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A Complementarity Framework for Forward Contracting Under Uncertainty

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We consider a particular instance of a stochastic multi-leader multi-follower equilibrium problem in which players compete in the forward and spot markets in successive periods. Proving the existence of such equilibria has proved difficult, as has the construction of globally convergent algorithms for obtaining such points.

By conjecturing a relationship between forward and spot decisions, we consider a variant of the original game and relate the equilibria of this game to a related simultaneous stochastic Nash game where forward and spot decisions are made simultaneously. We characterize the complementarity problem corresponding to the simultaneous Nash game and prove that it is indeed solvable. Moreover, we show that an equilibrium to this Nash game is a local Nash equilibrium of the conjectured variant of the multi-leader multi-follower game of interest. Numerical tests reveal that the difference between equilibrium profits between the original and constrained games are small.

Under uncertainty, the equilibrium point of interest is obtainable as the solution to a stochastic mixed-complementarity problem. Based on matrix-splitting methods, a globally convergent decomposition method is suggested for such a class of problems. Computational tests show that the effort grows linearly with the number of scenarios. Further tests show that the method can address larger networks as well. Finally, some policy-based insights are drawn from utilizing the framework to model a two-settlement six-node electricity market.

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1. Introduction

The first attempts at deregulation in the electricity sector were observed in Chile, England, and New Zealand in the early 1990s (Green and Newbery 1992, Chao and Huntington 1996). Similar efforts in the United States followed suit, particularly in states/control areas of California, New England, Pennsylvania-Jersey-Maryland (PJM) interchange, and New York. One of the difficulties in this process has always been deciding between a variety of possible market designs, primarily because simulating the impact of such designs is difficult (Schweppe et al. 1988, Wilson 2002). Arguably, equilibrium models provide a useful means of dealing with the oligopolistic structure prevalent in such markets (Hobbs 1986, Cardell et al. 1997). In addition, the multiple settlement structure and uncertainty have made these models large and complex (Kamat and Oren 2004, Yao et al. 2008).

The present work focuses on networked oligopolistic power markets characterized by a sequence of two settlements. Of these, the first is a financial settlement in which firms, with generation assets, enter into financial contracts; it is referred to as the forward market. Subject to these contracts, firms compete in the real-time or spot market, which is a physical market in which equilibrium generation, sales, and transmission decisions aid in defining prices and flows. Note that transmission decisions are made by the grid operator, referred to as the independent system operator (ISO).[1] This paper considers the Nash game played in the forward market where each firm is strategic with respect to the spot-market, focusing on the resulting subgame-perfect Nash equilibrium. Our work, while couched in the context of networked electricity markets, is aimed at general two-period equilibrium problems under uncertainty with the goal of addressing of two questions: First, can one provide existence results for equilibria when at least some of the agents possess some *market power*? Second, given the existence of such equilibria, are there efficient methods for determining such points? This paper presents existence and uniqueness results for an approximation to such equilibria and presents an efficient convergent algorithm for finding such equilibria if they do indeed exist.

Research in networked equilibrium problems has its roots in the efforts by Takayama and Judge on spatial price equilibria (SPE) (Takayama and Judge 1964). In particular, under the assumptions of competition and a linear demand function, the authors show that the problem can be posed as a convex quadratic program (QP). The equality between price and marginal cost is an important assumption of an SPE and was shown not to hold in the presence of spatial oligopolies as discussed by Harker (1984).

The work in two-settlement markets is founded on the seminal research by Allaz (1992). One of the findings of this line of research is that under specified conjectural variations, the presence of forward markets increases economic efficiency. In Allaz (1992), the modeling structure is that firms trade in forward contracts in the first period and subsequently trade on the spot-market in the second period, in the presence of uncertain demand. It emerges that if one of the firms does not have access to the forward market, then the other agents can use their forward positions to improve their respective profits. However, if all firms have access to forward contracts, then the profits for all firms tend to fall. Allaz and Vila (1993) discuss the infinite-horizon case and show that in the absence of uncertainty, market prices tend toward marginal costs.

Our modeling framework is more general than that adopted by Allaz (1992) in that we allow for quadratic costs of generation and inequality constrained problems in the spot-market. As a consequence, the resulting spot-market equilibrium problems are complementarity problems instead of linear constraints, and consequently, the agent problems fall under the category of hard nonconvex problems called mathematical programs with complementarity constraints (MPCCS) (Luo et al. 1996). Such constraints are generally of the form $0 \leqslant G(x) \perp H(x) \geqslant 0$, where $\perp$ implies that $[G(x)]_i[H(x)]_i = 0$ for all i. But defining a Nash equilibrium in forward decisions is a far more difficult proposition because it requires solving for an equilibrium in which each agent solves an MPCC. The resulting equilibrium is called a Nash-Stackelberg equilibrium because it arises from a Nash game between a collection of Stackelberg agents (Sherali et al. 1983), and the corresponding problem is termed as an equilibrium problem with complementarity constraints (EPCC). In general, an equilibrium to the resulting multi-leader multi-follower game is neither guaranteed to exist nor known to be unique; yet, there have been efforts to show that equilibria to such games do indeed exist. The seminal paper by Allaz (1992), was preceded by work by Sherali (1984) where the existence and uniqueness statements were provided in a class of multi-leader multi-follower games in which a Cournot game was played at both levels. Su (2007) extended the existence result to accommodate weaker assumptions on the costs and discussed algorithmic schemes. Under a compactness assumption on strategy sets, Hu and Ralph (2007) showed the existence of an equilibrium to a multi-leader multi-follower game arising in power markets, while de Miguel and Xu (2009) proved existence and uniqueness in a similar regime under uncertainty. Algorithms for obtaining such problems are notoriously dependent on an initial solution (Cardell et al. 1997, Hobbs et al. 2000, Yao et al. 2008) and currently possess no global convergence theory.

Extant theoretical and algorithmic research in the study of EPCCs is characterized by the absence of algorithms that are both globally convergent and scalable with respect to the size of the underlying probability measure. The present work is motivated by both of these challenges. An integral part of this paper's focus lies in addressing *uncertainty* in the articulation of the spot-market problem. This might arise from randomness in the spot prices, or in uncertain availability of the generation units, or even from variability in fuel prices. Consequently, the resulting agent problems become stochastic MPCCs. The efficient solution of such problems has gained some attention (see Shanbhag 2006). Most heuristic approaches for solving deterministic versions of such problems have relied on a Jacobi or a Gauss-Seidel technique. Such approaches require iterating across agent problems until there is negligible change in the equilibrium decision (Cardell et al. 1997, Hobbs et al. 2000). In contrast, a centralized approach involves determining a feasible solution to the collection of first-order optimality conditions (complementarity problems) of each of the agents (Su 2005, Leyffer and Munson 2005, Hu and Ralph 2007, Su 2007). None of the aforementioned methods currently possesses any global convergence theory. Yao et al. (2008) also study an active-set approach for solving equilibrium problems with complementarity constraints and apply it to a multi-settlement model in electricity markets. Our contributions overlap with those of Su (2007) from the standpoint of existence where the author provides a stronger existence result for a class of spot-forward games. Additionally, the work by Su (2005) constructs a sequential method for solving this class of problems via the solution of a sequence of complementarity problems.

This paper makes the following contributions:

1. First, we consider a constrained form of the original multi-leader multi-follower game in a two-node setting and relate it to a simultaneous stochastic Nash equilibrium problem, in which the first- and second-period decisions are taken *simultaneously*. An analysis of the mixed-complementarity problem corresponding to the Nash game reveals its solvability. Furthermore, a characterization of the mapping associated with this problem allows one to claim the key result that any equilibrium to the simultaneous Nash game is also an equilibrium to the conjectured variant of the multi-leader multi-follower game. A networked generalization of this setting is examined in the context of a power market where similar characterizations and relationships are observed.

2. A key benefit of the complementarity-based approach lies in greater accessibility to the equilibrium point, given by a solution a stochastic mixed-complementarity problem. Unfortunately, because the size of the problem can

grow with the size of Ω, the sample-space, standard techniques for solving such problems (such as pivoting methods, projection-based methods, and Newton-based schemes) cannot be employed. Instead, we develop a globally convergent matrix-splitting method for solving such a class of equilibrium problems. Our scheme is seen to possess the key property that the computational effort scales linearly with the size of underlying sample space.

3. Insights from a six-node electricity network are also provided and correspond well with those from the literature.

The remainder of this paper has five sections. Section 2 introduces the forward market model proposed by Allaz (1992). In particular, we provide existence theory for the equilibria associated with the simultaneous stochastic Nash problems and develop both a characterization of such equilibria and how the equilibria relate to conjectured Nash-Stackelberg equilibria (NSE). This class of equilibria refers to Nash equilibria between a set of agents, some of whom could be Stackelberg leaders with respect to some set of followers. Section 3 extends the formulation to the electricity market venue, where extensions of the aforementioned results are obtained. In §4, a splitting-based decomposition method is presented for solving such a class of stochastic equilibrium problems along with some computational results. Since the discussion of forward markets has been carried out in the context of electricity markets, some insight is provided from a six-node electricity market in §5. We conclude in §6.

2. Modeling Spot-Forward Markets

Consider a simple two-node market in which there are n generating firms operating at node 1. It is assumed that each firm sells as much as it generates. Each firm may sell its power at node 2 across a transmission line connecting the two nodes. The price of power at node 2 is given by

$$p^\omega := a^\omega - \sum_i g_i^\omega,$$

where g_i^ω is the generation level of firm i for a realization ω, and a^ω is the random intercept of the price function. Specifically, ω lies in the sample-space Ω. The generation cost for firm i is assumed to be linear and is given by $c_i g_i^\omega$. If firm i's forward position is denoted by f_i, then its profit π_i (given by the sum of the forward-market profit and the expected spot-market profit) is given by

$$\pi_i := p^f f_i + \mathbb{E}(p_s^\omega(g_i^\omega - f_i) - c_i g_i^\omega), \tag{1}$$

where p^f is the price in the forward-market. If we assume perfect foresight in the specification of forward prices, then we have $p^f = \mathbb{E}p^\omega$, and the resulting profit function π_i can be written as $\mathbb{E}((p^\omega - c_i)g_i^\omega)$. This is a commonly employed assumption in such settings (cf. Pieper 2001, Yao et al. 2008, Su 2007), and one basis for such an assumption lies in the belief that there are enough risk-neutral arbitrageurs that will trade to remove any possible profit opportunities that exist between the forward and expected spot prices. We recap the variables and parameters of the model in Table 1 and note that when addressing the two-node problem, the nodal subscript is suppressed when specifying the price function.

Table 1. List of variables and parameters.

	Definition
i	Firm index
ω	Scenario index
f_i	Forward position of firm i (single-node case)
g_i^ω	Generation level of firm i under scenario ω
p_j^f	Price in forward-market at node j
a_j^f	Linear intercept of forward price function at node j
p_j^ω	Price in spot-market at node j under scenario ω
a_j^ω	Linear intercept of spot price function at node j under scenario ω
c_i^ω	Linear cost of generation of firm i under scenario ω
d_i^ω	Quadratic cost of generation of firm i under scenario ω
s_{ij}^ω	Sales level from firm i to node j under scenario ω
f_{ij}	Forward position of firm i at node j
G_i^ω	Capacity of generator i under scenario ω
C_{ij}^ω	Line capacity of link (i, j) under scenario ω
w_{ij}^ω	Transmission price across link (i, j) under scenario ω
y_{ij}^ω	Transmission flow across link (i, j) under scenario ω
K	Number of scenarios
p^{ω_j}	Probability of scenario j
I, e	Identity matrix and column of ones

2.1. The Spot-Market Equilibrium

In the spot-market, under realization ω, agent i maximizes his profit given forward positions f_i and the generation decisions of all other agents in scenario ω, often compactly denoted by $g^{-i,\omega}$, as shown by the following parameterized optimization problem:

$$\text{AgSpot}_i^\omega(g^{-i,\omega}) \quad \underset{g_i^\omega \geqslant 0}{\text{maximize}}\,(p_s^\omega(g_i^\omega - f_i) - c_i g_i^\omega).$$

We define a scenario-specific spot-market game and its associated equilibrium as follows.

Definition 1 (Scenario-Specific Spot-Market Game). Given the forward positions of firms $1, \ldots, n$ denoted by $(f_1, \ldots, f_n)$, consider a game $\mathcal{G}_{Spot}^\omega$ in the spot-market associated with scenario ω where the ith firm solves the parameterized optimization problem $(\text{AgSpot}_i^\omega(g^{-i,\omega}))$. Then the associated scenario-specific spot-market equilibrium is given by $\{g_i^*\}_{i=1}^n$, where $g_i^* = (g_i^\omega)^*$ for all $\omega \in \Omega$, and $(g_i^\omega)^*$ solves $(\text{AgSpot}_i^\omega(g_{-i}^{*,\omega}))$.

Because this is a convex problem in g_i^ω, the equilibrium point is given by a linear complementarity problem LCP_1^ω, for each $\omega \in \Omega$:

$$\text{LCP}_1^\omega \quad 0 \leqslant g_i^\omega \perp 2g_i^\omega + \sum_{j \neq i} g_j^\omega - f_i + (c_i^\omega - a^\omega) \geqslant 0, \quad \forall\, i.$$

The existence and uniqueness of this scenario-specific spot-market equilibrium can be supported by Proposition 2.

PROPOSITION 2. *Consider the scenario-specific spot-market game given by $\mathcal{G}^{\omega}_{Spot}$. Then, given a set of forward decisions f, G^{ω}_{Spot} admits a unique equilibrium.*

PROOF. This follows from observing that the LCP^{ω}_1 can be written as

$$0 \leqslant g^{\omega} \perp M^{\omega} g^{\omega} + q^{\omega} \geqslant 0,$$

where $M^{\omega} = I + ee^T$, I denotes the identity matrix, e denotes the column of ones, $g^{\omega} = (g_1^{\omega}, \ldots, g_n^{\omega})^T$, and $q^{\omega} = (c_1^{\omega} - a^{\omega} - f_1, \ldots, c_n^{\omega} - a^{\omega} - f_N)^T$. Because M is a positive definite matrix, it follows from Cottle et al. (1992, Theorem 3.1.6) that a unique solution to the LCP^{ω}_1 always exists. □

2.2. Nash-Stackelberg Equilibria (NSE)

A Nash-Stackelberg equilibrium in this setting refers to an equilibrium in forward contracts subject to equilibrium in the spot-market (as specified by LCP^{ω}_1 for all $\omega \in \Omega$). The ith firm then solves (AgFor_i) in g and f_i while taking all the other forward positions f^{-i} as parameters:

$$\begin{aligned} \text{AgFor}_i(f^{-i}) \quad & \underset{g\geqslant 0,\, f_i}{\text{maximize}} \quad \mathbb{E}((p_s^{\omega} - c_i) g_i^{\omega}), \\ & \text{subject to} \quad 0 \leqslant g_i^{\omega} \perp 2g_i^{\omega} + \sum_{j\neq i} g_j^{\omega} - f_i \\ & \qquad\qquad + (c_i^{\omega} - a_s^{\omega}) \geqslant 0, \quad \forall\, i, \omega. \end{aligned}$$

The problem (AgFor_i) is a mathematical program with complementarity (or equilibrium) constraints or MPCC (or MPEC) (Luo et al. 1996). Apart from the complementarity constraint being nonconvex, it also lacks an interior, implying that the Mangasarian-Fromovitz constraint qualification (Bertsekas 1999) does not hold at any feasible point. The resulting Nash-Stackelberg equilibrium is defined as follows.

DEFINITION 3 (FORWARD-MARKET GAME). Consider a game in the forward-market, denoted by $\mathcal{G}_{For}$, where given f^{-i}, the ith firm solves the parameterized optimization problem ($\text{AgFor}_i(f^{-i})$) for all $i = 1, \ldots, n$; and suppose its associated equilibrium problem be denoted by (EPCC_1). Then the associated Nash-Stackelberg equilibrium of $\mathcal{G}_{For}$ in forward decisions is a tuple $\{f_i^*\}_{i=1}^n$, such that for all $i = 1, \ldots, n$, (g, f_i^*) is a solution to the ith agent's Stackelberg problem ($\text{AgFor}_i(f^{-i,*})$).

Such a game is a multi-leader multi-follower game and recently has been termed an equilibrium problem with equilibrium (complementarity) constraints or an EPEC (or EPCC). The nonconvexity and ill-posedness of each generator's problem in the previous approach is problematic and implies that existence and uniqueness questions are less easily answered. Furthermore, there are currently no globally convergent algorithms available for obtaining such equilibrium points.

Commonly used approaches include a Gauss-Seidel approach that solves each generator's problem and passes the solution to the next generator in hope that the iterates converge to an equilibrium point (Cardell et al. 1997, Hobbs et al. 2000, Pieper 2001). Scholtes (2001) shows that such approaches could result in cycling. Apart from the lack of convergence theory supporting such an approach, the stochasticity in the problems requires the employment of stochastic nonlinear programming methods for solving the agent problems (see Shanbhag 2006 for recent work on the solution of stochastic MPCCs). More recently, algorithmic work on such problems has centered on complementarity-based approaches (Leyffer and Munson 2005; Su 2005, 2007; Hu and Ralph 2007) and active-set approaches Yao et al. (2008). In particular, the work by Su (2005) employs a novel scheme that solves a sequence of complementarity problems. Each complementarity problem is parameterized by a regularization parameter that corresponds to a relaxation of the complementarity constraints of each agent's problem (an MPCC). By driving this regularization parameter to zero, this algorithm constructs a sequence that is shown to converge to an equilibrium point, under the assumption that the starting point is in a neighborhood of the solution. This is one of the first algorithms that provides local convergence properties. The same algorithm has been shown to be successful in the solution of EPCCs arising from two-period spot-forward markets (Su 2007). The work by Hu and Ralph (2007) uses a similar framework in which the complementarity formulation of the EPCC is solved via PATH (Dirkse and Ferris 1993).

