$ .

### 3. PROOFS

We assume throughout that  $P(Z_1 > 0) > 0$  because the case in which  $Z_1 = 0$  a.s. is trivial to treat.

**Proof of Proposition 2.1.** We shall employ a “coupling” argument to prove this result. Given  $(Z_n : n \geq 1)$  and  $x \geq 0$ , set  $S_0(x) = x$  and

$$S_{n+1}(x) + aS_{n+1}(x)^b = S_n(x) + Z_{n+1}$$

for  $n \geq 0$ . Put  $S(x) = (S_n(x) : n \geq 0)$  and note that  $P_x(S \in \cdot) = P(S(x) \in \cdot)$ . If  $0 \leq x \leq y$ , the monotonicity of  $S(x)$  in  $x$  implies that  $S_n(x) \leq S_n(y)$  for  $n \geq 0$ . Furthermore,

$$\begin{aligned} 0 &\leq S_{n+1}(y) - S_{n+1}(x) \\ &\leq S_{n+1}(y) + aS_{n+1}^b(y) - S_{n+1}(x) - aS_{n+1}^b(x) \\ &= (S_n(y) + Z_{n+1}) - (S_n(x) + Z_{n+1}) = S_n(y) - S_n(x), \end{aligned}$$

so  $(|S_n(y) - S_n(x)| : n \geq 0)$  is always a non-increasing sequence. It follows that if  $f$  is uniformly continuous on  $\mathbb{R}_+$ , then

$$\begin{aligned} |E_y f(S_n) - E_x f(S_n)| &= |E f(S_n(y)) - E f(S_n(x))| \\ &\leq \sup_{\delta \leq |x-y|, z \geq 0} (|f(z + \delta) - f(z)|), \end{aligned}$$

which can be made arbitrarily small (uniformly in  $n$ ) by choosing  $|x - y|$  small enough. Hence (i) follows immediately.

For (ii), note that, for  $x, y \in \mathbb{R}_+$ ,

$$|S_{n+1}(y) + aS_{n+1}^b(y) - S_{n+1}(x) - aS_{n+1}^b(x)| = |S_n(y) - S_n(x)|,$$

so that

$$(3.1) \quad |S_{n+1}(y) - S_{n+1}(x)| = \frac{1}{1 + ab\xi_{n+1}^{b-1}} |S_n(y) - S_n(x)|,$$

where  $\xi_{n+1}$  lies in the interval between  $S_{n+1}(x)$  and  $S_{n+1}(y)$ . Let  $\phi(\cdot)$  be the inverse function to  $x + ax^b$ , so that  $\phi(x + ax^b) = x$ . Note that, for  $z \geq 0$ ,

$$S_{n+1}(z) + aS_{n+1}^b(z) = S(z) + Z_{n+1} \geq Z_{n+1},$$

so that  $S_{n+1}(z) \geq \phi(Z_{n+1})$ . Hence, if  $b \geq 1$ , evidently  $\xi_{n+1}^{b-1} \geq \phi(Z_{n+1})^{b-1}$ , so that

$$|S_{n+1}(y) - S_{n+1}(x)| \leq (1 + ab\phi(Z_{n+1})^{b-1})^{-1} |S_n(y) - S_n(x)|.$$

So, we may conclude that  $S_{n+1}(y) - S_{n+1}(x) \rightarrow 0$  a.s. as  $n \rightarrow \infty$  whenever

$$(3.2) \quad \prod_{j=1}^n (1 + ab\phi(Z_j)^{b-1})^{-1} \rightarrow 0 \quad \text{a.s.}$$

as  $n \rightarrow \infty$ . But  $P(Z_1 \geq c_0) > 0$  for some  $c_0 > 0$ , so  $\phi(Z_j)^{b-1} \geq \phi(c_0)^{b-1}$  infinitely often a.s., since  $b - 1 > 0$ . This implies that the product (3.2) goes to zero a.s. It follows that, for  $x$  fixed and  $f$  bounded and uniformly continuous,

$$(3.3) \quad E_y f(S_n) - E_x f(S_n) \rightarrow 0$$

as  $n \rightarrow \infty$ . By integrating (3.3) over  $y$ , the bounded convergence theorem then implies that

$$E_\mu f(S_n) - E_x(S_n) \rightarrow 0$$

and

$$E_\nu f(S_n) - E_x(S_n) \rightarrow 0$$

as  $n \rightarrow \infty$ , proving (ii).