It should be emphasized that apart from the absence of rigorous global convergence theory for obtaining solutions to equilibrium problems with equilibrium constraints (EPECs), the schemes are characterized by an inability to cope with the size of the underlying problems arising from the scenario-based model. Therefore, in considering the development of an appropriate solution methodology, we concentrate on *convergent scalable schemes*, and this represents the focus of §4.

In general, the model suggested above does not prescribe a functional specification for forward prices. Our intent is to construct a modified Nash-Stackelberg equilibrium problem in which each agent's optimization problem is further constrained by the risk-neutrality constraint in a setting where forward prices are determined by an affine function

$$p^f = a^f - \sum_{i=1}^{n} f_i,$$

and the resulting risk-neutrality constraint is given by

$$\left(a^f - \sum_{i=1}^{n} f_i\right) = \mathbb{E}_{\omega}\left(a^{\omega} - \sum_{i=1}^{n} g_i^{\omega}\right). \tag{2}$$

Consequently, in the forward-market, the ith agent solves the following modification of $\text{Agfor}^i(f^{-i})$:

$$\text{AgFor}_i^{\text{con}}(f^{-i})$$

$$\begin{aligned} \underset{g\geqslant 0,\, f_i}{\text{maximize}} \quad & \mathbb{E}((p_s^\omega - c_i)g_i^\omega) \\ & 0 \leqslant g_i^\omega \perp 2g_i^\omega + \sum_{j\neq i} g_j^\omega - f_i + (c_i^\omega - a_s^\omega) \geqslant 0, \\ & \qquad\qquad\qquad\qquad\qquad\qquad \forall\, i, \omega, \\ \text{subject to} \quad & \left(a^f - \sum_{i=1}^n f_i\right) = \mathbb{E}_\omega\left(a^\omega - \sum_{i=1}^n g_i^\omega\right). \end{aligned}$$

The resulting *conjectured* Nash-Stackelberg game is defined as follows.

Definition 4 (Conjectured Forward-Market Game). Consider a game in the forward-market, denoted by $\mathscr{G}_{For}^{con}$, where given f^{-i}, the ith firm solves the parameterized optimization problem $(\text{AgFor}_i^{con}(f^{-i}))$ for all $i = 1, \ldots, n$; and suppose its associated equilibrium problem is denoted by (EPCC_2). Then the associated Nash-Stackelberg equilibrium of $\mathscr{G}_{For}^{con}$ in forward decisions is a tuple $\{f_i^*\}_{i=1}^n$, such that for all $i = 1, \ldots, n$, (g, f_i^*) is a solution to the ith agent's Stackelberg problem $(\text{AgFor}_i^{con}(f^{-i,*}))$.

This modification can be motivated in several ways, two on which we elaborate.

1. Affine forward price functions: In certain settings, market designers might have more information regarding the nature of forward prices, particularly that these prices are given by an affine relationship with forward positions. Adding the constraint (2) ensures that such a relationship is respected. In such cases, our constrained model would indeed be appropriate.

2. Conjectural approximation: In the absence of such information, making an affine assumption on forward prices allows for providing existence theory and scalable algorithms (as the remainder of the paper shows). The results would then pertain to a *constrained* or *conjectural* version of the original Nash-Stackelberg equilibrium problem and represent an approximation.

The remainder of this section provides some evidence that the solutions to the Nash-Stackelberg problem and its conjectural variant are very close in some settings. Table 2 represents a comparison of equilibria obtained via the two models for the model described in this section. We assume that a^ω is given by $\mathcal{N}(100, 2)$ where $\mathcal{N}(0, 1)$ is a normally distributed random variable with mean zero and variance one. The random cost is given by a random variable of $\mathcal{N}(10, 5)$. The intercept of the forward price function is varied to evaluate the impact on the forward positions and the equilibrium profits. Table 2 shows the difference in equilibria for $a^f = 1$, 10, and 20. For each value of a^f, the

Table 2. Comparison of Nash-Stackelberg equilibria.

$\lvert\Omega\rvert$	a^f	$\lVert f^{con} - f\rVert_\infty/1 + \lVert f\rVert_\infty$	$(\pi_1^{con} - \pi_1)/1 + \lVert\pi_1\rVert_\infty$	$(\pi_3^{con} - \pi_3)/1 + \lVert\pi_3\rVert_\infty$	$(\pi_6^{con} - \pi_6)/1 + \lVert\pi_6\rVert_\infty$	$\lVert\pi^{con} - \pi\rVert_\infty/1 + \lVert\pi\rVert_\infty$
1	1	1.0e−02	−2.8e−05	1.5e−05	−8.7e−08	2.8e−05
2	1	1.1e−01	6.2e−04	−1.0e−03	−1.3e−03	3.2e−03
3	1	2.7e−01	8.0e−04	−5.6e−03	−6.6e−03	1.3e−02
4	1	3.2e−01	−1.3e−02	−9.5e−03	−4.9e−03	1.3e−02
5	1	5.4e−03	2.7e−05	2.1e−04	1.4e−04	2.6e−04
6	1	7.8e−04	−1.6e−05	−2.8e−07	1.5e−06	2.5e−05
7	1	2.0e−01	−5.1e−03	−8.0e−03	−4.5e−03	8.3e−03
8	1	4.4e−01	−1.3e−03	−2.3e−02	−9.9e−03	3.7e−02
9	1	3.1e−01	5.9e−03	−5.1e−03	−1.7e−02	1.7e−02
10	1	5.7e−01	−2.4e−03	−4.3e−02	−1.1e−02	4.7e−02
1	10	9.4e−03	−2.1e−05	1.8e−05	−7.9e−08	2.2e−05
2	10	3.7e−01	1.1e−02	2.9e−04	−7.6e−03	1.1e−02
3	10	4.8e−01	−7.0e−03	−3.4e−02	−1.4e−02	3.4e−02
4	10	6.1e−01	−1.9e−02	−1.6e−02	−1.3e−02	3.8e−02
5	10	6.2e−03	4.0e−05	3.3e−04	1.6e−04	3.3e−04
6	10	2.2e−01	3.5e−03	−2.8e−03	−7.1e−03	1.0e−02
7	10	6.0e−01	9.5e−03	−1.6e−02	−3.1e−02	3.7e−02
8	10	5.6e−01	−4.8e−03	−2.5e−02	−1.4e−02	4.0e−02
9	10	4.6e−01	1.2e−02	−4.3e−03	−3.0e−02	3.0e−02
10	10	6.2e−01	−2.7e−03	−4.6e−02	−1.2e−02	4.9e−02
1	20	5.5e−02	1.2e−04	−1.1e−04	2.2e−07	1.6e−04
2	20	7.1e−01	−7.3e−03	−4.5e−02	−1.4e−02	4.5e−02
3	20	6.1e−01	−8.3e−04	5.0e−03	−1.3e−02	1.3e−02
4	20	9.9e−01	−2.7e−03	−4.7e−02	−5.5e−02	7.3e−02
5	20	1.3e−01	−2.0e−03	−1.8e−03	−3.2e−03	3.2e−03
6	20	5.0e−01	8.1e−03	−8.6e−03	−1.9e−02	2.4e−02
7	20	7.4e−01	6.6e−03	−3.7e−02	−4.6e−02	5.7e−02
8	20	8.3e−01	3.5e−03	−5.0e−02	−3.8e−02	7.6e−02
9	20	7.8e−01	−2.0e−02	−6.4e−02	−4.0e−02	6.4e−02
10	20	7.6e−01	4.0e−03	−4.7e−02	−2.4e−02	5.4e−02

number of scenarios is raised from 1 to 10, to examine the impact of uncertainty. We find that there is no clear trend that relates relative difference to growth in uncertainty. Across the three choices of a^f, the relative difference in forward positions does not change significantly. Importantly, the deviation in equilibrium profits (in the ∞-norm) stays reasonably modest reaching a maximum level of 7%. The deviation in forward positions, expectedly, is a little higher but does not exceed 10%. What can also be observed is that while agent-specific profits differ slightly, the changes are often in the same direction. If they are in opposite directions, then the norms of such a change are generally small (as seen with $\|\Omega\| = 2$, $a^f = 1$, for instance). It should be emphasized that these insights pertain to this particular model and more definitive statements can be made only by providing theoretical underpinnings.

2.3. Simultaneous Stochastic Nash Equilibria (SSNEs)

The previous subsection presented a modified Nash-Stackelberg equilibrium problem, which we will refer to as an NSE. In this section, we construct a stochastic Nash game, in which the first- and second-period decisions are taken *simultaneously* with the intent of relating equilibria arising from this class of games to those arising from the original multi-leader multi-follower game. The resulting equilibrium, referred to as a simultaneous stochastic Nash equilibrium or an SSNE, is defined next.

Definition 5 (Simultaneous Stochastic Nash Equilibrium (SSNE)). Consider the simultaneous stochastic Nash game, denoted by $\mathscr{G}_{SSNE}$, in which the ith firm solves the parameterized optimization problem $\text{AgSSN}_i(f, g^{-i})$:

$$\text{AgSSN}_i(f, g^{-i}) \quad \underset{g_i^\omega \geqslant 0,\, \forall \omega \in \Omega}{\text{maximize}} \; \mathbb{E}(p_s^\omega (g_i^\omega - f_i) - c_i^\omega g_i^\omega),$$

while forward decisions f are specified as per the risk-neutrality constraint (2). Then the associated simultaneous stochastic Nash equilibrium (SSNE) is given by a tuple $(g_i^*, f_i^*)_{i=1}^n$, where firm i solves $(\text{AgSSN}_i(f, g^{-i}))$ while the forward decisions are constrained by (2).

The equilibrium conditions corresponding to $\mathscr{G}_{SSNE}$ are given by

$$0 \leqslant g^\omega \perp M^\omega g^\omega - f + (c^\omega - a^\omega e) \geqslant 0, \quad \forall \omega \in \Omega,$$
$$\left(a^f - \sum_{i=1}^n f_i\right) = \mathbb{E}_\omega \left(a^\omega - \sum_{i=1}^n g_i^\omega\right).$$

By creating $n-1$ identical copies of the risk-neutrality constraints, the resulting equilibrium conditions can be compactly restated as a square mixed linear-complementarity problem (SSNE-CP):

$$\text{SSNE-CP} \quad 0 \leqslant g \perp Mg + Nf + (b-a) \geqslant 0, \quad Wf = \hat{W} g + d,$$

where $W = ee^T, \quad \hat{W} = \begin{pmatrix} p_{\omega_1} W & \dots & p_{\omega_K} W \end{pmatrix},$

$$d = \begin{pmatrix} (a^f - \mathbb{E}a^\omega) \\ \vdots \\ (a^f - \mathbb{E}a^\omega) \end{pmatrix}, \quad (b-a) = \begin{pmatrix} \bar{b}^1 - a^{\omega_1} e \\ \vdots \\ \bar{b}^K - a^{\omega_K} e \end{pmatrix},$$

$$\bar{b}^k = \begin{pmatrix} c_1^{\omega_1} \\ \vdots \\ c_n^{\omega_1} \end{pmatrix},$$

$$M = \begin{pmatrix} M^{\omega_1} & & \\ & \ddots & \\ & & M^{\omega_K} \end{pmatrix} = \begin{pmatrix} I + ee^T & & \\ & \ddots & \\ & & I + ee^T \end{pmatrix},$$

and

$$N = -\begin{pmatrix} I \\ \vdots \\ I \end{pmatrix}. \tag{3}$$

The equilibrium conditions of this problem are identical to the joint feasibility set of each agent's problem in the Nash-Stackelberg equilibrium problem when constrained by an affine forward price conjecture, as the next result specifies.

Lemma 6. *Consider a solution (g, f) to the mixed-complementarity problem (SSNE-CP). Then for $i = 1, \dots, n$, the tuple (g, f_i) is a feasible solution to $(AgFor_i^{\text{con}}(f^{-i}))$.*

Proof. Follows immediately. □

Clearly, all NSEs arising from $\mathscr{G}_{For}^{con}$ are SSNEs. Expectedly, the reverse characterization does not always hold. In attempting to prove precisely such a relationship, the question of the existence of an SSNE will be approached by showing that the equilibrium conditions (sufficient) admit a solution. This may be attempted through a null-space approach. This relies on the elimination of (2), which can be compactly written as

$$W_f f = \hat{W}_f g + a^f - \mathbb{E}_\omega a^\omega,$$

where $\quad W_f = e^T$ and $\hat{W}_f = -\begin{pmatrix} p^{\omega_1} e^T & \dots & p^{\omega_K} e^T \end{pmatrix}.$

Let Y_W denote an orthonormal basis for the range-space of W_f^T while Z_W denotes an orthonormal basis for the null-space of W_f^T, respectively. It follows that f can be expressed as $Z_W f_Z + Y_W f_Y$. By noting that $(W_f Y_W)$ is a square positive definite matrix, $f_Y = (W_f Y_W)^{-1}(Y_W(\hat{W}_f g + a^f - \mathbb{E}_\omega a^\omega))$. It follows that the complementarity constraint in (SSNE-CP) is given by

$$0 \leqslant g \perp \underbrace{(M + NY_W(W_f Y_W)^{-1}\hat{W}_f)}_{\hat{M}_1} g + \underbrace{(NZ_W f_Z + d + NY_W(W_f Y_W)^{-1}(a^f - \mathbb{E}_\omega a^\omega))}_{q_1} \geqslant 0. \tag{4}$$

Because $W_f = e^T$, Y_W is given by $(1/\sqrt{n})e$, and $W_f Y_W = (1/\sqrt{n})e^T e = \sqrt{n}$. It follows that $NY_W(W_fY_W)^{-1}\hat{W}$ and $\hat{M}_1$ are given by

$$NY_W(W_fY_W)^{-1}\hat{W}_f = -\frac{1}{n}\begin{pmatrix} I \\ \vdots \\ I \end{pmatrix} e\begin{pmatrix} p^{\omega_1}e^T & \dots & p^{\omega_K}e^T \end{pmatrix}$$

$$= -\frac{1}{n}\begin{pmatrix} p^{\omega_1}ee^T & \dots & p^{\omega_K}ee^T \\ \vdots & \ddots & \vdots \\ p^{\omega_1}ee^T & \dots & p^{\omega_K}ee^T \end{pmatrix},$$

$$\hat{M}_1 := \begin{pmatrix} I+ee^T & & \\ & \ddots & \\ & & I+ee^T \end{pmatrix} - \frac{1}{n}\begin{pmatrix} p^{\omega_1}ee^T & \dots & p^{\omega_K}ee^T \\ \vdots & & \vdots \\ p^{\omega_1}ee^T & \dots & p^{\omega_K}ee^T \end{pmatrix}, \quad \text{respectively.} \tag{5}$$

It should be emphasized that the above discussion would have been far simpler if we had used the explicit forms for Y_W and Z_W. However, our goal was to provide an analysis that could conceivably be extended to accommodate general price conjectures. We may now analyze whether an SSNE can be shown to always exist by analyzing the complementarity problem given by (4).

Proposition 7 (Existence of an SSNE). *Consider the simultaneous stochastic Nash game denoted by $\mathcal{G}_{SSNE}$. Then $\mathcal{G}_{SSNE}$ always admits an equilibrium.*

Proof. The sufficient equilibrium conditions of $\mathcal{G}_{SSNE}$ are given by (SSNE-CP), and it suffices to show that this mixed-complementarity problem always admits a solution. But this mixed CP may be reduced to a pure LCP, denoted by (4), and it remains to show that for any f_Z the LCP$(q_1(f_Z), \hat{M}_1)$ is solvable where $\hat{M}_1$ is given by (5). If $e^T x_i$ is denoted by $\bar{x}_i$ for all i, then the positive definiteness of $\hat{M}_1$ can be expressed as follows:

$$x^T\hat{M}_1x = \begin{pmatrix} x_1 \\ \vdots \\ x_K \end{pmatrix}^T \left(\begin{pmatrix} I+ee^T & & \\ & \ddots & \\ & & I+ee^T \end{pmatrix} - \frac{1}{n}\begin{pmatrix} p^{\omega_1}ee^T & \dots & p^{\omega_K}ee^T \\ \vdots & & \vdots \\ p^{\omega_1}ee^T & \dots & p^{\omega_K}ee^T \end{pmatrix}\right)\begin{pmatrix} x_1 \\ \vdots \\ x_K \end{pmatrix}$$

$$= \sum_{i=1}^K(\|x_i\|^2 + \bar{x}_i^2) - \frac{1}{n}\sum_{i=1}^K p^{\omega_i}\bar{x}_i^2 - \frac{1}{n}\sum_{i=1}^K\sum_{j\neq i}(p^{\omega_i}+p^{\omega_j})\bar{x}_i^T\bar{x}_j$$

$$\geqslant \sum_{i=1}^K((\|x_i\|^2 + \bar{x}_i^2) - \frac{1}{n}\sum_{i=1}^K p^{\omega_i}\bar{x}_i^2 - \frac{1}{2n}\sum_{i=1}^K\sum_{j\neq i}(p^{\omega_i}+p^{\omega_j})(\bar{x}_i^2+\bar{x}_j^2),$$

because $\bar{x}_i^2 + \bar{x}_j^2 \geqslant 2\bar{x}_j\bar{x}_i$. Further analysis reveals that the expression on the right can be simplified as follows:

$$\sum_{i=1}^K(\|x_i\|^2+\bar{x}_i^2) - \frac{1}{n}\sum_{i=1}^K p^{\omega_i}\bar{x}_i^2 - \frac{1}{2n}\sum_{i=1}^K\sum_{j\neq i}(p^{\omega_i}+p^{\omega_j})(\bar{x}_i^2+\bar{x}_j^2)$$

$$= \sum_{i=1}^K\left(\|x_i\|^2 + \bar{x}_i^2 - \frac{1}{n}p^{\omega_i}\bar{x}_i^2 - \frac{1}{2n}\left((K-1)p^{\omega_i} + \sum_{j\neq i}p^{\omega_j}\right)\bar{x}_i^2\right).$$

Further simplification shows that the right-hand side can be expressed as

$$= \sum_{i=1}^K\left(\|x_i\|^2 + \bar{x}_i^2 - \frac{1}{n}p^{\omega_i}\bar{x}_i^2 - \frac{1}{2n}((K-2)p^{\omega_i}+1)\bar{x}_i^2\right)$$

$$= \sum_{i=1}^K\left(\|x_i\|^2 + \bar{x}_i^2 - \frac{1}{n}p^{\omega_i}\bar{x}_i^2 - \frac{(1+(K-2)p^{\omega_i})}{2n}\bar{x}_i^2\right)$$

$$= \sum_{i=1}^K\left(\|x_i\|^2 + \bar{x}_i^2 - \frac{(1+(K-4)p^{\omega_i})}{2n}\bar{x}_i^2\right).$$

Therefore, we have that

$$\sum_{i=1}^K\|x_i\|^2 + \sum_{i=1}^K\left(1 - \frac{(1+(K-4)p^{\omega_i})}{2n}\right)\bar{x}_i^2$$

$$\geqslant \sum_{i=1}^K\|x_i\|^2 + \left(\min_{i\in\{1,\dots,N\}}\bar{x}_i^2\right)\underbrace{\sum_{i=1}^K\left(1 - \frac{(1+(K-4)p^{\omega_i})}{2n}\right)}_{\text{Term 1}}.$$

Because $p^{\omega_i} \leqslant 1$ for $i = 1, \dots, K$, term 1 can be shown to be nonnegative as per

$$\sum_{i=1}^K\left(1 - \frac{(1+(K-4)p^{\omega_i})}{2n}\right) \geqslant K\frac{(2n-1)}{2n} - \frac{(K-4)}{2n}$$

$$= \frac{2K(n-1)+4}{2n} \geqslant 0.$$

It follows that

$$\sum_{i=1}^K\|x_i\|^2 + \left(\min_{i\in\{1,\dots,N\}}\bar{x}_i^2\right)\sum_{i=1}^K\left(1 - \frac{(1+(K-4)p^{\omega_i})}{2n}\right)$$

$$\geqslant \sum_{i=1}^K\|x_i\|^2 > 0,$$

implying that $\hat{M}_1$ is positive definite and LCP$(q_1(f_Z), M_1)$ is solvable for all f_Z by Theorem 3.1.6 of Cottle et al. (1992). It follows that (SSNE-CP) admits a solution, and the simultaneous stochastic Nash equilibrium problem is solvable. □

We conclude this subsection with a characterization of the mapping pertaining to complementarity problem (SSNE-CP). This problem can be viewed as a complementarity problem CP($\mathbf{C}, F$) where $\mathbf{C}$, a closed convex cone, and F, the associated mapping, are defined as

$$\mathbf{C} := \mathbb{R}_+^{Kn} \times \mathbb{R}^n \quad \text{and} \quad F(g, f) := \begin{pmatrix} Mg + Nf + (b-a) \\ -\hat{W} + Wf - d \end{pmatrix}. \tag{6}$$

The complementarity problem CP($\mathbf{C}, F$) requires an x satisfying

$$\mathbf{C} \ni x \perp F(x) \in \mathbf{C}^*, \tag{7}$$

where $\mathbf{C}^* = \{U\colon\ u^T x \geqslant 0,\ x \in \mathbf{C}\}$. Note that by showing that characterizing F as a $\mathbf{P}_0$ mapping is of particular relevance in the next section, where we attempt to relate SSNEs to NSEs.

Lemma 8. *Consider the complementarity problem CP*($\mathbf{C}, F$), *corresponding to* (*SSNE-CP*), *where* $\mathbf{C}$ *and* F *are defined in* (6). *Then* F *is a* $\mathbf{P}_0$ *mapping over* $\mathbf{C}$.

Proof. Recall from Proposition 3.5.9 of Facchinei and Pang (2003) that F is a $\mathbf{P}_0$ mapping over a Cartesian set $\mathbf{C}$ if $\nabla F(x)$ is a $\mathbf{P}_0$ matrix for all $x \in \mathbf{C}$. Note that the cartesian nature of $\mathbf{C}$ follows trivially. Showing that ∇F is a $\mathbf{P}_0$ matrix requires proving that every principal minor of ∇F has a nonnegative determinant. By the definition of F, the Jacobian ∇F is given by

$$\nabla F = \begin{pmatrix} M_1 & & & -I \\ & \ddots & & \vdots \\ & & M_K & -I \\ -p^{\omega_1} W & \ldots & -p^{\omega_K} W & W \end{pmatrix},$$

where $M_i \in \mathbb{R}^{n \times n}$ for $i = 1, \ldots, n$ and $W \in \mathbb{R}^{n \times n}$. Consider any principal submatrix denoted by $G_{\alpha\alpha}$ where $\alpha = \bigcup_{i=1}^{K+1} \alpha_i \subseteq \{1, \ldots, (K+1)n\}$, and α_i are nonoverlapping index sets. We prove the required result by proving the nonnegativity of the determinants corresponding to the following mutually exclusive cases:

1. Suppose $\alpha_{K+1} = \varnothing$, implying that $\alpha \subseteq \{1, \ldots, nK\}$ and $G_{\alpha\alpha}$ is a submatrix of a positive definite matrix. Consequently, $G_{\alpha\alpha}$ is positive definite and $\det(G_{\alpha\alpha}) > 0$.
2. Suppose $\bigcup_{i=1}^{K} \alpha_i = \varnothing$ and $\alpha \subseteq \{nK+1, \ldots, n(K+1)\}$. Then $G_{\alpha\alpha}$ is a principal submatrix of W, a positive semidefinite matrix, and $\det(G_{\alpha\alpha}) \geqslant 0$.
3. Suppose α is such that $G_{\alpha\alpha}$ contains multiple rows of the system

$$\begin{pmatrix} -p^{\omega_1} W & \ldots & -p^{\omega_K} W & W \end{pmatrix}, \tag{8}$$

or $|\alpha_{K+1}| > 1$. Then, at least two rows of $G_{\alpha\alpha}$ are identical and $\det(G_{\alpha\alpha}) = 0$. It remains to show that $\det(G_{\alpha\alpha}) \geqslant 0$ when $G_{\alpha\alpha}$ contains a single row from (8) or $|\alpha_{K+1}| = 1$. Suppose $M_{\alpha_j\alpha_j}$ represents the principal submatrix of M_j with index set α_j and $W_{\alpha_{K+1}\alpha_j}$ represents the submatrix of W specified by row and column index sets specified by α_{K+1} and α_j. If $I_{\alpha_j\alpha_{K+1}}$ is defined analogously, then the determinant of $G_{\alpha\alpha}$ is specified by

$$\det(G_{\alpha\alpha}) = \prod_{i=1}^{K} \det(M_{\alpha_i\alpha_i}) \cdot \left(1 - \sum_{j=1}^{K} p^{\omega_j} (W_{\alpha_{K+1},\alpha_j} (M_{\alpha_j\alpha_j})^{-1} (I_{\alpha_j\alpha_{K+1}})\right).$$

Note that $I_{\alpha_j\alpha_{K+1}}$ is a column of the identity matrix and $M_{\alpha_j\alpha_j}^{-1} = (I_j - (1/(1+n_j))e_j e_j^T)$, where I_j and e_j denote the identity matrix and the column of ones in $\mathbb{R}^{n_j \times n_j}$ and $\mathbb{R}^{n_j}$, respectively, where $n_j = |\alpha_j|$. Furthermore, by recalling that $W_{\alpha_{K+1}\alpha_j} = e_j^T$ and $(A)_i$ denotes the ith column of A, then it follows that

$$\begin{aligned} &\sum_{j=1}^{K} p^{\omega_j} W_{\alpha_{K+1},\alpha_j} (M_{\alpha_j\alpha_j})^{-1} (I_{\alpha_j\alpha_{K+1}}) \\ &\quad = \sum_{j=1}^{K} p^{\omega_j} e_j^T \left(I_j - \frac{1}{n_j+1} e_j e_j^T\right)_{\beta_j} \\ &\quad = \sum_{j=1}^{K} p^{\omega_j} \left(1 - \frac{n_j}{n_j+1}\right) < 1. \end{aligned}$$

Finally, $\det(G_{\alpha\alpha}) > 0$ because $\det(M_{\alpha_j\alpha_j}) > 0$ for all $j = 1, \ldots, K$. □

2.4. SSNEs and NSEs

In the previous subsection we showed that simultaneous stochastic Nash equilibria exist. Yet the relationship between SSNEs and NSEs requires a closer examination. In particular, we consider the natural question: when is an SSNE, corresponding to $\mathcal{G}_{SSNE}$, an NSE of $\mathcal{G}_{For}^{con}$? Recall that an NSE requires that every agent solves a Stackelberg problem or a mathematical program with complementarity constraints. The remainder of this section focuses on proving when an *SSNE is indeed an NSE*.

Recall that the SSNE requires the solution of the following mixed-complementarity problem while the NSE is given by an equilibrium problem with complementarity constraints denoted by (EPCC$_2$). In particular, the EPCC of interest represents an equilibrium problem in forward decisions f_i in which agent i solves $\text{AgFor}_i^{\text{con}}(f^{-i})$. It might be recalled from Lemma 6 that a feasible point to all the agent problems in ($\text{AgFor}_i^{\text{con}}(f^{-i})$) is given by a solution to (SSNE-CP).

The main result of this section states that any equilibrium of $\mathcal{G}_{SSNE}$ is an equilibrium to $\mathcal{G}_{For}^{Con}$. This requires an intermediate result that relates to the complementarity problem arising in each agent's problem. This agent-specific complementarity problem is denoted by CP($\mathbf{C}_i, F_i$), where

$$F_i(x_i; x^{-i}) := \begin{pmatrix} M & N_i \\ -e^T & 1 \end{pmatrix} \begin{pmatrix} g \\ f_i \end{pmatrix} + \begin{pmatrix} b - a + N\bar{f}^{-i} \\ -\sum_{j\neq i} f_j - d \end{pmatrix},$$
$$x_i := \begin{pmatrix} g \\ f_i \end{pmatrix}, \quad \mathbf{C}_i := \mathbb{R}^{Kn}_+ \times \mathbb{R}, \quad \bar{f}^{-i}_j = \begin{cases} f_j, & j \neq i \\ 0, & j = i \end{cases}, \tag{9}$$

and N_i is the ith column on N. We begin by providing a characterization of the agent-specific feasible region arising in AgFor$_i^{con}(f^{-i})$.

Lemma 9. *Consider the parameterized agent-specific problem (AgFor$_i^{con}(f^{-i})$) where the feasible region is given by the complementarity problem CP($\mathbf{C}_i, F_i$), defined in* (9) *and* (7). *Then F_i is a $\mathbf{P}-$mapping over $\mathbf{C}_i$.*

Proof. We employ a proof similar to that provided in Lemma 8. We begin by recalling that every principal submatrix of ∇F_i is a principal submatrix of ∇F, defined in the proof of Lemma 8. However, we observe from Lemma 8 that every principal submatrix has a nonnegative determinant. It suffices to show that every principal submatrix has a strictly positive determinant because every index set α is given by $\bigcup_{i=1}^{K+1} \alpha_i$ and α_{K+1} has a cardinality of one. In effect, when considering case (2) from the proof of Lemma 8, the submatrix is 1×1 matrix given by 1 and trivially has a positive determinant. Because all other principal submatrices lead to positive determinants, the required result follows. □

Utilizing this result, we now prove that every equilibrium of $\mathscr{G}_{ssne}$ is an equilibrium of $\mathscr{G}^{con}_{For}$.

Theorem 10 (Existence of an NSE to $\mathscr{G}^{con}_{For}$). *Suppose (g, f) is an equilibrium of $\mathscr{G}_{ssne}$. Then (g, f) is a local Nash equilibrium of $\mathscr{G}^{con}_{For}$ and a solution of (EPCC$_2$). Furthermore, an equilibrium of $\mathscr{G}^{con}_{For}$ always exists.*

Proof. This follows from noting that for $i = 1, \ldots, n$, given f^{-i}, the feasible region of (AgFor$^i_{con}(f^{-i})$) is a singleton. In particular, if this holds, then (g, f_i) is trivially a local minimizer of (AgFor$_i^{con}(f^{-i})$). It remains to show that given f^{-i}, the feasible region of each of the agent problems is indeed a singleton. But, given an f^{-i}, the complementarity problem corresponding to each agent has a $\mathbf{P}$-mapping. It follows from (Facchinei and Pang 2003, Theorem 3.5.10) that at most one solution to this problem exists. However, we know that (g, f_i) is a feasible solution to this problem, given f^{-i} because (g, f) is a solution to (SSNE-CP). It can therefore be concluded that the feasible region of each agent's problem is indeed a singleton. Because (SSNE-CP) is always solvable, the existence of an equilibrium to $\mathscr{G}^{con}_{For}$ is always guaranteed. □

3. Networked Electricity Spot-Forward Markets

Section 2 introduced a two-node spot-forward market model in a simple setting with n producers and infinite capacities. In this section, we place the problem in the context of electricity markets and consider a two-settlement problem, in which participants trade in the forward- and spot-markets in subsequent periods. Generally, only physical linkages between nodes or buses are considered, and this node-linkage specification is denoted by the node-admittance matrix. The admittance characteristics of the linkages are articulated through the branch-admittance matrix. Our analysis is restricted to *high-voltage* transmission systems, allowing us to assume that the voltage angles are small and the voltage magnitudes are constant. Moreover, the losses are considered to be negligible. The resulting power flow equations are often termed *DC load flow equations*. Further details can be found in Schweppe et al. (1988). Throughout our analysis, we use DC load flow analysis to specify flows.

3.1. Spot-Market Equilibrium

Consider an n-node network with a firm at each node. We assume that firm i has a generator at node i but might sell to all other nodes in the network (we assume a fully connected grid but this assumption is without loss of generality). The sales by firm i (housed at node i) to node j are denoted by s_{ij}. We collectively denote the sales decisions by firm i by $s_{i,.} = (s^{\omega}_{i1}, \ldots, s^{\omega}_{in})$. In addition, s^{ω}_{-i} refers to the sales decisions of all agents except i, namely, $(s^{\omega}_{j,.}, j \neq i)$. Suppose that the nodal demand function at node j under realization ω is given by

$$p^{\omega}_j(s_{.,j}) := a^{\omega}_j - \sum_i s^{\omega}_{ij}. \tag{10}$$

Suppose firm i generates g^{ω}_i units of power and sells s^{ω}_{ij} units of power to node j under realization ω. We also denote the forward purchases of firm i at node j by f_{ij}. Also, the capacity on sales and generation is denoted by C^{ω}_{ij} and G^{ω}_i. The capacity and conservation constraints are given by $g^{\omega}_i \leqslant G^{\omega}_i$ and $g^{\omega}_i = \sum_j s^{\omega}_{ij}$, respectively. We may eliminate the generation variable g^{ω}_i by using the conservation constraints to obtain a reduced model:

$$\text{nAgSpot}^{\omega}_i(s^{\omega}_{-i,.}, f) \quad \begin{array}{ll} \text{maximize}_{s^{\omega}_{i,.}} & h(s^{\omega}_{i,.}) \\ & G^{\omega}_i - \sum_j s^{\omega}_{ij} \geqslant 0\colon\ \psi^{\omega}_i, \\ \text{subject to} & s^{\omega}_{ij} \geqslant 0\colon\ \gamma^{\omega}_{ij} \quad \forall\, j \\ & C^{\omega}_{ij} - s^{\omega}_{ij} \geqslant 0\colon\ \alpha^{\omega}_{ij}, \quad \forall\, j, \end{array}$$

where $h(s^{\omega}_{i,.})$ is defined as

$$h(s^{\omega}_{i,.}) := \underbrace{p^{\omega}_i(s_{i,.})(s^{\omega}_{ii} - f_{ii})}_{\text{Net revenue from local sales}} + \underbrace{\sum_{k\neq i}(p^{\omega}_k(s^{\omega}_{k,.}) + w^{\omega}_{ik})(s^{\omega}_{ik} - f_{ik})}_{\text{Net revenue from networked sales with transmission costs}}$$

$$\underbrace{-\; c_i^\omega \sum_j s_{ij}^\omega - \frac{1}{2} d_i^\omega \Big(\sum_j s_{ij}^\omega\Big)^2}_{\text{Linear and quadratic costs of generation}} .$$

Note that w_{ik}^ω is sign-unconstrained, and firms may obtain revenue or pay a charge contingent on its sign. The optimality conditions of this problem are given by

$$\begin{aligned}
&0 \leqslant s_{ii}^\omega \perp (2+d_i)s_{ii}^\omega \\
&\qquad + \sum_{k\neq i} s_{ki}^\omega - f_{ii} + \psi_i^\omega + \alpha_{ii}^\omega + c_i^\omega - a_i^\omega \geqslant 0, \quad \forall\, i\\
&0 \leqslant s_{ij}^\omega \perp (2+d_i)s_{ij}^\omega \\
&\qquad + \sum_{k\neq i} s_{kj}^\omega - w_{ij}^\omega - f_{ij} + \psi_i^\omega + \alpha_{ij}^\omega + c_i^\omega - a_i^\omega \geqslant 0, \quad \forall\, j \neq i, \forall\, i,\\
&0 \leqslant \psi_i^\omega \perp G_i^\omega - \sum_j s_{ij}^\omega \geqslant 0, \quad \forall\, i\\
&0 \leqslant \alpha_{ij}^\omega \perp C_{ij}^\omega - s_{ij}^\omega \geqslant 0, \quad \forall\, j\neq i, \forall\, i
\end{aligned}$$

for all $\omega \in \Omega$. The net flow across linkage (ij) during realization ω is given by

$$y_{ij}^\omega = s_{ij}^\omega - s_{ji}^\omega, \quad \forall\, j, i. \tag{11}$$

A transmission provider is now introduced into the framework. He maximizes transmission revenue subject to meeting transmission constraints as shown in the transmission provider's problem $T(s)$, where s collectively refers to the sales decisions (see below). Note that the firms *pay* the provider for transmission of electricity; consequently, the transmission provider can be seen as maximizing its transmission revenue.