To prove (iii), we again use (3.1), and note that

$$S_{n+1}(x) \leq \phi(S_n(x) + Z_{n+1}).$$

Consequently, for  $y \geq 0$ ,  $\xi_{n+1} \leq \phi(S_n(x) + Z_{n+1}) + |x - y|$ . Thus, for  $0 < b < 1$ , we have

$$\begin{aligned} & |S_{n+1}(x) - S_{n+1}(y)| \\ & \leq \left(1 + ab\left(\phi(S_n(x) + Z_{n+1}) + |x - y|\right)^{b-1}\right)^{-1} |S_n(x) - S_n(y)|, \end{aligned}$$

so that

$$(3.4) \quad |S_n(x) - S_n(y)| \leq \prod_{j=0}^{n-1} \left(1 + ab\left(\phi(S_j(x) + Z_{j+1}) + |x - y|\right)^{b-1}\right)^{-1} |x - y|.$$

Let  $N_1 = \inf\{n \geq 0 : S_n(x) \leq c\}$  and  $N_{j+1} = \inf\{n > N_j : S_n(x) \leq c\}$  for  $j \geq 0$ . On  $\{\liminf_{n \rightarrow \infty} S_n(x) < c\}$ , the  $N_j$ 's are finite a.s. Furthermore, on  $\{N_j < \infty\}$ , we obtain

$$\begin{aligned} & P\left(\left(1 + ab\left(\phi(S_{N_j}(x) + Z_{N_j+1}) + |x - y|\right)^{b-1}\right)^{-1}\right. \\ & \quad \left. \leq \left(1 + ab\left(\phi(c + c_0) + |x - y|\right)^{b-1}\right)^{-1} |Z_0, \dots, Z_{N_j}\right) \\ & = P\left(\left(1 + ab\left(\phi(S_{N_j}(x) + Z_{N_j+1}) + |x - y|\right)^{b-1}\right)^{-1}\right. \\ & \quad \left. \leq \left(1 + ab\left(\phi(c + c_0) + |x - y|\right)^{b-1}\right)^{-1} |S_{N_j}\right) \\ & \geq P(Z_{N_j+1} \leq c_0 | S_{N_j}) = P(Z_1 \leq c_0). \end{aligned}$$

The conditional Borel–Cantelli lemma (see, for example p. 324 of [7]) therefore implies that

$$\begin{aligned} & \left(1 + ab\left(\phi(S_{N_j}(x) + Z_{N_j+1}) + |x - y|\right)^{b-1}\right)^{-1} \\ & \leq \left(1 + ab\left(\phi(c + c_0) + |x - y|\right)^{b-1}\right)^{-1} \end{aligned}$$

infinitely often a.s. on  $\{\liminf_{n \rightarrow \infty} S_n(x) < c\}$ . So,

$$\begin{aligned} & P(S_n(x) - S_n(y) \rightarrow 0 \text{ as } n \rightarrow \infty) \\ & \geq P(S_n(x) - S_n(y) \rightarrow 0 \text{ as } n \rightarrow \infty, \liminf_{n \rightarrow \infty} S_n(x) < c) \\ & = P(\liminf_{n \rightarrow \infty} S_n(x) < c). \end{aligned}$$

Sending  $c \rightarrow \infty$ , we conclude that  $S_n(x) - S_n(y) \rightarrow 0$  a.s. as  $n \rightarrow \infty$ . As in part (ii), we find that  $E_\mu f(S_n) - E_\nu f(S_n) \rightarrow 0$  as  $n \rightarrow \infty$  for all probabilities  $\mu$  and  $\nu$  on  $\mathbb{R}_+$ , proving (iii). ■

**Proof of Proposition 2.2.** Suppose that

$$\delta = 1 - P_x(\liminf_{n \rightarrow \infty} S_n < \infty) = P_x(\lim_{n \rightarrow \infty} S_n = \infty) > 0.$$

Choose  $c_0$  large enough so that  $P_\pi(S_0 > c_0) < \delta/4$ . Then

$$\begin{aligned} (3.5) \quad P_\pi(S_n > 2c_0 + x) &\leq \int_{\{y: y \leq c_0\}} \pi(dy) P_y(S_n > 2c_0 + x) + P_\pi(S_0 > c_0) \\ &\leq \int_{\{y: y \leq c_0\}} \pi(dy) P_y(S_n > c_0 + |x - y|) + \delta/4 \\ &\leq \int_{\{y: y \leq c_0\}} \pi(dy) P_y(S_n > c_0) + \delta/4 \\ &\leq P_\pi(S_n > c_0) + \delta/4 \leq \delta/2. \end{aligned}$$