Let the price of transmitting a unit across link (ij) for realization ω be given by w_{ij}^ω with the corresponding flow being denoted by y_{ij}^ω. The linkage capacity during realization ω is given by C_{ij}^ω.

$$\begin{aligned}
\text{nISO}^\omega(s^\omega) \quad & \text{maximize} \quad \sum_{i,j} (w_{ij}^\omega)^T y_{ij}^\omega,\\
& \text{subject to} \quad C_{ij}^\omega - y_{ij}^\omega \geqslant 0\colon \lambda_{ij}^\omega\\
& \qquad\qquad\quad C_{ij}^\omega + y_{ij}^\omega \geqslant 0\colon \lambda_{ji}^\omega, \quad \forall\, i, j.
\end{aligned}$$

The transmission provider's optimality conditions are given by

$$\begin{aligned}
&w_{ij}^\omega = \lambda_{ij}^\omega - \lambda_{ji}^\omega,\\
&0 \leqslant \lambda_{ij}^\omega \perp C_{ij}^\omega - s_{ij}^\omega + s_{ji}^\omega \geqslant 0,\\
&0 \leqslant \lambda_{ji}^\omega \perp C_{ij}^\omega + s_{ij}^\omega - s_{ji}^\omega \geqslant 0,
\end{aligned}$$

for all $\omega \in \Omega$. We define the scenario-specific spot-market Nash equilibrium as follows.

Definition 11 (Scenario-Specific Networked Spot-Market Game). Consider a game $\mathcal{G}_{nSpot}^\omega$ in the spot-market associated with scenario ω and given forward positions f_{ij}, $\forall\, i, j = 1, \ldots, n$ where the ith firm solves the parameterized optimization problem $(\text{nAgSpot}_i^\omega(s_{-i,.}^\omega))$ for $i = 1, \ldots, n$ and the ISO solves $(\text{nISO}^\omega(s^\omega))$. Then the Nash equilibrium in spot-market decisions is given by the tuples $\{s_{i,.}^*\}_{i=1}^n$ and $\{y_{1,.}^*, \ldots, y_{n,.}^*\}_{i=1}^n$, where $(s_{i,.}^\omega)^*$ and $(y^\omega)^*$ are solutions of $(\text{nAgSpot}_i^\omega(s_{-i}^{\omega,*}, f))$ and $(\text{nISO}^\omega(s^{\omega,*}))$, respectively, for all $i = 1, \ldots, n$ and for all $\omega \in \Omega$.

Specifically, given f_{ij}, $\forall\, i, j$, if s^ω, ψ^ω, α^ω, λ^ω, $\bar{M}^\omega$, $\bar{E}$, and $\bar{I}$ are defined as

$$s^\omega := \begin{pmatrix} s_{1,.}^\omega \\ \vdots \\ s_{n,.}^\omega \end{pmatrix}, \quad \psi^\omega := \begin{pmatrix} \psi_1^\omega \\ \vdots \\ \psi_n^\omega \end{pmatrix}, \quad \alpha^\omega := \begin{pmatrix} \alpha_{1,.}^\omega \\ \vdots \\ \alpha_{n,.}^\omega \end{pmatrix},$$

$$\lambda^\omega = \begin{pmatrix} \lambda_{1,.}^\omega \\ \vdots \\ \lambda_{n,.}^\omega \end{pmatrix}, \quad \bar{M}^\omega := \operatorname{diag}(r^\omega) + \begin{pmatrix} I & \ldots & I \\ \vdots & \ddots & \vdots \\ I & \ldots & I \end{pmatrix}, \tag{12}$$

$$\bar{E} := \begin{pmatrix} -e^T & & \\ & \ddots & \\ & & -e^T \end{pmatrix}, \quad \bar{I} := \begin{pmatrix} -I & & \\ & \ddots & \\ & & -I \end{pmatrix},$$

and $r_i^\omega = (2 + d_i^\omega)$ for $i = 1, \ldots, n$, then the equilibrium conditions to scenario-based Nash equilibrium problem are given by the complementarity problem

$$0 \leqslant z^\omega \perp \hat{M}^\omega z^\omega + Nf + q^\omega \geqslant 0, \tag{13}$$

$$\text{where} \quad z^\omega = \begin{pmatrix} s^\omega \\ \psi^\omega \\ \alpha^\omega \\ \lambda^\omega \end{pmatrix}, \quad \hat{M}^\omega = \begin{pmatrix} \bar{M}^\omega & -\bar{E}^T & -\bar{I} & -\bar{F}^T \\ \bar{E} & & & \\ \bar{I} & & & \\ \bar{F} & & & \end{pmatrix},$$

$$N = -\begin{pmatrix} I \\ \\ \\ \\ \end{pmatrix}, \quad q^\omega = \begin{pmatrix} q_s^\omega \\ q_\psi^\omega \\ q_\alpha^\omega \\ q_\lambda^\omega \end{pmatrix}, \tag{14}$$

$$q_s^\omega = (c_i^\omega - a_i^\omega)_{i\in\mathcal{N}}, \quad q_\psi^\omega = (G_i^\omega)_{i\in\mathcal{N}}, \quad q_\alpha^\omega = (C_{ij}^\omega)_{i,\,j\in\mathcal{N}},$$

$$q_\lambda^\omega = \begin{pmatrix} C_{ij}^\omega \\ C_{ij}^\omega \end{pmatrix}_{i,\,j\in\mathcal{N}}, \tag{15}$$

and the kth row of $\bar{F}$ corresponds to link (i, j) and $\bar{F}_{k,\,i*n+j} = -\bar{F}_{k,\,j*n+i} = 1$.

It should be further remarked that the transmission prices, denoted by w_{ij}^ω, cannot be set independently. The equilibrium conditions of the game dictate that these prices are given by the difference between the Lagrange multipliers on the line. Specifically, we have that $w_{ij}^\omega = \lambda_{ij}^\omega - \lambda_{ji}^\omega$

for all $i, j,\ i \neq j$. However, we must emphasize that the unboundedness cannot be done away with in general but requires examining if indeed there are equilibria at infinity. In fact, the need to characterize equilibria is founded on precisely such concerns: can there be equilibria with arbitrarily high prices? In fact, while we show that equilibria exist and are given by a solution to a complementarity problem with a $\mathbf{P}_0$ mapping, we observe that the trajectory of equilibria is unbounded (see end of §3.3). Note that a regularized problem introduces well-posedness and leads to unique equilibria.

Proposition 12. *Consider the spot-market game denoted by $\mathscr{G}^{\omega}_{nSpot}$. Then given a set of forward decisions f, $\mathscr{G}^{\omega}_{nSpot}$ admits a Nash equilibrium.*

Proof. The matrix $\hat{M}^{\omega}$ is a positive semidefinite matrix because $\bar{M}^{\omega}$ is a symmetric positive definite matrix. Recall that for an LCP(q, M), if M is positive semidefinite and there exists a $z \geqslant 0$ such that $Mz + q \geqslant 0$, then LCP(q, M) is solvable. In effect, it suffices to show that (13) admits a feasible solution, given an arbitrary set of forward positions denoted by f. Any solution z^{ω} to this problem is given by $(s^{\omega}, \psi^{\omega}, \alpha^{\omega}, \lambda^{\omega})$. Suppose $s^{\omega}, \lambda^{\omega} \equiv \mathbf{0}$. It follows from the nonnegativity of q^{ω}_{ψ}, q^{ω}_{α}, and q^{ω}_{λ} for all $\omega \in \Omega$ that the following three constraints are feasible:

$$\bar{E}s^{\omega} + q^{\omega}_{\psi} = q^{\omega}_{\psi} \geqslant 0, \quad \bar{I}s^{\omega} + q^{\omega}_{\alpha} = q^{\omega}_{\alpha} \geqslant 0,$$
$$\bar{F}s^{\omega} + q^{\omega}_{\lambda} = q^{\omega}_{\lambda} \geqslant 0;$$

and it remains to show that

$$-f - \bar{E}^T\psi^{\omega} - \bar{I}\alpha^{\omega} + q^{\omega}_s \geqslant 0.$$

It can be seen that if $-f + q^{\omega}_s$ is nonnegative, then ψ^{ω}, $\alpha^{\omega} \equiv 0$ forms the remainder of the feasible solution. If not, then either ψ^{ω} or α^{ω} can be made sufficiently positive to ensure that

$$-f - \bar{E}^T\psi^{\omega} - \bar{I}^T\alpha^{\omega} + q^{\omega}_s \geqslant 0,$$

where $\bar{E}, \bar{I}$ have at least one strictly negative entry in every column. This concludes the proof, and the scenario-specific spot-market equilibrium problem admits a solution. □

Agents compete in the forward market subject to equilibrium in the spot-market. Because, the specification of the spot-market is uncertain, a scenario-based characterization is used. The resulting Nash-Stackelberg game is a networked extension of $\mathscr{G}_{For}$ and is the focus of the next section.

3.2. A Nash-Stackelberg Equilibrium

The Nash-Stackelberg equilibrium in forward decisions involves a set of agents in which the ith generator maximizes his expected profit from forward and spot positions subject to the complementarity constraint

$$\text{LCP}_i \quad 0 \leqslant z^{\omega} \perp \hat{\mathscr{M}}^{\omega} z^{\omega} - N_i f_i + q^{\omega}_i \geqslant 0 \quad \forall\,\omega,$$

where $q^{\omega}_i = q^{\omega} - \sum_{j\neq i} N_j f_j$, where N_i is an appropriately defined submatrix of N. In an effort to maintain consistent notation, agent i's optimization problem is denoted by $(\text{nAgFor}_i(f^{-i}))$:

$$\begin{aligned} &\text{nAgFor}_i(f^{-i})\\ &\quad \underset{f_i,\, z}{\text{maximize}} \quad \mathbb{E}(\tfrac{1}{2} z^{\omega,T} Q^{\omega} z^{\omega} + (r^{\omega})^T z^{\omega}),\\ &\quad \text{subject to} \quad 0 \leqslant z^{\omega} \perp \hat{\mathscr{M}}^{\omega} z^{\omega} - Nf + q^{\omega} \geqslant 0 \quad \forall\,\omega, \end{aligned}$$

where the profit function is defined by

$$\begin{aligned} &\frac{1}{2}(z^{\omega})^T Q^{\omega} z^{\omega} + (r^{\omega})^T z^{\omega}\\ &\quad := p^{\omega}_i(s_{i,\cdot})s^{\omega}_{ii} + \sum_{k\neq i}(p^{\omega}_k(s^{\omega}_{jk}) + w^{\omega}_{ik})s^{\omega}_{ik}\\ &\qquad - c^{\omega}_i g^{\omega}_i - \frac{1}{2} d^{\omega}_i (g^{\omega}_i)^2. \end{aligned} \tag{16}$$

The corresponding Nash-Stackelberg equilibrium is defined as follows.

Definition 13 (Networked Forward-Market Game). Consider a game in the forward-market, denoted by $\mathscr{G}_{nFor}$, where given f^{-i}, the ith firm solves the parameterized optimization problem $(\text{nAgFor}_i(f^{-i}))$ for $i = 1, \ldots, n$, and suppose its associated equilibrium problem is denoted by (EPCC$_3$). Then the associated Nash-Stackelberg equilibrium of $\mathscr{G}_{nFor}$ in forward decisions is a tuple $\{f^*_i\}^n_{i=1}$ such that for all $i = 1, \ldots, n$, (z, f_i) is a solution to the ith agent's Stackelberg problem $(\text{nAgFor}_i(f^{-i,*}))$.

For the reasons described in §2, the analysis of such equilibrium problems is difficult, particularly from the standpoint of showing the existence of equilibria. In the single-node setting, we introduced a conjecture and showed that a related simultaneous stochastic Nash game was indeed an equilibrium to the conjectured variant of the Nash-Stackelberg game. The latter leads to a modified set of agent problems, defined for the ith player as

$$\begin{aligned} &\text{nAgFor}^{\text{con}}_i(f^{-i})\\ &\quad \underset{f_i,\, z=(s,\psi,\alpha,\lambda)}{\text{maximize}} \quad \mathbb{E}(\tfrac{1}{2} z^{\omega,T} Q^{\omega} z^{\omega} + (r^{\omega})^T z^{\omega}),\\ &\quad \text{subject to} \quad 0 \leqslant z^{\omega} \perp \hat{\mathscr{M}}^{\omega} z^{\omega} - Nf + q^{\omega} \geqslant 0 \quad \forall\,\omega,\\ &\qquad\qquad\qquad a^f_j - \sum_i f_{ij} = \mathbb{E}_{\omega}\left(a^{\omega}_j - \sum_i s^{\omega}_{ij}\right), \quad \forall\, j, \end{aligned}$$

allowing us to define the conjectured multi-leader multi-follower game.

Definition 14 (Conjectured Networked Forward-Market Game). Consider a game in the forward-market, denoted by $\mathscr{G}^{con}_{nFor}$, where given f^{-i}, the ith firm solves the parameterized optimization problem $(\text{nAgFor}^{con}_i(f^{-i}))$ for $i = 1, \ldots, n$, and suppose its associated equilibrium problem is denoted by (EPCC_4). Then the associated Nash-Stackelberg equilibrium of $\mathcal{G}^{con}_{nFor}$ in forward decisions is a tuple $\{f_i^*\}_{i=1}^n$ such that for all $i = 1, \ldots, n$, (z, f_i) is a solution to the ith agent's Stackelberg problem $(\text{nAgFor}_i^{con}(f^{-i,*}))$.

Again, this conjecture corresponds with a perturbation of a risk-neutrality constraint relating forward prices at a node with the expected spot prices. Our goal is to examine whether one may provide an existence statement for equilibria pertaining to $\mathcal{G}^{con}_{nFor}$ and possibly relate its equilibria to those arising from the networked analogues of SSNEs.

3.3. Networked Simultaneous Stochastic Nash Equilibria (SSNE)

As in §2, we observe that we may construct a stochastic Nash game whose equilibrium conditions are given by the joint set of feasibility conditions of the agent problems. In fact, we showed for the two-node problem, that an equilibrium of this game is an equilibrium of the conjectured variant of the Nash-Stackelberg game. We pursue a similar question in the networked setting: First, we analyze whether networked SSNE indeed exist and whether they are unique; and we subsequently relate the obtained SSNEs to the Nash-Stackelberg equilibria of interest. We begin by defining the networked simultaneous stochastic Nash game.

Definition 15 (Simultaneous Stochastic Nash Equilibrium SSNE). Consider the simultaneous stochastic Nash game, denoted by $\mathcal{G}_{nSSNE}$, in which the ith firm solves the parameterized optimization problem $\text{nAgSpot}_i(f, s^{\omega}_{-i})$ for $i = 1, \ldots, n$, and the ISO solves $(\text{nISO}(s^{\omega}))$, while forward decisions f are specified by the risk-neutrality constraint:

$$a_j^f - \sum_i f_{ij} = \mathbb{E}_{\omega}\left(a_j^{\omega} - \sum_i s_{ij}^{\omega}\right), \quad \forall j.$$

Extending the framework in §2, the equilibrium conditions of the simultaneous stochastic Nash game are given by the stochastic mixed linear-complementarity problem, denoted by (SSNE-CPN).

$$\text{SSNE-CPN} \quad \begin{aligned} &0 \leqslant z \perp \hat{M} z + N f + q \geqslant 0,\\ &W_f f = \hat{W}_f z + d, \end{aligned}$$

where $W_f = (I \quad \ldots \quad I)$ and

$$\hat{W}^r = (p^{\omega_1} W_f \quad \ldots \quad p^{\omega_K} W_f \quad \mathbf{0} \quad \ldots \quad \mathbf{0}). \tag{17}$$

As discussed in §2, one avenue for analyzing the existence of a mixed-complementarity problem is through the elimination of linear equality constraint (Cottle et al. 1992), an option that is available only if W_f is square and nonsingular. In the current setting, W_f is a rectangular full row-rank system, and we present a null-space approach for reducing the problem to a pure LCP. This requires an orthonormal basis for the range-space of $(W^f)^T$, as given by the following result.