On  $\{\lim_{n \rightarrow \infty} S_n = \infty\}$ , we have

$$\frac{1}{n} \sum_{j=0}^{n-1} I(S_j > 2c_0 + x) \rightarrow 1$$

as  $n \rightarrow \infty$ . Hence Fatou's lemma yields

$$\begin{aligned} (3.6) \quad \liminf_{n \rightarrow \infty} E_x \frac{1}{n} \sum_{j=0}^{n-1} I(S_j > 2c_0 + x) \\ \geq E_x \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} I(S_j > 2c_0 + x) \geq P_x(\lim_{n \rightarrow \infty} S_n = \infty) = \delta. \end{aligned}$$

But (3.5) shows that

$$\frac{1}{n} \sum_{j=0}^{n-1} E_x I(S_j > 2c_0 + x) = \frac{1}{n} \sum_{j=0}^{n-1} P_x(S_j > 2c_0 + x) \leq \delta/2,$$

which contradicts (3.6), proving (i).

To prove part (ii), let  $f$  be bounded and uniformly continuous. If  $b \geq 1$ , then  $E_\mu f(S_n) - E_\nu f(S_n) \rightarrow 0$  for any two probabilities  $\mu$  and  $\nu$  as a consequence of part (ii) of Proposition 2.1, whereas, if  $b < 1$ , then  $E_\mu f(S_n) - E_\nu f(S_n) \rightarrow 0$  as a result of part (i) of Proposition 2.2 (proved above) and part (iii) of Proposition 2.1. Setting  $\nu = \pi$  and using the stationarity of  $\pi$  establishes that  $E_\mu f(S_n) \rightarrow E_\pi f(S_0)$  for all bounded continuous  $f$ , proving that  $P_\mu(S_n \in \cdot) \Rightarrow P_\pi(S_0 \in \cdot)$  as  $n \rightarrow \infty$ .

Finally, for part (iii), suppose  $\mu \neq \pi$  is a stationary probability distribution. Then, part (iii) (of Proposition 2.2) guarantees that  $P_\mu(S_0 \in \cdot) = P_\mu(S_n \in \cdot) = P_\pi(S_0 \in \cdot)$ , showing that  $\mu = \pi$ . ■

**Proof of Proposition 2.3.** Suppose first that  $E_c\tau(c) < \infty$ . Because  $S_n(x) < S_n(c)$  for  $0 \leq x \leq c$  and  $n \geq 0$ , evidently  $E_x\tau(c) \leq E_c\tau(c)$  for  $0 \leq x \leq c$ , so that

$$\sup\{E_x\tau(c) : 0 \leq x \leq c\} = E_c\tau(c) < \infty.$$

Theorem 12.4.3 (part iv.) and Theorem 12.1.2 (part i.) of [15] then yield the existence of a stationary distribution  $\pi$ . For the converse, suppose that a stationary distribution  $\pi$  exists. According to Proposition 2.2,  $\pi$  is necessarily the unique stationary distribution of  $S$ . The uniqueness of  $\pi$  ensures that  $S$  is an ergodic stationary sequence under  $P_\pi$  (p. 118 of [2]). Choose  $c$  large enough so that  $\pi([0, c]) > 0$ . Then, Proposition 6.3.8 of [2] proves that

$$\pi([0, c]) = \frac{1}{E_{\pi_c}\tau(c)},$$

where  $\pi_c(\cdot) = \pi(\cdot \cap [0, c])/\pi([0, c])$ . It follows that  $E_{\pi_c}\tau(c) < \infty$ . This ensures the existence of  $x \in [0, c]$  for which  $E_x\tau(c) < \infty$ . Suppose now that  $\|Z_1\| = \sup\{x \geq 0 : P(Z_1 \leq x) < 1\} = \infty$ . Then