Lemma 16. *Consider the matrix W_f defined by $W_f = (I \ \ldots \ I)$ where $I \in \mathbb{R}^{n\times n}$ and $W_f \in \mathbb{R}^{n^2 \times n}$. Then an orthonormal basis for the range space of W_f^T is given by Y_W where Y_W is defined as*

$$Y_W = -\frac{1}{\sqrt{n}}(I \quad \ldots \quad I)^T.$$

Proof. Proof omitted. □

Next, we show that (SSNE-CPN) can be transformed to a linear-complementarity problem. If Z_W denotes a basis for the null-space of W_f, then f can be expressed as $Z_W f_Z + Y_W f_Y$ where $W_f Y_W$ is a square nonsingular matrix. It follows that we have

$$\hat{W}_f z - W_f Y_W f_Y + d = 0, \quad \text{implying that}$$

$$f_Y = (W_f Y_W)^{-1}(\hat{W}_f z + d),$$

allowing us to express (SSNE-CPN) as the following linear-complementarity problem:

$$\begin{aligned} 0 \leqslant z \perp &\underbrace{(\hat{M} + N Y_W (W_f Y_W)^{-1} \hat{W}_f)}_{\hat{M}_2} z \\ &+ \underbrace{N Z_W f_Z + (q + N Y_W (W_f Y_W)^{-1} d)}_{q_2(f_Z)} \geqslant 0, \end{aligned}$$

where f_Z is a null-space component of f. Furthermore, $W_f Y_W$ is given by

$$W_f Y_W = -\frac{1}{\sqrt{n}} n I, \quad \text{implying that } (W_f Y_W)^{-1} = -\frac{1}{\sqrt{n}} I.$$

Because $Y_W = -(1/\sqrt{n}) W_f^T$, it can be concluded that

$$\begin{aligned} &Y_W (W_f Y_W)^{-1} \hat{W}_f \\ &\quad = -\frac{1}{\sqrt{n}}(-W_f^T)\frac{1}{\sqrt{n}} I (p^{\omega_1} W_f \quad \ldots \quad p^{\omega_K} W_f \quad \mathbf{0} \quad \ldots \quad \mathbf{0}) \\ &\quad = \frac{1}{n}(p^{\omega_1} W \quad \ldots \quad p^{\omega_K} W \quad \mathbf{0} \quad \ldots \quad \mathbf{0}), \end{aligned}$$

where $W = \begin{pmatrix} I & \ldots & I \\ \vdots & \ddots & \vdots \\ I & \ldots & I \end{pmatrix}$.

It follows that $\hat{M}_2$ is given by

$$\hat{M}_2 = \begin{pmatrix} M_r & -A^T \\ A & \end{pmatrix}, \text{ where } M_r = \begin{pmatrix} \bar{M}_1 & & \\ & \ddots & \\ & & \bar{M}_K \end{pmatrix} - \frac{1}{n}\begin{pmatrix} p^{\omega_1} W & \ldots & p^{\omega_K} W \\ \vdots & \ddots & \vdots \\ p^{\omega_1} W & \ldots & p^{\omega_K} W \end{pmatrix}. \tag{18}$$

If M_r can be shown to be positive definite, then $\hat{M}_2$ can be claimed to be positive semidefinite.

Lemma 17. *Consider the matrices M_r and $\hat{M}_2$ defined by* (18). *For $n \geqslant 2$, M_r is positive definite and $\hat{M}_2$ is positive semidefinite.*

Proof. It suffices to show that M_r is positive definite. This requires showing that $\mathbf{z}^T M \mathbf{z} > 0$ for all nonzero $\mathbf{z}$ where M_r is given by

$$M_r = \underbrace{\begin{pmatrix} r_d^{\omega_1} I & & \\ & \ddots & \\ & & r_d^{\omega_K} I \end{pmatrix}}_{H_1} + \underbrace{\begin{pmatrix} W & & \\ & \ddots & \\ & & W \end{pmatrix} - \frac{1}{n}\begin{pmatrix} p^{\omega_1} W & \dots & p^{\omega_K} W \\ \vdots & \ddots & \vdots \\ p^{\omega_1} W & \dots & p^{\omega_K} W \end{pmatrix}}_{H_2}, \qquad (19)$$

where $r_d^{\omega_j} = \operatorname{diag}(r^{\omega_j})$. Because $r_d^{\omega_j}$ are nonnegative for $j = 1, \dots, K$, it suffices to show that H_2 is positive semidefinite. We begin by observing that W can be expressed as follows:

$$W = \begin{pmatrix} I & \dots & I \\ \vdots & \ddots & \vdots \\ I & \dots & I \end{pmatrix} = \begin{pmatrix} I \\ \vdots \\ I \end{pmatrix} \begin{pmatrix} I & \dots & I \end{pmatrix} = UU^T.$$

If $U^T z_i$ is denoted by $\bar{z}_i$ for all i, then the positive definiteness of M_r may be expressed as follows:

$$\begin{aligned} z^T H_2 z &= \begin{pmatrix} z_1 \\ \vdots \\ z_K \end{pmatrix}^T \left(\begin{pmatrix} W & & \\ & \ddots & \\ & & W \end{pmatrix} - \frac{1}{n} \begin{pmatrix} p^{\omega_1} W & \dots & p^{\omega_K} W \\ \vdots & & \vdots \\ p^{\omega_1} W & \dots & p^{\omega_K} W \end{pmatrix} \right) \begin{pmatrix} z_1 \\ \vdots \\ z_K \end{pmatrix} \\ &= \sum_{i=1}^K \|\bar{z}_i\|^2 - \frac{1}{n} \sum_{i=1}^K p^{\omega_i} \|\bar{z}_i\|^2 - \frac{1}{n} \sum_{i=1}^K \sum_{j \neq i} (p^{\omega_i} + p^{\omega_j}) \bar{z}_i^T \bar{z}_j \\ &\geqslant \sum_{i=1}^K \|\bar{z}_i\|^2 - \frac{1}{n} \sum_{i=1}^K p^{\omega_i} \|\bar{z}_i\|^2 \\ &\quad - \frac{1}{2n} \sum_{i=1}^K \sum_{j \neq i} (p^{\omega_i} + p^{\omega_j}) (\|\bar{z}_i\|^2 + \|\bar{z}_j\|^2), \end{aligned}$$

because $\|\bar{z}_i\|^2 + \|\bar{z}_j\|^2 \geqslant 2\bar{z}_j^T \bar{z}_i$. Further analysis reveals that the expression on the right can be simplified as follows:

$$\begin{aligned} &\sum_{i=1}^K \|\bar{z}_i\|^2 - \frac{1}{n} \sum_{i=1}^K p^{\omega_i} \|\bar{z}_i\|^2 \\ &\quad - \frac{1}{2n} \sum_{i=1}^K \sum_{j \neq i} (p^{\omega_i} + p^{\omega_j}) (\|\bar{z}_i\|^2 + \|\bar{z}_j\|^2) \\ &= \sum_{i=1}^K \left(\|\bar{z}_i\|^2 - \frac{1}{n} p^{\omega_i} \|\bar{z}_i\|^2 - \frac{1}{2n} ((K-1) p^{\omega_i} + \sum_{j \neq i} p^{\omega_j}) \|\bar{z}_i\|^2 \right). \end{aligned}$$

Finally, the expression on the right-hand side can be further expressed as

$$\begin{aligned} &\sum_{i=1}^K \left(\|\bar{z}_i\|^2 - \frac{1}{n} p^{\omega_i} \|\bar{z}_i\|^2 - \frac{1}{2n} ((K-1) p^{\omega_i} + \sum_{j \neq i} p^{\omega_j}) \|\bar{z}_i\|^2 \right) \\ &= \sum_{i=1}^K \left(\|\bar{z}_i\|^2 - \frac{1}{n} p^{\omega_i} \|\bar{z}_i\|^2 - \frac{1}{2n} ((K-2) p^{\omega_i} + 1) \|\bar{z}_i\|^2 \right) \\ &= \sum_{i=1}^K \left(\|\bar{z}_i\|^2 - \frac{1}{n} p^{\omega_i} \|\bar{z}_i\|^2 - \frac{(1 + (K-2) p^{\omega_i})}{2n} \|\bar{z}_i\|^2 \right) \\ &= \sum_{i=1}^K \left(\|\bar{z}_i\|^2 - \frac{(1 + (K-4) p^{\omega_i})}{2n} \|\bar{z}_i\|^2 \right). \end{aligned}$$

Therefore, we have that

$$\begin{aligned} &\sum_{i=1}^K \left(1 - \frac{(1 + (K-4) p^{\omega_i})}{2n} \right) \|\bar{z}_i\|^2 \\ &\geqslant \left(\min_{i \in \{1, \dots, N\}} \|\bar{z}_i\|^2 \right) \underbrace{\sum_{i=1}^K \left(1 - \frac{(1 + (K-4) p^{\omega_i})}{2n} \right)}_{\text{Term 1}}. \end{aligned}$$

Because $p^{\omega_i} \leqslant 1$ for $i = 1, \dots, K$, term 1 can be shown to be nonnegative as per

$$\begin{aligned} \sum_{i=1}^K \left(1 - \frac{(1 + (K-4) p^{\omega_i})}{2n} \right) &\geqslant K \frac{(2n-1)}{2n} - \frac{(K-4)}{2n} \\ &= \frac{2K(n-1) + 4}{2n} \geqslant 0. \end{aligned}$$

It follows that

$$\sum_{i=1}^K \left(\min_{i \in \{1, \dots, N\}} \|\bar{z}_i\|^2 \right) \sum_{i=1}^K \left(1 - \frac{(1 + (K-4) p^{\omega_i})}{2n} \right) \geqslant \sum_{i=1}^K \|\bar{z}_i\|^2 > 0,$$

implying that M_r is positive definite.

Finally, for a nonzero vector $z = (u, v)$, the product $z^T \hat{M}_2 z$ can be expressed either as (1) or (2) based on whether $u \neq 0$ or $u \equiv 0$:

$$(1) \quad \begin{pmatrix} u \\ v \end{pmatrix}^T \begin{pmatrix} M_r & -A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = u^T M_r u > 0,$$

$$(2) \quad \begin{pmatrix} 0 \\ v \end{pmatrix}^T \begin{pmatrix} M_r & -A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} = 0,$$

leading to the immediate conclusion that $\hat{M}_2$ is positive semidefinite. □

Therefore, for a given f_Z, the complementarity problem (SSNE-CPN) is a monotone complementarity problem, and its solvability can be concluded through Proposition 18.

Proposition 18 (Existence of Equilibrium to $\mathcal{G}_{nSSNE}$). *Consider the simultaneous stochastic Nash equilibrium problem denoted by $\mathcal{G}_{nSSNE}$. Then an equilibrium to this problem is given by a triple (z, f_Y, f_Z), where*

$$z \in \mathrm{SOL}(q(f_Z), M_2), \qquad f_Y = (W_f Y_W)^{-1} (\hat{W}_f z + d),$$

and f_Z is an arbitrary vector. Furthermore, the LCP$(q(f_Z), M_2)$ is a monotone LCP that admits a solution for all f_Z.

Proof. From the earlier discussion, we observe that a solution to (SSNE-CPN) is given by (z, f_Z, f_Y) where f_Z and f_Y are the null-space and range-space components of f. Additionally, f_Y can be uniquely derived from z by noting that $f_Y = (W_f Y_W)^{-1}(\hat{W}_f z + d)$.

It remains to show that $\mathrm{LCP}(M_2, q_2(f_Z))$ is solvable for all f_Z. First, we note that by the positive definiteness of M^r, the positive semidefiniteness of M_2 follows. From Cottle et al. (1992, Theorem 3.1.2), the feasibility of $\mathrm{LCP}(q_2(f_Z), M_2)$ suffices for solvability; in effect, if there exists a $z \geqslant 0$ such that $Mz + q \geqslant 0$, $\mathrm{LCP}(q_2(f_Z), M_2)$ is solvable. Recall that $z = (s, \psi, \alpha, \lambda)$ and let $s, \lambda \equiv 0$. From the nonnegativity of q_ψ^ω, q_α^ω, and q_λ^ω, we have that

$$\bar{E}s^\omega + q_\psi^\omega = q_\psi^\omega \geqslant 0, \quad \bar{I}s^\omega + q_\alpha^\omega = q_\alpha^\omega \geqslant 0,$$

$$\bar{F}s^\omega + q_\lambda^\omega = q_\lambda^\omega \geqslant 0, \quad \forall\, \omega \in \Omega.$$

Feasibility of z follows by ascertaining if one can determine a nonnegative ψ^ω and α^ω such that

$$M_2 s^\omega - \bar{E}^T\psi^\omega - \bar{I}^T\alpha^\omega + q_s^\omega = -\bar{E}^T\psi^\omega - \bar{I}^T\alpha^\omega + q_s^\omega \geqslant 0, \quad \forall\, \omega \in \Omega.$$

Clearly, if $q_s^\omega \geqslant 0$, $\psi, \alpha \equiv 0$ will suffice; if not, by noting that $\bar{E}^T$ and $\bar{I}^T$ are matrices with a positive entry on each row, it follows that by raising either ψ or α to a sufficiently large positive level, we obtain feasibility. This concludes the proof. □

Motivated by a need to characterize (SSNE-CPN), we observe that this problem is equivalent to the square-complementarity system denoted by $\mathrm{CP}(\mathbf{C}, F)$ where

$$F(z, f) = \begin{pmatrix} \hat{M}z + Nf + q \\ \hat{W}_f z + W_f f + d \end{pmatrix} \quad \text{and} \quad \mathbf{C} = \mathbb{R}_+^{K(n^2+m^2)} \times \mathbb{R}^{n^2}, \tag{20}$$

where $z \in \mathbb{R}^{n^2+m^2}$. Furthermore, we show as in §2 that F is a $\mathbf{P}_0$ mapping over the cone $\mathbf{C}$. In an effort to simplify the exposition, we consider a setting where transmission constraints are relaxed and only capacity constraints persist. Consequently, the mapping ∇F is given by

$$\nabla F = \begin{pmatrix} \hat{M}_1 & & & N \\ & \ddots & & \vdots \\ & & \hat{M}_K & N \\ -p^{\omega_1}\hat{W} & \dots & -p^{\omega_K}\hat{W} & W \end{pmatrix},$$

$$\text{where } \hat{M}_j = \begin{pmatrix} M_j + r_d^{\omega_j} & I \\ -I & \end{pmatrix}, \; N_j = \begin{pmatrix} -I \\ \end{pmatrix}, \tag{21}$$

$$\hat{W} = (W \quad \mathbf{0}), \tag{22}$$

where M_j is defined in (12). Note that $\hat{M}_j$ and its inverse are given by

$$\hat{M} = \begin{pmatrix} \bar{M}_j & I \\ -I & \end{pmatrix} \quad \text{and} \quad \hat{M}_j^{-1} = \begin{pmatrix} & -I \\ I & \bar{M}_j \end{pmatrix}, \text{ respectively.}$$

Lemma 19. *Consider the complementarity problem $CP(\mathbf{C}, F)$ defined in* (20). *Then the mapping $F(z, f)$ is a $\mathbf{P}_0$ mapping over the cone $\mathbf{C}$.*

Proof. We prove that $F(z, f) \in \mathbf{P}_0$ by showing that $\nabla F(z, f)$ is a $\mathbf{P}_0$ matrix for all $(z, f) \in \mathbf{C}$. A matrix $M \in \mathbb{R}^{n\times n}$ belongs to the class of $\mathbf{P}_0$ matrices, if all principal submatrices have nonnegative determinants. This prompts such an evaluation for all submatrices $G_{\alpha\alpha}$

$$G_{\alpha\alpha} = \begin{pmatrix} \hat{M}_{\alpha_1\alpha_1} & & & N_{\alpha_1\alpha_{K+1}} \\ & \ddots & & \vdots \\ & & \hat{M}_{\alpha_K\alpha_K} & N_{\alpha_K\alpha_{K+1}} \\ -p^{\omega_1}\hat{W}_{\alpha_{K+1}\alpha_1} & \dots & -p^{\omega_K}\hat{W}_{\alpha_{K+1}\alpha_K} & W_{\alpha_{K+1}\alpha_{K+1}} \end{pmatrix},$$

where $\alpha = \bigcup_{i=1}^{K+1}\alpha_i \subseteq \{1, \dots, K(m^2+n^2)+n^2\}$ and $\alpha_i \cap \alpha_j = \varnothing$, $i \neq j$. We consider the following set of mutually exclusive cases and show in each case that the principal submatrix has a nonnegative determinant.

(i) If $\alpha_{K+1} = \varnothing$, then $G_{\alpha\alpha}$ is any submatrix of $\bar{G}$ where

$$\bar{G} = \begin{pmatrix} \hat{M}_{\alpha_1\alpha_1} & & \\ & \ddots & \\ & & \hat{M}_{\alpha_K\alpha_K} \end{pmatrix},$$

then $\det(G_{\alpha\alpha}) \geqslant 0$ because $\bar{G}$ is a positive semidefinite matrix, implying that it is a $\mathbf{P}_0$ matrix.