$$\infty > E_x\tau(c) \geq \int_{(c, \infty)} P_x(S_1 \in dy) E_y\tau(c)$$

guarantees the existence of  $y > c$  for which  $E_y\tau(c) < \infty$ . But the monotonicity of  $S_n(\cdot)$  implies that  $E_c\tau(c) \leq E_y\tau(c)$  for  $y \geq c$ , so that  $E_c\tau(c) < \infty$ . If  $\|Z_1\| < \infty$ , note that  $S_n(x) < s_n^*(x)$ , where  $(s_n^*(x) : n \geq 0)$  satisfies the recursion  $s_0^*(x) = x$  and  $s_{n+1}^*(x) + as_{n+1}^*(x)^b = \|Z_1\| + s_n^*(x)$  for  $n \geq 0$ . It is easily seen that  $\lim_{n \rightarrow \infty} s_n^*(x) = (\|Z_1\|/a)^{1/b}$ . Let us put  $c = (\|Z_1\|/a)^{1/b} + 1$ . Since  $(s_n^*(c) : n \geq 0)$  hits the interval  $[0, c]$  in a finite number of iterations,  $\tau(c)$  is bounded above by a finite deterministic quantity, proving that  $E_c\tau(c) < \infty$ . ■

**Proof of Proposition 2.4.** Because  $s_n(x)$  decreases monotonically and deterministically to zero as  $n \rightarrow \infty$ , it follows that  $t_{c_1}(x) = t_{c_2}(x) + O(1)$  as  $x \rightarrow \infty$  (for  $c_1, c_2$  positive). Thus, it is sufficient to establish, for each  $\epsilon > 0$ , the existence of  $c = c(\epsilon)$  such that  $\liminf_{x \rightarrow \infty} t_c(x)/h(x) \geq 1 - \epsilon$ . Assume first that  $b \in (0, 1)$ . Fix  $x$  and set  $s_n = s_n(x)$ . Put  $y_n = (s_n + as_n^b)^{1-b}$ . For  $z \geq 0$ ,

$$(1 + z)^{1-b} \leq 1 + (1 - b)z.$$

Hence

$$\begin{aligned} y_n - y_{n-1} &= (s_n + as_n^b)^{1-b} - (s_{n-1} + as_{n-1}^b)^{1-b} \\ &= s_{n-1}^{1-b} - (s_{n-1} + as_{n-1}^b)^{1-b} \\ &= s_{n-1}^{1-b} (1 - (1 + as_{n-1}^{b-1})^{1-b}) \\ &\geq s_{n-1}^{1-b} (1 - 1 - (1 - b)as_{n-1}^{b-1}) = -(1 - b)a. \end{aligned}$$

Set  $c = 1$ . It follows that

$$y_{t_c(x)} - y_0 \geq -(1 - b)at_c(x),$$

so that

$$t_c(x) \geq \frac{(x + ax^b)^{1-b} - (1 + a)^{1-b}}{(1 - b)a},$$

from which we conclude that

$$\liminf_{x \rightarrow \infty} t_c(x)/x^{1-b} \geq [(1 - b)a]^{-1}.$$

If  $b = 1$ , we find that  $s_n = (1 + a)^{-n}x$ , so that if  $c = 1$ ,  $t_c(x) \geq \log x / \log(1 + a)$  for  $x > 1$ , proving the required result.

Finally, for  $b > 1$ , we let  $y_n = \log \log(e + s_n + as_n^b)$ . For  $\epsilon > 0$ ,

$$\frac{\log(e + x)}{\log(e + z + az^b)} \geq \frac{1}{b}(1 - \epsilon)$$

for  $z \geq z(\epsilon)$ . Put  $c = z(\epsilon)$ . Note that if  $y_{n-1} \geq c$ , then

$$\begin{aligned} y_n - y_{n-1} &= \log \log(e + s_{n-1}) - \log \log(e + s_{n-1} + as_{n-1}^b) \\ &= \log \left\{ \frac{\log(e + s_{n-1})}{\log(e + s_{n-1} + as_{n-1}^b)} \right\} \\ &\leq -\log b + \log(1 - \epsilon). \end{aligned}$$

It follows that, for  $x > c$ ,

$$y_{t_c(x)} - y_0 \geq (-\log b + \log(1 - \epsilon))t_c(x),$$

so that

$$t_c(x) \geq \frac{\log \log(e + x + ax^b) - \log \log(e + c + ac^b)}{\log b - \log(1 - \epsilon)}.$$

This lower bound is easily seen to imply the required result. ■

**Proof of Proposition 2.5.** Recall that  $\phi(\cdot)$  is the inverse function to  $x + ax^b$ . For  $b < 1$ ,  $\phi(x) \sim x$  as  $x \rightarrow \infty$ , whereas for  $b = 1$ ,  $\phi(x) = x/(1 + a)$ . Finally, for  $b > 1$ ,  $\phi(x) \sim (x/a)^{1/b}$  as  $x \rightarrow \infty$ . Note that  $E_ch(S_1) = E(h \circ \phi)(c + Z_1)$ . Given the asymptotics for  $\phi$  and the monotonicity of both  $\phi$  and  $h$ , we need to check that  $E(c + Z_1)^{1-b} < \infty$  if and only if  $EZ_1^{1-b} < \infty$  (for  $0 < b < 1$ ), whereas for  $b = 1$ , we must verify that  $E(c + Z_1) < \infty$  if and only if  $EZ_1 < \infty$ . Finally, for  $b > 1$ , we must show that  $E \log \log(e + ((c + Z_1)/a)^{1/b}) < \infty$  if and only if  $E \log \log(e + Z_1) < \infty$ . All three cases can be easily established by elementary arguments. ■

**Proof of Proposition 2.6.** We use appropriately defined (stochastic) Lyapunov functions to prove the result. For  $b \in (0, 1)$ , let  $v(x) = (x + ax^b)^{1-b}$ . Then,

$$\begin{aligned} & v(S_1(x)) - v(x) \\ &= (x + Z_1)^{1-b} - v(x) \\ &= x^{1-b}(1 + Z_1/x)^{1-b} - x^{1-b}(1 + ax^{b-1})^{1-b} \\ &= x^{1-b}(1 + (1-b)Z_1/x + o(1/x)) - x^{1-b}(1 + a(1-b)x^{b-1} + o(x^{b-1})) \\ &\rightarrow -a(1-b) \end{aligned}$$

as  $x \rightarrow \infty$ . Furthermore,

$$|v(S_1(x)) - v(x)| \leq x^{1-b}(1^{1-b} + (Z_1/x)^{1-b}) - x^{1-b} = Z_1^{1-b},$$

so that the dominated convergence theorem proves that

$$Ev(S_1(x)) - v(x) = E_x v(S_1) - v(x) \rightarrow -a(1-b)$$

as  $x \rightarrow \infty$ . Given  $\epsilon > 0$ , it follows from Theorem 11.3.4 of [15] that there exists  $c(\epsilon)$  such that

$$E_x \tau(c(\epsilon)) \leq \frac{v(x)(1+\epsilon)}{a(1-b)}$$

for  $x > c(\epsilon)$ . Choose  $c = c(1)$ . Note that for  $x > c(\epsilon)$ ,  $E_x \tau(c) \leq E_x \tau(c(\epsilon)) + E_{c(\epsilon)} \tau(c)$ , proving that

$$\limsup_{x \rightarrow \infty} E_x \tau(c)/v(x) \leq (1+\epsilon)/(a(1-b)).$$

Since  $\epsilon > 0$  was arbitrary, this proves the required result for  $b < 1$ . For  $b = 1$ , choose  $v(x) = \log(1+x)$ . Then,

$$\begin{aligned} & v(S_1(x)) - v(x) = \log(1 + (x + Z_1)/(1+a)) - \log(1+x) \\ &= \log \left\{ \frac{1 + (x + Z_1)/(1+a)}{1+x} \right\} \\ &\rightarrow -\log(1+a) \end{aligned}$$

as  $x \rightarrow \infty$ . In addition,

$$\begin{aligned} |v(S_1(x)) - v(x)| &= \log \left\{ \left( \frac{1}{1+a} \right) \left( \frac{1+x+Z_1}{1+x} \right) + \frac{a}{1+a} \right\} \\ &\leq \log(1 + Z_1 + a/(1-a)) \\ &\leq \log(1 + Z_1) + \log(1 + a/(1-a)). \end{aligned}$$

The dominated convergence theorem again applies as  $x \rightarrow \infty$ , and the same argument as for  $b < 1$  proves the required result.