(ii) If $\bigcup_{i=1}^K \alpha_i = \varnothing$, then $G_{\alpha\alpha}$ is a principal submatrix of W, a positive semidefinite matrix, implying that $\det(G_{\alpha\alpha}) \geqslant 0$.

(iii) If α is chosen arbitrarily and $|\alpha_{K+1}| > n$, then $G_{\alpha\alpha}$ has a zero determinant because at least two of the rows are identical. It remains to show that $\det(G_{\alpha\alpha}) \geqslant 0$ when $|\alpha_{n+1}| \leqslant n$. Without loss of generality, we assume that $|\alpha_{K+1}| = n$, implying that $W_{\alpha_{K+1}\alpha_{K+1}}$ is an identity matrix of size n. Then $\det(G_{\alpha\alpha})$ is given by the following:

$$\det(G_{\alpha\alpha}) = \prod_{i=1}^K \det(\hat{M}^i_{\alpha_i\alpha_i}) \cdot \det\left(W_{\alpha_{K+1}\alpha_{K+1}} - \sum_{i=1}^K(-p^{\omega_i}\hat{W}_{\alpha_{K+1}\alpha_i}(\hat{M}^i_{\alpha_i\alpha_i})^{-1}N_{\alpha_i\alpha_{K+1}})\right). \tag{23}$$

The remainder of our proof is twofold:

(a) First, we show that $\det(\hat{M}^i_{\alpha_i\alpha_i})$ is positive. We begin by noting that $\bar{M}^i_{\alpha_i\alpha_i}$, a principal submatrix of a positive definite matrix, namely $\bar{M}^i$, is also positive definite. Consequently, $(\bar{M}^i_{\alpha_i\alpha_i})^{-1}$ is also positive definite. Finally, the structure of $\hat{M}_i$ allows one to express $\det(\hat{M}^i_{\alpha_i\alpha_i})$ as

$$\det(\hat{M}^i_{\alpha_i\alpha_i}) = \det(\bar{M}^i_{\alpha_i\alpha_i})\det((\bar{M}^i_{\alpha_i\alpha_i})^{-1}),$$

where the second term in the product is the Schur complement. The positivity of the determinant follows from the positive definiteness of $(\bar{M}^i_{\alpha_i\alpha_i})^{-1}$.

(b) Next, we show that the second term in (23), namely the Schur complement, has a nonnegative determinant. This can be concluded by observing that

$$\begin{aligned}(-p^{\omega_i}\hat{W}_{\alpha_{K+1}\alpha_i}(\hat{M}^i_{\alpha_i\alpha_i})^{-1}N_{\alpha_i\alpha_{K+1}})\\ = p^{\omega_i}\begin{pmatrix}W & \mathbf{0}\end{pmatrix}\begin{pmatrix}\mathbf{0} & -I\\ I & \bar{M}_i\end{pmatrix}\begin{pmatrix}-I\\ \mathbf{0}\end{pmatrix} = \mathbf{0}, \quad \text{forall } i=1,\ldots,K.\end{aligned}$$

Because $W_{\alpha_{K+1}\alpha_{K+1}}$ is an identity matrix, it follows that

$$\begin{aligned}\det\left(W_{\alpha_{K+1}\alpha_{K+1}} - \sum_{i=1}^{K}(-p^{\omega_i}\hat{W}_{\alpha_{K+1}\alpha_i}(\hat{M}^i_{\alpha_i\alpha_i})^{-1}N_{\alpha_i\alpha_{K+1}})\right)\\ = \det(W_{\alpha_{K+1}\alpha_{K+1}}) = 1. \quad \square\end{aligned}$$

A characterization of the mapping as a $\mathbf{P}_0$ mapping has immediate relevance from the standpoint that it allows for claiming uniqueness of a perturbed problem. Specifically, the complementarity problem $\text{CP}(\mathbf{C}, F+\delta\mathbf{I})$ admits a unique solution where $\nabla\mathbf{I} = I$. This has relevance in developing a relationship between NSEs and SSNEs, a question that will be probed in the next subsection.

Proposition 20. *Consider the complementarity problem $CP(\mathbf{C}, F)$. Then the perturbed problem $CP(\mathbf{C}, F + \delta\mathbf{I})$ admits a unique solution for all $\delta > 0$.*

Proof. This follows immediately from noting that F is a continuous $\mathbf{P}_0$ function over a cone $\mathbf{C}$ that can be expressed as a cartesian product. It follows from Facchinei and Pang (2003, Theorem 3.5.15) that the perturbed complementarity problem admits a unique solution. $\square$

It would be natural to expect that if (z_δ, f_δ) represents the unique solution to $\text{CP}(\mathbf{C}, F+\delta\mathbf{I})$, that

$$\lim_{\delta\to 0}(z_\delta, f_\delta) = (z, f),$$

where (z, f) is a solution of $\text{CP}(\mathbf{C}, F)$. This technique, termed as a Tikhonov regularization scheme (Facchinei and Pang 2003), leads to a unique trajectory that converges to the least-norm solution of $\text{CP}(\mathbf{C}, F)$ if F is a monotone map. However, our mapping is weaker in that it belongs to the class $\mathbf{P}_0$. By leveraging (Facchinei and Pang (2003, Theorem 12.2.8), a sufficiency condition for the Tikhonov trajectory to converge to a solution of the original problem is that the solution set of $\text{CP}(\mathbf{C}, F)$ is bounded. Yet, from Proposition 18, we show that a solution ray can be constructed along which f_Z, the null-space component of f, is made arbitrarily large. Consequently, the solution set of $\text{CP}(\mathbf{C}, F)$ is not bounded, and one may not directly claim that the Tikhonov trajectory converges. Furthermore, it remains unclear if the limit point of this trajectory will indeed be an NSE.

3.4. Networked SSNEs and NSEs

Our goal in this section is to derive a relationship between simultaneous stochastic Nash equilibria and Nash-Stackelberg equilibria. The first observation we make is that a perturbed SSNE, given by the solution to $\text{CP}(\mathbf{C}, F+\delta\mathbf{I})$, is an equilibrium of a perturbed variant of the conjectured Nash-Stackelberg equilibrium problem. The perturbed game, denoted by $\mathscr{G}^{con,\delta}_{nFor}$, is a Nash game in which the ith agent solves $(\text{nAgFor}_i^{\text{con},\delta}(f^{-i}))$:

$$\begin{aligned}&\text{nAgFor}_i^{\text{con},\delta}(f^{-i})\\ &\underset{f_i,\, z=(s,\psi,\alpha,\lambda)}{\text{maximize}} \quad \mathbb{E}(\tfrac{1}{2}z^{\omega,T}Q^\omega z^\omega + (r^\omega)^T z^\omega),\\ &\text{subject to} \quad 0 \leqslant z^\omega \perp (\hat{\mathscr{M}}^\omega + \delta I)z^\omega - Nf + q^\omega \geqslant 0 \quad \forall\,\omega,\\ &\qquad a_j^f - \sum_{i\neq j} f_{ij} - f_{jj}(1+\delta) = \mathbb{E}_\omega\Big(a_j^\omega - \sum_i s_{ij}^\omega\Big), \quad \forall\, j.\end{aligned}$$

Note the crucial difference between $(\text{nAgFor}_i^{\text{con}}(f^{-i}))$ and its variant, defined above, lies in the perturbation of the complementarity problem specifying the feasible region. This regularization is crucial in ensuring that the feasible region of every agent problem, given a collection of competitive forward decisions, is indeed a singleton, as clarified by Proposition 21.

Proposition 21. *Given a $\delta > 0$, suppose (z, f) is a solution of $\mathscr{G}^\delta_{nSSNE}$. Then (z, f) is a local Nash-Stackelberg equilibrium of $(\mathscr{G}^\delta_{nFor})$. Furthermore, the local Nash-Stackelberg equilibrium always exists.*

Proof. We proceed as in Theorem 10. First, given a solution f^{-i}, the feasible region of $(\text{nAgFor}_i^{\text{con},\delta}(f^{-i}))$ is given by a complementarity problem $\text{CP}(\mathbf{C}, F_\delta)$ where

$$F_\delta(z, f) = \begin{pmatrix}(\hat{M} + \delta I)z + Nf\\ \hat{W}_f z + (W_f + \delta I)f\end{pmatrix}.$$

But F is a continuous $\mathbf{P}_0$ mapping over $\mathbf{C}$, implying that its regularization—namely $F+\delta\mathbf{I}$—leads to a complementarity problem that has a unique solution (see Facchinei and Pang 2003, Th. 3.5.15). It follows that the feasible region of the agent problem is a singleton and (z, f_i) is trivially a local minimizer of $(\text{nAgFor}_i^{\text{con},\delta}(f^{-i}))$ for all $i = 1, \ldots, n$. It follows that $(z, f_i)_{i=1}^n$ is a local Nash-Stackelberg equilibrium of (EPCC_4^δ). Finally, because the simultaneous stochastic Nash equilibrium, corresponding to $(\mathscr{G}^\delta_{nSSNE})$ both exists and is unique, it follows that $\mathscr{G}^\delta_{nFor}$ always admits an equilibrium. $\square$

It must be remarked that if $\delta \to 0$, it remains unclear if we can show that the feasible regions of the agent problems remain singletons, given f^{-i}. Consequently, we employ a fixed positive regularization parameter δ.

In this section, we have shown that a solution to a specific complementarity problem provides at least one solution to the original Nash-Stackelberg game. Additionally,

Table 3. Comparison of accuracy and effort: Conjectured games vs. EPECs.

	Conjecture with $\delta = 1e-4$				Conjecture with $\delta = 1e-8$				Gauss-Seidel EPEC solver			
K	$\Delta\pi$	Δp_f	iter	t^{cpu}	$\Delta\pi$	Δp_f	iter	t^{cpu}	$\Delta\pi$	Δp_f	iter	t^{cpu}
1	0.000	0.000	11	0.0	0.000	0.000	11	0.0	0.000	0.000	273	2.1
2	0.000	0.000	16	0.1	0.000	0.000	17	0.1	0.000	0.000	482	5.5
3	0.000	0.000	33	0.6	0.000	0.000	15	0.2	0.000	0.000	305	5.9
4	0.000	0.000	22	0.8	0.000	0.000	22	0.7	0.000	0.000	302	8.3
5	0.008	0.005	17	0.8	0.008	0.005	17	0.8	0.016	0.000	478	12.5
6	0.004	0.003	14	1.0	0.004	0.003	14	1.0	0.009	0.000	272	8.8
7	0.000	0.000	19	2.6	0.000	0.000	17	2.2	0.000	0.000	333	12.6
8	0.027	0.017	15	2.4	0.027	0.017	15	2.4	0.054	0.000	1,993	57.6
9	0.001	0.001	30	8.0	0.001	0.001	33	9.0	0.002	0.000	4,550	229.7
10	0.007	0.005	22	7.2	0.007	0.005	22	7.2	0.015	0.000	3,629	231.5

we showed that an approximation of this complementarity problem is always solvable. Next, we provide a quantitative comparison of both the accuracy and computation effort associated with using the conjectured framework and draw a comparison with a more standard EPEC approach. The approaches are employed on a six-node electricity market model with a varying number of scenarios.

In Table 3 we provide a comparison of both the accuracy and effort associated with using the Nash approach versus solving the EPEC, via a Gauss-Seidel scheme (Hu and Ralph 2007). Using the solution to the mixed-complementarity problem as a basis, we compute the solutions of the conjectured problems and the EPECs using $\Delta\pi$ and Δp_f as a basis for comparison, where $\Delta\pi$ and Δp_f are defined as

$$\Delta\pi = \frac{\|\pi - \pi^*\|}{\|\pi^*\|} \quad \text{and} \quad \Delta p_f = \frac{\|p_f - p_f^*\|}{\|p_f^*\|}.$$

The effort is specified in terms of the number of minor iterations (iter) and CPU time (t^{cpu}). Recall that our conjecture is a perturbation of the affine price function by δ, and results are provided for $\delta = 1e-4$ and $\delta = 1e-8$. It should also be noted that for all the schemes, feasibility and optimality tolerances were set at $1e-8$, and KNITRO was employed for solving both the complementarity problems and the MPCCs (in the EPCC solution framework). The Gauss-Seidel EPCC solver is terminated when the change $\Delta f_k = \|f_k - f_{k-1}\|/\|f_{k-1}\| \leqslant 1e-5$.

From an accuracy perspective, the perturbed Nash games provide very accurate solutions even for $\delta = 1e-4$. Importantly, as the size of the problem increases, the accuracy does *not* degenerate in the perturbed Nash setting.

In terms of effort, the Nash problems are obviously much easier to solve than the EPCCs because they are LCPs. It is observed that the number of minor iterations tends to stay between 20 and 40 for both values of δ. The EPCC solver cycles through each agent problem, solving six MPCCs for each cycle. As a consequence, the total number of minor iterations over all cycles can be quite large. In particular, we see a steep incline in the number of minor iterations as the number of scenarios grows. If one measures CPU time, the growth is even more severe, suggesting that Gauss-Seidel schemes for large-scale problems might be inadvisable.

4. A Decomposition-Based Splitting Algorithm

The earlier two sections showed that in both a two-node and a networked setting, the solution to the conjectured variant of the multi-leader multi-follower game is obtainable through the solution of a simultaneous stochastic Nash game. The latter leads to a stochastic complementarity problem, and under the assumption of a discrete distribution with an arbitrarily large support, the size of this problem might grow to astronomical levels. As a consequence, direct approaches for the solution of such problems are inadvisable. In fact, we provide some computational evidence to support that the growth in effort is exponential, implying the need for a decomposition method for such a class of problems.

This section is devoted toward developing a scalable approach for solving the obtained class of stochastic mixed-complementarity problems. Methods for the solution of LCPs range from interior-point methods (Ralph and Wright 2000, Facchinei and Pang 2003, Cottle et al. 1992), splitting methods (Cottle et al. 1992), to Newton-based approaches (Ferris and Munson 2000, Cottle et al. 1992). Extensions to the stochastic case have been dealt with by Lin et al. (2003).

We present a splitting-based decomposition (referred to as the DS method) method based on solving the mixed-LCP through the solution of a sequence of LCPs. Each LCP is stochastic in nature and can be arbitrarily large. In §4.1, we present the DS method along with convergence theory. The computational burden can be lightened considerably by the use of sampling, and these ideas are discussed in §4.2. In §4.3 we provide a description of the performance of the method and compare it with solving the problem directly using KNITRO (Byrd et al. 1999).

4.1. The DS Algorithm

We solve the stochastic complementarity problem arising from $\mathcal{G}_{nSSNE}$ by a scenario-decomposition approach that relies on the ideas of matrix splitting methods (cf. Cottle et al. 1992). This ensures that when the number of scenarios grows, the problem could still be solved efficiently. However, (SSNE-CPN) is not immediately scenario-separable because the equality constraints contain an expectation term and the complementarity constraints contain the first-stage forward decisions f. Our problem of interest is a regularized form of (SSNE-CPN), denoted by (SSNE-CPN$_\delta$) and defined as

$$\text{SSNE-CPN}_\delta \quad 0 \leqslant z \perp (\hat{M}+\delta I)z + Nf + q \geqslant 0$$
$$\hat{W}_f z + (W_f + \delta I) f + d = 0.$$

Because $W_f + \delta I$ is nonsingular, we may eliminate f, leading to a linear complementarity problem:

$$\text{SSNE-LCPN}_\delta \quad 0 \leqslant z \perp \mathbf{M}z + \mathbf{q} \geqslant 0,$$

where

$$z = \begin{pmatrix} s \\ \psi \\ \alpha \\ \lambda \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} \mathbf{M}_\delta + \delta I & -\hat{E}^T & -\hat{I}^T & -\widehat{F}^T \\ \hat{E} & \delta I & & \\ \hat{I} & & \delta I & \\ \widehat{F} & & & \delta I \end{pmatrix},$$

$$\mathbf{q} = \begin{pmatrix} q_s - (W_f^\delta)^{-1} d \\ q_\psi \\ q_\alpha \\ q_\lambda \end{pmatrix}, \quad \hat{E}^T = \begin{pmatrix} \bar{E}^T & & \\ & \ddots & \\ & & \bar{E}^T \end{pmatrix},$$

$$\hat{I}^T = \begin{pmatrix} \bar{I}^T & & \\ & \ddots & \\ & & \bar{I}^T \end{pmatrix}, \quad \hat{F}^T = \begin{pmatrix} \bar{F}^T & & \\ & \ddots & \\ & & \bar{F}^T \end{pmatrix},$$

$$\mathbf{M}_\delta = \begin{pmatrix} \bar{M}^{\omega_1} - p^{\omega_1}(W_f^\delta)^{-1}W & -p^{\omega_2}(W_f^\delta)^{-1}W & \dots & -p^{\omega_K}(W_f^\delta)^{-1}W \\ -p^{\omega_1}(W_f^\delta)^{-1}W & \bar{M}^{\omega_2} - p^{\omega_2}(W_f^\delta)^{-1}W & \dots & -p^{\omega_K}(W_f^\delta)^{-1}W \\ \vdots & & & \\ -p^{\omega_1}(W_f^\delta)^{-1}W & -p^{\omega_2}(W_f^\delta)^{-1}W & \dots & \bar{M}^{\omega_K} - p^{\omega_K}(W_f^\delta)^{-1}W. \end{pmatrix}. \tag{24}$$

If I_d, I_c, and W denote

$$I_d = \delta \begin{pmatrix} I & & \\ & \ddots & \\ & & I \end{pmatrix} \quad \text{and} \quad I_c = \begin{pmatrix} I \\ \vdots \\ I \end{pmatrix}, \quad \text{and}$$

$$W = \begin{pmatrix} I & \dots & I \\ \vdots & \ddots & \vdots \\ I & \dots & I \end{pmatrix},$$

then the $\mathbf{M}_\delta$ can be shown to be positive definite in the next result. This is an important step because the monotonicity of the complementarity problem in the reduced problem provides us with an avenue for developing a decomposition scheme.

Proposition 22. *If $\delta > 0$ and W_f^δ, W, and $\mathbf{M}_\delta$ are defined as* (17) *and* (24)*, then the matrix $\mathbf{M}_\delta$ is positive definite.*

Proof. The inverse of W_f^δ is analytically obtainable by recalling that

$$(I_d + I_c I_c^T)^{-1} = I_d^{-1} - I_d^{-1} I_c (I + I_c^T I_d^{-1} I_c)^{-1} I_c^T I_d^{-1}.$$

Therefore, $(W_f^\delta)^{-1}$ is given by

$$(W_f^\delta)^{-1} = I_d^{-1} - I_d^{-1} I_c \left(I + \frac{n}{\delta} I\right)^{-1} I_c^T I_d$$
$$= I_d^{-1} - \frac{\delta}{\delta+n} I_d^{-1} I_c I_c^T I_d^{-1} = \frac{1}{\delta} I_d - \frac{1}{\delta(\delta+n)} W.$$

After some simplification, $(W_f^\delta)^{-1}\hat{W}$ may be expressed as $\big(p^{\omega_1}(W_f^\delta)^{-1}W \quad \dots \quad p^{\omega_K}(W_f^\delta)^{-1}W\big)$, where

$$(W_f^\delta)^{-1} W = \left(\frac{1}{\delta} W - \frac{n}{\delta(\delta+n)} W\right) = \frac{1}{\delta+n} W. \tag{25}$$

It follows that M_δ, after some simplification, is given by

$$M_\delta := \underbrace{\begin{pmatrix} r_d^{\omega_1} I & & \\ & \ddots & \\ & & r_d^{\omega_K} \end{pmatrix}}_{\mathbf{H}_1} + \underbrace{\begin{pmatrix} W & & \\ & \ddots & \\ & & W \end{pmatrix} - \frac{1}{n+\delta} \begin{pmatrix} p^{\omega_1} W & \dots & p^{\omega_K} W \\ \vdots & \ddots & \vdots \\ p^{\omega_1} W & \dots & p^{\omega_K} W \end{pmatrix}}_{\mathbf{H}_2^\delta},$$

where $r_d^{\omega_j} = \operatorname{diag}(r^{\omega_j})$. From Lemma 17 and the positivity of δ we have for any $z \neq 0$

$$z^T (H_1 + H_2^\delta) z \geqslant z^T (H_1 + H_2) z > 0,$$

where H_1 and H_2 are specified in (19). $\square$

This transformation has several implications. First, it leads to a monotone linear-complementarity problem because M_δ is positive definite. However, in eliminating the forward decisions, we witness the loss of diagonal decomposability, often a crucial component for developing scalable schemes. Yet, as the remainder of this subsection shows, we utilize matrix-splitting methods to recover decomposability, and therefore scalability.

While the existence and uniqueness of a solution to (SSNE-CPN$_\delta$) follows from Proposition 20, what is not clear is whether a scalable splitting method can be employed for its solution. Specifically, we are interested in solving a sequence of problems denoted by (SSNE-CPN$_\delta^j$) to obtain a solution to (SSNE-CPN$_\delta$) where the latter is defined as

$$\text{SSNE-CPN}_\delta^j \quad 0 \leqslant z^j \perp B z^j + q^j \geqslant 0,$$

where

$$q^j := (\mathbf{q} + Cz^{j-1}), \quad B = \begin{pmatrix} \hat{M} & -U^T \\ U & I \end{pmatrix},$$

$$C = \begin{pmatrix} N(W^f)^{-1}\hat{W} & 0 \\ 0 & (\delta - 1)I \end{pmatrix}, \tag{26}$$

and $\mathbf{M} = B + C$. The choice of B is essential in ensuring that (SSNE-CPN$_\delta^j$) can be decomposed into a set of scenario-specific LCPs. While this appears challenging given the structure of B, the diagonally decomposable structure of $\hat{M}$ and the decomposable structure of U allow for precisely such a decomposability. It remains to show that the overall matrix splitting scheme does indeed converge.

Note that the positive definiteness of B ensures the uniqueness of each iterate. Such *splitting* methods are discussed extensively in Cottle et al. (1992) in the context of LCPs and are not guaranteed to work in general. In this case, the matrix $\mathbf{M}$ is positive definite while B is positive definite (but not necessarily symmetric). Given a positive definite matrix B, we define $\hat{B}$ such that $\hat{B}^T\hat{B} = \frac{1}{2}(B + B^T)$ where $\bar{B} = \frac{1}{2}(B + B^T)$. Without loss of generality, it suffices to assume all quadratic components of generation cost are zero and $\bar{B}$ is given by

$$\bar{B} = \begin{pmatrix} W+I & & & \\ & \ddots & & \\ & & W+I & \\ & & & I \end{pmatrix} \quad \text{and} \quad \hat{B} = \begin{pmatrix} D & & & \\ & \ddots & & \\ & & D & \\ & & & I \end{pmatrix},$$

$$D = \begin{pmatrix} aI & bI & \dots & bI \\ bI & aI & \ddots & \vdots \\ \vdots & \ddots & \ddots & bI \\ bI & \dots & bI & aI \end{pmatrix}, \tag{27}$$

where[2] $a = (1 + b)$ and $b = (\sqrt{n+1} - 1)/n$. It follows that D^{-1} is given by

$$\begin{aligned} &(I_d + (\sqrt{b}I_c)(\sqrt{b}I_c)^T)^{-1} \\ &\quad = (I_d - bI_dI_c(I + bI_c^TI_c)^{-1}I_c^TI_d) \\ &\quad = \left(I_d - bI_dI_c\left(\frac{1}{nb+1}(I_c^TI_c)\right)I_c^TI_d\right) = \left(I_d - \frac{b}{nb+1}W\right). \end{aligned}$$

This allows us to prove the following condition on the spectral radius.

Lemma 23. *Given a* $\delta \in (0, 1)$, *and let* $\hat{B}$ *and* C *be as defined in* (27) *and* (26), *and* $b = (\sqrt{n+1} - 1)/n$. *Then the following hold*:

(a) *If the size of the sample-space* K *is bounded as*

$$K < \left(\frac{(n+\delta)(nb+1)^2}{n}\right)^2,$$

then the spectral radius $\rho(\hat{B}, C) = \|(\hat{B}^{-1})^T C \hat{B}^{-1}\|_2 < 1$.

(b) *If* $p^{\omega_j} = 1/K$ *for all* $j = 1, \dots, K$, *then* $\|(\hat{B}^{-1})^T C \hat{B}^{-1}\|_2 < 1$.

Proof. The matrix $\hat{B}^{-T}C\hat{B}$ is given by

$$\begin{aligned} &= \frac{1}{n+\delta}\begin{pmatrix} \left(I_d - \frac{b}{nb+1}W\right) & & & \\ & \ddots & & \\ & & \left(I_d - \frac{b}{nb+1}W\right) & \\ & & & I \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} p^{\omega_1}W & \dots & p^{\omega_K}W & \\ \vdots & \vdots & \vdots & \\ p^{\omega_1}W & \dots & p^{\omega_K}W & \\ & & & (\delta-1)I \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} \left(I_d - \frac{b}{nb+1}W\right) & & & \\ & \ddots & & \\ & & \left(I_d - \frac{b}{nb+1}W\right) & \\ & & & I \end{pmatrix}, \\ &= \frac{1}{n+\delta}\begin{pmatrix} \left(I_d - \frac{b}{nb+1}W\right) & & & \\ & \ddots & & \\ & & \left(I_d - \frac{b}{nb+1}W\right) & \\ & & & I \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} p^{\omega_1}\frac{1}{nb+1}W & \dots & p^{\omega_K}\frac{1}{nb+1}W & \\ \vdots & \vdots & \vdots & \\ p^{\omega_1}\frac{1}{nb+1}W & \dots & p^{\omega_K}\frac{1}{nb+1}W & \\ & & & (\delta-1)I \end{pmatrix}, \\ &= \frac{1}{n+\delta}\begin{pmatrix} p^{\omega_1}\frac{1}{(nb+1)^2}W & \dots & p^{\omega_K}\frac{1}{(nb+1)^2}W & \\ \vdots & \vdots & \vdots & \\ p^{\omega_1}\frac{1}{(nb+1)^2}W & \dots & p^{\omega_K}\frac{1}{(nb+1)^2}W & \\ & & & (\delta-1)I \end{pmatrix} \\ &= \frac{1}{(n+\delta)(nb+1)^2}A, \end{aligned}$$

$$\text{where } A := \begin{pmatrix} p^{\omega_1}W & \dots & p^{\omega_K}W & \\ \vdots & \vdots & \vdots & \\ p^{\omega_1}W & \dots & p^{\omega_K}W & \\ & & & (\delta-1)I \end{pmatrix}.$$

Finally, by recalling that $(1 - \delta) < 1 \leqslant n$, we note that $\|A\|_2$ can be bounded as per $\|A\|_2 \leqslant \sqrt{\|A\|_1\|A\|_\infty}$, where the $\|\cdot\|_\infty$ and $\|\cdot\|_1$ are given by the maximum absolute row and column sum, respectively. The structure of A allows us to obtain explicit bounds for $\|A\|_\infty$, $\|A\|_1$, and $\|A\|_2$:

$$\|A\|_\infty = \sum_{j=1}^{K} np^{\omega_j} = n,$$

$$\|A\|_1 = \max_j (Knp^{\omega_j}) \leqslant nK, \quad \|A\|_2 \leqslant n\sqrt{K}.$$

Finally, the required spectral condition $\|\hat{B}^{-T}C\hat{B}\| < 1$ holds when

$$\frac{n\sqrt{K}}{(n+\delta)(nb+1)^2} < 1 \quad \text{or } K < \left(\frac{(n+\delta)(nb+1)^2}{n}\right)^2.$$

If we further assume that $p^{\omega_j} = 1/K$ for all $j = 1, \ldots, K$, it then follows that $\|A\|_1 = n$ and $\|A\|_2 \leqslant n$, implying that K can be chosen to be arbitrarily large. □

Note that for arbitrarily probability measures, the result does not require that K be bounded but merely requires that it should not grow significantly faster than n^2. Moreover, $\|A\|_1$ is large when one scenario has a high likelihood. Expectedly, when n is small, then the resulting systems are well within direct solvers, and the real challenges lie for large n and K. Note that the more likely setting is that the probability distribution is uniform (as arising from a sampled problem), and in this setting the choice of K can be arbitrary. Based on this bound, the convergence of the splitting method can be established, as Theorem 24 states.

Theorem 24 (Theorem 5.3.12 From Cottle et al. 1992). *Suppose B is a positive definite matrix. Let $\hat{B}$ be a positive definite matrix such that $\hat{B}^T\hat{B} = \frac{1}{2}(B + B^T)$. Suppose also that*

$$\rho(\hat{B}, C) = \|\hat{B}^{-T}C\hat{B}^{-1}\| < 1.$$

Then for any starting $z \geqslant 0$, the uniquely defined sequence of iterates $\{z_k\}$ converges to a unique solution of (SSNE-CPN$_\delta$).

Algorithm 1 provides an outline of the splitting method. Specifically, the scheme constructs a sequence $\{z^j\}$ that converges to a solution to (SSNE-CPN$_\delta$)) where the iterate z^j is computed in a scenario-specific fashion, providing $(z^j)^\omega$ for $\omega \in \Omega$.

Algorithm 1 (Decomposition and splitting method)

Initial $\delta, \hat{\epsilon} > 0,\ j = 1$
while $\|\operatorname{diag}(z^j)((\mathbf{M})z^j + \mathbf{q})\|_\infty > \hat{\epsilon}$ **do**
  $z^j \in SOL(B, q^j(\epsilon_k))$
  for $\omega = 1{:}\ |\Omega|$ **do**
    Let $(z^j)^\omega \in SOL(B_j^\omega, (q^j)^\omega)$
  $q^j := q + Cz^j$
  $j := j + 1$

As a final note, we observe that there could be other splitting-based approaches that are less reliant on spectral bounds. Yet these are less susceptible to separability. For instance, given a problem $0 \leqslant z \perp Mz + q \geqslant 0$, where M is a row-sufficient matrix (that is not necessarily symmetric), then z solves the given LCP if and only if z solves the convex quadratic program

$$\min\ z^T(q + (M + M^T)z),$$

$$Mz + q \geqslant 0, \quad z \geqslant 0.$$

Specifically, $M + M^T$ could be split as $B + C$, allowing for decomposing the objective function by scenario (see Cottle et al. 1992). However, the constraints are still not immediately separable by scenario, implying that the quadratic program, although convex, is not immediately separable.

4.2. Introducing Sampling

In this section, we consider how one can further ease the computational burden by using a sample of the distribution, instead of using the original distribution at each iteration of the matrix splitting method. Consequently, at each iteration of the DS method we solve

$$\widehat{\text{SSNE-CPN}}_\delta^j$$

$$0 \leqslant z_j^\omega \perp (\bar{M}^\omega + \delta I)z_j^\omega + Nf_j + \bar{q}^\omega \geqslant 0 \quad \forall\, \omega \in \Omega_k$$

$$\frac{1}{|\Omega_j|}\sum_{\omega \in \Omega_j} W^\omega z_j^\omega - W_f^\delta f_j + q^f = 0.$$

In effect, at the jth iteration, the problem size is proportional to $n_j = |\Omega_j|$. If the sequence n_j increases fast enough to K, such a scheme is seen to converge in practice. An important question is which distribution to use in the construction of the sample. We construct residuals based on $r_j^\omega = (z_j^\omega)^T(\bar{M}^\omega z^\omega + \bar{q}_j^\omega)$. The physical interpretation of this residual vector is that scenarios with large residuals are further away from the solution than scenarios with smaller residuals. This allows for two approaches that are inspired by ideas of inexact-Newton methods. In this class of methods, the Newton direction is computed with increasing exactness as one approaches the solution, the benefit being the savings in computational effort. In a similar fashion, away from the solution, we solve an approximation of the complementarity problem by selecting or sampling a set of scenarios using the set of residuals to guide this choice. Two schemes are suggested. The first merely sorts the residual vector and chooses the largest n_j scenarios (from the standpoint of residuals) and is denoted by `SORT` The second technique biases the true distribution by the normalized residual vector. In effect, this raises the likelihood of choosing a scenario if the residual associated with it is large. We refer to this strategy as `MC`. Finally, the approach using all the scenarios is denoted by `FULL`.

4.3. Computational Results

The algorithm was implemented on Matlab 7.0 on a Linux platform with 2GB of RAM. The subproblem solver employed was the nonlinear programming solver KNITRO (Byrd et al. 1999).

Comparison with standard solvers: First we discuss our computational experience with direct methods for solving stochastic complementarity problems and contrast the growth in effort with that seen when using decomposition methods.

Table 4 provides a comparison between a direct solution of the problem and a splitting-based approach, on the basis of CPU time. We observe that the growth in CPU time for KNITRO is significantly higher than the parallelized effort associated with the splitting methods. The parallelized time was calculated by assuming that each scenario problem was solved in parallel. It was not obtained through an actual implementation but obtained by as an estimate. Specifically, the estimate used the maximum time taken to solve any of the scenario problems as the time taken for the entire set of scenarios. In fact, for sample sizes as small as 50, the direct approach has already grown in effort by a factor of 100, when compared with a sample size of 10.

Scalability of the method with network size: Table 5 shows how the algorithm scales with the size of the network. While a clear trend is not evident, the number of major and minor iterations does not grow rapidly with the size of the network. For instance, when the network size is raised from 3 to 18 nodes with 25 scenarios, the number of minor iterations increases from 325 to 925. Each subproblem does take longer to solve because the corresponding complementarity problems have grown. In fact, the key barrier in solving the problem for larger networks lies in the need to construct the matrix of the linear-complementarity problem (LCP). Solving the scenario problems can be done effectively for sizes well into the tens of thousands of variables. For addressing large-scale networks, we would not construct such a matrix directly but would work instead with the scenario blocks.

Scalability of method and sampled variants with $|\Omega|$: Table 6 compares the behavior of the DS method. Our basis of comparison is a set of equilibrium problems based on a three-node network with s scenarios. The resulting deterministic problems are of the order of $n^2 s$. In fact, even with such small networks, the deterministic complementarity problem is of the order of 20,000 variables. The termination criterion in the DS methods are based on when the complementarity residual is sufficiently small, or namely $z^T(Mz+q) \leqslant \epsilon$. The initial values for the forward and spot positions are zero. Moreover, the sampling extensions are started at 40% of $|\Omega|$ and are incremented by 1.1 at the end of each major iteration. When comparing KNITRO to the iterative methods, we use CPU time as a basis of comparison. Note that the CPU time only accounts for the calls to the solver and not for linear algebra operations. Moreover, all calls to KNITRO are with default options in terms of optimality criteria. However, when comparing the iterative methods, we use the number of major and minor iterations. The minor iterations would essentially correspond to the total number of complementarity problems solved. This is analogous to using the number of function and gradient evaluations for first-derivative optimization methods.