Finally, for  $b > 1$ , choose  $v(x) = \log \log(e + x + ax^b)$ . Here,

$$v(S_1(x)) - v(x) = \log \log(e + x + Z_1) - \log \log(e + x + ax^b) \rightarrow -\log b$$

as  $x \rightarrow \infty$ . Also,

$$\begin{aligned} |v(S_1(x)) - v(x)| &= \log \left\{ \frac{\log(e + x + Z_1)}{\log(e + x + ax^b)} \right\} \\ &\leq \log \left\{ \frac{\log(e + x + ax^b) + \log(e + Z_1)}{\log(e + x + ax^b)} \right\} \\ &= \log(1 + \log(e + Z_1)) = 1 + \log \log(e + Z_1). \end{aligned}$$

As above, the dominated convergence theorem applies as  $x \rightarrow \infty$ , completing the proof. ■

**Proof of Theorem 2.2.** For  $b = 1$ , we use the fact that

$$\pi(\cdot) = P\left(\sum_{i=0}^{\infty} (1+a)^{-i} Z_i \in \cdot\right),$$

so that

$$E_{\pi} S_0^p = E\left(\sum_{i=0}^{\infty} (1+a)^{-i} Z_i\right)^p.$$

But  $(\sum_{i=0}^{\infty} (1+a)^{-i} Z_i)^p \geq Z_0^p$ , proving the necessity of the moment condition. For sufficiency, we use the bound

$$\left(\sum_{i=0}^{\infty} (1+a)^{-i} Z_i\right)^p \leq \sum_{i=0}^{\infty} (1+a)^{-ip} Z_i^p$$

when  $0 \leq p \leq 1$ ; and when  $p > 1$ , use the bound

$$\left(\sum_{i=0}^{\infty} (1+a)^{-i} Z_i\right)^p \leq \left(\frac{1+a}{a}\right)^p \sum_{i=0}^{\infty} \left(1 - \frac{1}{1+a}\right)^i (1+a)^{-i} Z_i^p.$$

The sufficiency for  $b = 1$  follows immediately from these bounds. To see how to obtain the bound for  $p > 1$ , define the random variable  $Y$  to be equal to  $Z_i$  with probability

$$\left(1 - \frac{1}{1+a}\right) \left(\frac{1}{1+a}\right)^i,$$

and observe that, by Hölder's inequality,  $(EY)^p \leq EY^p$ .

For the sufficiency proof for  $b < 1$ , we let  $v(x) = (x + ax^b)^{p+1-b}$  and note that conditional on  $S_0 = x$ ,

$$\begin{aligned} x^{-p}[v(S_1) - v(x)] &= x^{-p}[(x + Z_1)^{p+1-b} - (x + ax^b)^{p+1-b}] \\ &= x^{-p}x^{p+1-b}[(1 + Z_1/x)^{p+1-b} - (1 + ax^{b-1})^{p+1-b}] \\ &= x^{1-b}[(1 + (p+1-b)(Z_1/x) + o(1/x)) \\ &\quad - (1 + a(p+1-b)x^{b-1} + o(x^{b-1}))] \\ &= -a(p+1-b) + o(1) \end{aligned}$$

as  $x \rightarrow \infty$ . The inequality  $(1+x)^a \leq 1 + (a \vee 1)x^a$ , valid for  $x \geq 0$ ,  $a \geq 0$ , yields the bound

$$\begin{aligned} x^{-p}|v(S_1) - v(x)| &\leq x^{1-b}|(1 + Z_1/x)^{p+1-b} - 1| \\ &\leq x^{1-b}(Z_1/x)^{p+1-b}((p+1-b) \vee 1), \end{aligned}$$

so the dominated convergence theorem applies, and hence

$$E_x v(S_1) - v(x) = -a(p+1-b)x^p + o(x^p)$$

as  $x \rightarrow \infty$ . Hence, there exists  $c < \infty$  such that

$$E_x v(S_1) - v(x) \leq -\frac{a}{2}(p+1-b)x^p$$

for  $x \geq c$ . Since

$$E_\pi S_0^p = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=0}^N E_x S_j^p,$$

Theorem 14.2.2 of [15] implies that  $E_\pi S_0^p < \infty$ .