Table 4. CPU time comparison: direct vs. splitting method (parallelized time).

n	s	cputime_{direct}	$\|f_*^{DS} - f_*\|$	iter^{DS}	$\text{maj}-\text{iter}^{DS}$	cputime_{DS}
3	10	0.66	3.4e–07	80	11	0.31
3	20	4.66	6.8e–07	187	22	0.63
3	30	13.37	2.0e–06	74	9	0.25
3	40	35.35	4.0e–07	62	8	0.19
3	50	73.38	7.5e–07	63	8	0.19
3	60	149.45	5.2e–07	101	12	0.28

Table 5. Scalability of DS method with size of network.

n	s	minor-iter	maj-iter
3	25	325	4
4	25	350	4
5	25	225	4
6	25	450	4
7	25	550	4
8	25	500	4
9	25	825	4
10	25	775	4
11	25	700	4
12	25	725	4
13	25	725	4
14	25	800	4
15	25	775	4
16	25	775	4
17	25	750	4
18	25	925	4
19	25	850	4
20	25	900	4

Summary of findings: The main findings of our computational research were:

- The growth in effort when using splitting-based methods is approximately linear with s while direct approaches result in rapid exponential growth.
- The number of major iterations is approximately constant across different sample sizes and ranges from 8 to 11 when $n = 3$. For larger networks, the effort does not grow significantly, with the main challenge being the construction of the full complementarity matrix.
- While the sampling/sorting extensions often outperform the FULL implementation, a conclusive statement requires further research.

Remark on suitability of algorithm: We conclude this section with a short discussion on the suitability of our methodological approach. The problem of interest is a monotone linear complementarity problem and a host of

Table 6. DS method: `SORT`, `SAMP` and `FULL`.

s	$\|\Delta^{MC}\|$	`minor-iter`^{MC}	`maj-iter`^{MC}	$\|\Delta^{\mathtt{SORT}}\|$	`minor-iter`$^{\mathtt{SORT}}$	`maj-iter`$^{\mathtt{SORT}}$	`minor-iter`$^{\mathtt{Full}}$	`maj-iter`$^{\mathtt{Full}}$
50	2.90E–08	3,214	11	5.90E–09	3,226	11	3,014	10
70	1.40E–08	4,498	11	1.40E–08	4,020	10	4,202	10
90	1.30E–08	5,207	10	1.30E–08	5,208	10	5,439	10
110	1.80E–08	5,723	9	1.60E–08	6,468	10	6,757	10
130	1.40E–08	7,703	10	6.30E–08	7,699	10	7,160	9
150	1.10E–07	8,813	10	1.10E–07	8,816	10	9,152	10
170	6.40E–08	10,041	10	1.60E–08	10,049	10	10,471	10
190	1.50E–08	11,198	10	1.00E–08	11,199	10	10,360	9
210	7.10E–09	12,397	10	1.50E–08	12,390	10	12,918	10
230	3.20E–08	13,545	10	3.20E–08	13,556	10	12,564	9
250	9.70E–09	13,106	9	3.00E–08	13,102	9	15,402	10
270	7.30E–08	17,845	11	8.60E–08	14,141	9	16,664	10
290	7.50E–08	17,176	10	9.30E–08	17,178	10	15,906	9
310	7.40E–09	18,343	10	2.40E–08	18,379	10	19,126	10
330	4.20E–08	19,573	10	5.70E–08	17,298	9	18,101	9
350	1.10E–08	20,743	10	9.80E–08	20,713	10	21,595	10
370	2.30E–09	21,902	10	5.50E–08	21,906	10	22,791	10
390	8.80E–08	20,367	9	9.50E–08	23,065	10	23,985	10
410	6.90E–08	24,171	10	1.30E–07	22,469	9	26,459	10
430	2.40E–08	19,552	8	6.10E–08	26,630	10	27,701	10
450	5.50E–08	26,608	10	2.50E–08	27,778	10	28,947	10
470	2.70E–08	24,578	9	6.40E–08	25,611	9	30,166	10
490	3.10E–08	28,997	10	5.10E–08	30,157	10	31,377	10
510	3.10E–08	33,593	11	5.60E–08	27,736	9	32,640	10
530	6.10E–08	27,755	9	2.30E–08	32,585	10	33,912	10
550	2.50E–08	32,535	10	9.20E–08	33,756	10	31,245	9
570	5.90E–08	33,739	10	8.10E–08	30,942	9	36,404	10

schemes exist for such problems, such as pivoting methods, projection-based methods and matrix splitting schemes (cf. Cottle et al. 1992 for an overview of the schemes). Our problem, however, is a large-scale monotone LCP with a rather specific structure, arising from the agent-specific two-period stochastic programs. In particular, the size of the problem is directly proportional to the cardinality of Ω, the sample-space. Accordingly, we concentrate on the development of scalable schemes with an important characteristic: the computational effort should grow slowly with $|\Omega|$. Matrix-splitting methods provide one avenue for deriving such scalability because the splitting allows for the solution of $|\Omega|$ smaller LCPs at each major iteration. In fact, Table 6 shows that the effort grows linearly with the size of the sample-space. Finally, the construction of such schemes requires providing appropriate spectral properties, as seen in §4.1.

Note that other convergent decomposition schemes could also be constructed. For instance, an alternate approach could be through the use of interior point methods (Facchinei and Pang 2003, Ralph and Wright 2000), wherein the Newton direction is computed via a decomposition scheme—an avenue that has been investigated for solving stochastic nonlinear programs (Shanbhag 2006). In recent work, Kannan et al. (2011) employed a projection-based method for solving a stochastic game-theoretic problem. One of the challenges in such approaches is that the projection step requires the solution of a stochastic quadratic program (when the constraints are polyhedral), a challenge that is overcome through the use of scalable dual decomposition methods (Ruszczyński 2003, Shanbhag 2006).

5. An Electricity Market Model

In this section, we apply our two-period model to a six-node power market. We assume that each node houses an independent generator, and we assume full connectivity between the nodes. Each firm is faced with specifying forward positions in the first period. Subject to these positions and the realization of the uncertainty, the firms then compete in a spot-market. It is assumed that there are s possible realizations that the randomness can assume.

We restrict ourselves to a six-node model with 20 scenarios in the second period ($n = 6$, $s = 20$). Spot-market prices are specified based on a random demand function $p_i^\omega = a_i^\omega - \sum_j s_{ji}^\omega$, $i = 1, \ldots, 6$, while forward prices are similarly defined by $p_i^f = a_i^f - \sum_j f_{ij}$. Table 7 specifies the parameters associated with the price functions. The base case parameters allow for no uncertainty. Using the base

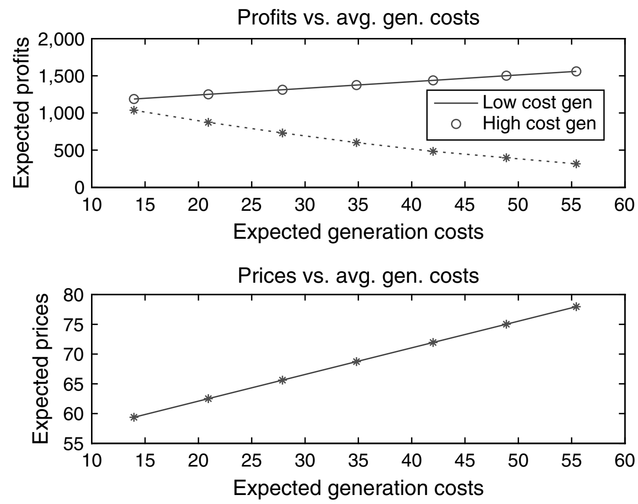
Table 7. Model parameters (base case).

a_ω^s	100
m_ω^s	1
a^f	100
m^f	1

Table 8. Generator details (base case).

i	C_i	c_i	d_i
1	20	0	0
2	50	10	1
3	60	12	1
4	70	14	1
5	80	16	1
6	90	18	1

Figure 1. Variability of equilibrium profits with costs.



scenario as a reference point, we examine the behavior of the market in a variety of settings.

Table 8 provides base case details of each of the six generators. In particular, the first generator has lower capacity of 20 MW with negligible operating costs and can be likened to a wind-based generator. The other five generators have quadratic costs of generation with a capacity levels ranging from 50 to 90 MW.

▷ *Base-case* (*Table* 9): In the base case, there is precisely one scenario in the future, implying that there is no uncertainty. The wind generator generates at maximum capacity while generators 2 and 3 also generate close to their capacity. Generators 4–6, however, do not use their entire capacity. In fact, as capacity increases in the face of modest cost increases, firms tend to have higher sales. Yet, as costs increase significantly, as seen with generators 4–6, generation levels are depressed. Because the transmission constraints are slack, the nodal prices are identical across the network.

▷ *High fuel prices* (*Table* 10): Next, we consider a setting where agents compete with the possibility of high fuel prices in the future. We assume that generator 6 has the highest proportion of fuel-fired generation while generator 2 has the lowest, and we further assume that the increase in costs are perfectly correlated for each generator, with the actual value of the increase being specified by a multiplier. The reference level of the random linear cost in the spot-market for each of the 20 scenarios is determined by a normally distributed random variable of mean zero and variance one. Using the previously specified multiplier, the corresponding firm-specific generation costs are determined for each scenario.

Expectedly, the price of electricity rises as generation is suppressed. Firm 1's profits increase because its costs are unchanged as a wind-generator but reaps the benefits of higher prices. We examine this prospect further by examining the impact of increasing costs on the profits and provide a schematic in Figure 1. Interestingly, as the costs increase even further, low-cost generators see a steady ascent in profits at the expense of high-cost generators. From Figure 1, it can also be seen that as costs become even larger, the price increases toward the maximum price while the market participation keeps reducing correspondingly.

▷ *Higher expected availability* (*Table* 11): If the capacity is assumed to be random in the spot-market with higher

Table 9. Base case.

i	Expected linear cost	Expected profits	Expected sales	Expected availab.	Price
1	0.00	1188.06	20.00	20.00	59.40
2	10.00	1219.54	48.14	50.00	59.40
3	12.00	1123.05	46.43	60.00	59.40
4	14.00	1030.48	44.72	70.00	59.40
5	16.00	941.83	43.00	80.00	59.40
6	18.00	857.10	41.29	90.00	59.40

Table 10. High fuel prices.

i	Expected linear cost	Expected profits	Expected sales	Expected availab.	Price
1	0.00	1,437.50	20.00	20.00	71.88
2	30.35	863.55	39.61	50.00	71.88
3	35.73	654.62	35.00	60.00	71.88
4	41.08	477.11	30.42	70.00	71.88
5	48.44	276.39	24.10	80.00	71.88
6	53.67	168.71	19.62	90.00	71.88

Table 11. High expected availability with quadratic costs.

i	Expected linear cost	Expected profits	Expected sales	Expected availab.	Price
1	0.00	2,803.94	49.82	49.82	56.29
2	10.00	1,071.22	45.92	80.12	56.29
3	12.00	980.71	44.21	89.91	56.29
4	14.00	894.12	42.49	99.69	56.29
5	16.00	811.44	40.78	110.15	56.29
6	18.00	732.69	39.06	119.89	56.29

Table 12. High expected availability with zero quadratic costs

i	Expected linear cost	Expected profits	Expected sales	Expected availab.	Price
1	0.00	934.30	49.82	49.82	18.76
2	10.00	613.88	70.12	70.12	18.76
3	12.00	539.85	79.91	79.91	18.76
4	1.00	1,592.52	89.69	89.69	18.76
5	4.00	1,477.75	100.15	100.15	18.76
6	16.00	269.65	97.78	109.89	18.76

Table 13. Highly constrained transmission lines.

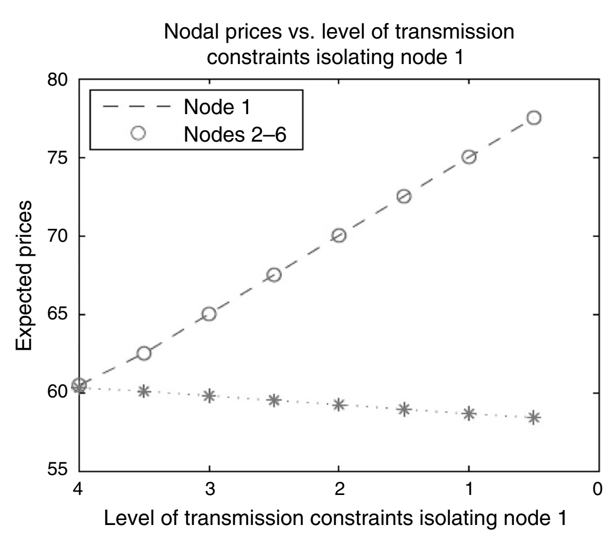
i	Expected linear cost	Expected profits	Expected sales	Expected availab.	Price
1	0.00	1,550.00	20.00	20.00	77.50
2	10.00	1,152.76	40.00	40.00	58.82
3	12.00	1,047.19	44.72	50.00	57.78
4	14.00	974.76	43.34	60.00	58.16
5	16.00	904.89	41.95	70.00	58.55
6	18.00	830.57	40.42	80.00	58.76

expected availability, the results are found to be interesting in that they do not appear to result in uniformly higher profits. Clearly, increased profits would be expected if capacity constraints are tight, as in the case of the wind-generator (generator 1), and that is observed here. However, in the case of generator 6, the expected sales increase by nearly 30 units from the base case, with a steep decline in profits. In effect, incremental generation from the base case comes at a significant loss.

A possible answer lies in the quadratic costs of generation. Incremental generation, despite its availability, comes at a significant loss. To ascertain if quadratic costs do contribute, we recomputed the equilibria under the assumption of zero quadratic costs, we find that the generation levels are at capacity—essentially the quadratic costs were keeping generation suppressed (see Table 12). Another possible explanation might be found in noting that increasing the availability leads to more intense competition. This manifests in higher generation and correspondingly lower prices and profits.

▷ *Highly constrained transmission lines* (*Table* 13): Constrained transmission lines are often pointed as being responsible for high nodal prices. In fact, if transmission capacities of lines leading to node 1 are assumed to reduce from 4 units to 0.5 units, then (see Figure 2) it can be seen that while nodal prices are very similar for higher levels of transmission levels, at lower levels of transmission the nodal price at node 1 jumps by more than 20 units as this node is effectively isolated. In fact, the price at that node is a directly related to available generation capacity at that node. Table 13 provides a summary of expected profits under the setting that capacity levels are 0.5. Interestingly, the isolated generator garners nearly 30% higher profits than the base case in a constrained transmission setting.

Figure 2. Sensitivity of nodal prices.



Insights from the model: Finally, some insights are provided from a six-node electricity market model with uncertain spot-prices, costs and capacities:

- In a regime of high fuel prices, firms with low-cost generation garner profits at the expense of firms with higher costs. As costs keep increasing, prices increase and participation falls.
- Higher expected availability does not manifest itself in increased profits, partially because increased generation is less profitable owing to quadratic costs and possibly from more intense competition. If the quadratic cost of generation is reduced to zero, we do observe that not only do firms make more profits, they generate at full capacity.
- Constrained transmission lines lead to price differences across the network. Particularly interesting is the increase in profits arising from high prices seen in nodes that have reduced access to the rest of the network.

6. Contributions and Future Research

This paper is motivated by the challenges in both proving the existence of a Nash-Stackelberg equilibrium in an uncertain multiperiod setting as well as by question of developing scalable convergent algorithms for obtaining such points. In this section, we summarize some of the main thrusts of our work.

1. We consider a spot-forward equilibrium problem in which agents compete in the forward-market subject to equilibrium in the spot-market. Additionally, we work under the setting that the forward prices are specified by an affine price function, and this is enforced by adding a linear risk-neutrality constraint to each agent's problem. This could be viewed as a consequence of a belief of how forward prices are set. Alternately, one could also view such a constraint as being a conjecture on forward price functions. In the current context, preliminary numerical tests show that equilibrium profits vary slightly between the conjectured and original models and suggest that these differences are relatively invariant to price function intercepts and uncertainty.

2. We construct a simultaneous stochastic Nash-equilibrium (SSNE) problem whose solution is shown to be a local Nash equilibrium of the conjectured version of the original multi-leader multi-follower game. Furthermore, we show that the SSNE always exists and is characterized by a complementarity problem with a $\mathbf{P}_0$ mapping in both two-node and more general networked settings.

3. The SSNE may be obtained as a solution to a stochastic mixed-complementarity problem. A *scalable* matrix splitting algorithm for solving large-scale stochastic problems is presented along with global convergence theory. Preliminary computational tests show that computational effort grows linearly with the size of the underlying distribution. Also, sampled variants of the algorithms are often seen to perform better. Further tests show that the number of major and minor iterations do not grow significantly with the size of the network.

4. We use our model to derive insights from a six-node spot-forward electricity market in which costs, prices, and capacities are uncertain in the second period. We observe that higher expected availability appears to result in higher profits in settings where the quadratic costs are modest. Furthermore, when transmission lines to a particular node or zone are constrained, it is seen that both prices and profits rise steeply in that region.

Endnotes

1. Note that throughout the paper we use the terms *firms*, *players*, and *generators* interchangeably. The latter in particular is employed when referring to physical production in the real-time market.
2. This calculation is obtainable through some simple algebra.

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