For the necessity, note that if  $Z_i = 0$  a.s., the above analysis continues to hold for  $v(S_n(x))$  replaced by  $v(s_n(x))$ , so that

$$v(s_1(x)) - v(x) \geq -a(p+1-b)x^p + o(x^p)$$

as  $x \rightarrow \infty$ . Hence, there exists  $c_1 < \infty$  such that  $\pi[0, c_1] > 0$  and

$$v(s_1(x)) - v(x) \geq -2a(p+1-b)x^p$$

for  $x \geq c_1$ . It follows that so long as  $s_{n-1}(x) \geq c_1$ , we have

$$v(s_n(x)) - v(s_{n-1}(x)) \geq -2a(p+1-b)s_{n-1}(x)^p.$$

Since the above inequality holds for  $n \leq t_{c_1}(x)$ , summing the above inequalities over  $n \leq t_{c_1}(x)$  gives

$$v(s_{t_{c_1}(x)}(x)) - v(x) \geq -2a(p+1-b) \sum_{j=0}^{t_{c_1}(x)} s_j(x)^p.$$

So, for non-negative  $Z_i$ 's, the domination of  $(s_n(x) : n \geq 0)$  by  $(S_n(x) : n \geq 0)$  leads to

$$E \sum_{j=0}^{\tau(c_1)-1} S_j(x)^p \geq \sum_{j=0}^{t_{c_1}(x)-1} s_j(x)^p \geq \frac{v(x) - v(c_1)}{2a(p+1-b)}.$$

Since the set  $[0, c_1]$  is recurrent, by Theorem 10.4.7 of [15], the measure

$$\pi_{c_1}^\circ(\cdot) = \int_{[0, c_1]} \pi(dx) E_x \sum_{j=1}^{\infty} \mathbf{1}\{\cdot; \tau(c_1) \geq j\}$$

is invariant, supported on  $[0, c_1]$ , and satisfies  $\pi_{c_1}^\circ(A) = \pi(A)$  for all Borel  $A \subseteq \{x : \tau_x([0, c_1]) < \infty\}$ . So the recurrence of the chain and the uniqueness of  $\pi$  imply that  $\pi = \pi_{c_1}^\circ$ , whence

$$\begin{aligned} E_\pi S_0^p &= E_{\pi_{c_1}^\circ} S_0^p = \int_{[0, c_1]} \pi(dx) \sum_{j=1}^{\infty} E_x[S_j^p; \tau(c_1) \geq j] \\ &= \int_{[0, c_1]} \pi(dx) E_x \sum_{j=1}^{\tau(c_1)} S_j^p = \int_{[0, c_1]} \pi(dx) E_x \sum_{j=0}^{\tau(c_1)-1} S_j^p, \end{aligned}$$

where the last equality follows from the invariance of  $\pi$ . Hence, if  $\pi_{c_1}$  is the restriction of  $\pi$  to  $[0, c_1]$ , which is well defined since  $\tau(c_1) < \infty$ , we have

$$E_\pi S_0^p = E_{\pi_{c_1}} \sum_{j=0}^{\tau(c_1)-1} S_j^p \cdot \pi[0, c_1],$$

so that

$$\begin{aligned} E_\pi S_0^p &\geq \int_{[0, c_1]} \pi(dx) E_x E_{S_1} \sum_{j=0}^{\tau(c_1)-1} S_j^p \\ &\geq \pi[0, c_1] \cdot E_0 \left( \frac{v(S_1) - v(c_1)}{2a(p+1-b)} \right) I(S_1 \geq c_1) \\ &= \pi[0, c_1] \cdot E(Z_1^{p+1-b} - v(c_1)) I(Z_1 \geq \phi(c_1)), \end{aligned}$$

proving the necessity of the moment condition.

The proof for  $b > 1$  is similar. Here, we set  $v(x) = (x + ax^b)^{p/b}$ . Note that conditional on  $S_0 = x$ ,

$$\begin{aligned} x^{-p}[v(S_1) - v(x)] &= x^{-p}[(x + Z_1)^{p/b} - (x + ax^b)^{p/b}] \\ &\leq x^{-p}[2^{p/b}(x^{p/b} + Z_1^{p/b}) - a^{p/b}x^p(1 + x^{1-b}/a)^{p/b}] \\ &\leq 2^{p/b}x^{-p(b-1)/b} + 2^{p/b}x^{-p}Z_1^{p/b} - a^{p/b} + o(1), \end{aligned}$$

so that in the presence of the moment condition  $EZ_1^{p/b} < \infty$ , we may use the dominated convergence theorem to conclude that there exists  $c < \infty$  such that  $x^{-p}(E_x v(S_1) - v(x)) \leq -a^{p/b}/2$  for  $x \geq c$ . The sufficiency then follows as for  $b < 1$ . The proof of necessity uses the current choice of  $v(\cdot)$  analogously to the argument employed for  $b < 1$ , and is therefore omitted. ■

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