On the Theoretical Comparison of Low-Bias Steady-State Estimators

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The time-average estimator is typically biased in the context of steady-state simulation, and its bias is of order $1/t$, where t represents simulated time. Several "low-bias" estimators have been developed that have a lower order bias, and, to first-order, the same variance of the time-average. We argue that this kind of first-order comparison is insufficient, and that a second-order asymptotic expansion of the mean square error (MSE) of the estimators is needed. We provide such an expansion for the time-average estimator in both the Markov and regenerative settings. Additionally, we provide a full bias expansion and a second-order MSE expansion for the Meketon–Heidelberger low-bias estimator, and show that its MSE can be asymptotically higher or lower than that of the time-average depending on the problem. The situation is different in the context of parallel steady-state simulation, where a reduction in bias that leaves the first-order variance unaffected is arguably an improvement in performance.

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1. INTRODUCTION

Let $X = (X(t) : t \geq 0)$ be a real-valued stochastic process, in which $X(t)$ represents the output of a simulation at (simulated) time t. Suppose that X has a steady-state, in the sense that there exists a (deterministic) constant α such that

$$\frac{1}{t}\int_0^t X(s)ds \Longrightarrow \alpha \tag{1}$$

as $t \to \infty$, where $\Longrightarrow$ denotes weak convergence. The quantity α is known as the steady-state mean of X (also known as the "time-average limit" of X), and the problem of computing α via simulation is called the steady-state simulation problem.

The law of large numbers (LLN) limit theorem (1) asserts that the time-average

$$\alpha(t) \triangleq \frac{1}{t}\int_0^t X(s)ds$$

is a consistent simulation-based estimator for α. One of the principal challenges in steady-state simulation is dealing with the fact that $\alpha(t)$ is typically a biased estimator of α. This bias is induced by initial transient effects associated with the initialization of X at time $t = 0$ using a distribution that is atypical of steady-state behavior (e.g., initializing a queue in the empty state).

In great generality, it can be shown that there exists a constant b for which

$$\mathrm{E}\,\alpha(t) = \alpha + \frac{b}{t} + o(1/t) \tag{2}$$

as $t \to \infty$, where $o(1/t)$ represents a function for which $t\,o(1/t) \to 0$ as $t \to \infty$; see, for example, Proposition 2.1 below. An estimator $\widetilde{\alpha}(t)$ is therefore said to be a "low-bias estimator" if

$$\mathrm{E}\,\widetilde{\alpha}(t) = \alpha + o(1/t) \tag{3}$$

as $t \to \infty$. Such low-bias estimators are believed to enjoy superior small-sample performance relative to the time-average $\alpha(t)$ (for small to moderate values of t).

Of course, the performance of an estimator depends on more than its bias. For example, $\mathrm{var}\,\widetilde{\alpha}(t)$ plays a key role in the large-sample behavior of the estimator. As a consequence, comparing the variance of $\widetilde{\alpha}(t)$ to that of $\alpha(t)$ must clearly enter into any theoretical analysis of a low-bias estimator's performance. It is known, in great generality, that

$$\mathrm{var}\;\alpha(t) = \frac{\sigma^2}{t} + o(1/t)$$

as $t \to \infty$; the quantity σ^2 is called the time-average variance constant. Hence, if it can be established that

$$\mathrm{var}\;\widetilde{\alpha}(t) = \frac{\sigma^2}{t} + o(1/t)$$

as $t \to \infty$, it follows that $\widetilde{\alpha}(t)$ (in the presence of (2) and (3)) has better bias behavior than does $\alpha(t)$, with no asymptotic degradation of variance. On this basis, and provided the cost of producing both estimators is the same, one might conclude one's theoretical comparison of $\widetilde{\alpha}(t)$ to $\alpha(t)$ by asserting $\widetilde{\alpha}(t)$'s superiority. This type of reasoning has implicitly appeared in several theoretical treatments of low-bias estimators. A key contribution of this article is to show that this analysis may be oversimplified: the question of theoretical comparison is more nuanced.

When the cost of producing two estimators is the same, a natural measure of an estimator's efficiency is its mean square error (MSE). But, under the bias and variance expansions above, the MSE of $\widetilde{\alpha}$ is the same as that of α up to order $1/t$, so that an efficiency criterion based on "first order" asymptotics will declare them equivalent; for example, α and $\widetilde{\alpha}$ typically have the same *asymptotic efficiency* in the framework of Glynn and Whitt [1992]. This suggests that comparing two such estimators may require a second order asymptotic analysis. We shall show in this paper that the variance of the time average typically enjoys a second order variance expansion of the form

$$\operatorname{var} \alpha(t) \;=\; \frac{\sigma^2}{t} + \frac{\nu}{t^2} + o(1/t^2),$$

whereas

$$\operatorname{var} \widetilde{\alpha}(t) \;=\; \frac{\sigma^2}{t} + \frac{\kappa}{t^2} + o(1/t^2).$$

Hence, it follows that

$$\operatorname{MSE}(\alpha(t)) \;=\; \frac{\sigma^2}{t} + \frac{\nu + b^2}{t^2} + o(1/t^2),$$

whereas

$$\operatorname{MSE}(\widetilde{\alpha}(t)) \;=\; \frac{\sigma^2}{t} + \frac{\kappa}{t^2} + o(1/t^2).$$

Thus, a low-bias estimator has asymptotically smaller mean square error than the time-average estimator if and only if $\kappa < \nu + b^2$. Hence, any comprehensive theoretical analysis of a low-bias estimator must necessarily include a development of its second order variance expansion. To build an appropriate theoretical framework for such comparisons, we compute the second order variance and MSE expansions for the most widely used steady-state estimator, namely the time-average. We provide expressions for the constants arising in the second-order expansion in both the Markov process setting and the regenerative setting. The Markov process expressions involve solutions to an equation known as Poisson's equation, whereas the regenerative expressions involve moments of certain random variables defined in terms of regenerative cycles. To assess how an alternative estimator performs against the time-average, one can then develop its MSE expansion, and compare it against the expressions given here. To illustrate our theory, we perform such a comparison for the Meketon–Heidelberger estimator (a widely studied low-bias estimator). Our theory shows that the Meketon–Heidelberger estimator neither dominates nor is dominated by the time-average estimator, in the sense of mean square error. Thus, no universal recommendation can be reached with regard to use of the Meketon–Heidelberger estimator vis-à-vis the time average in the single replications context. While we do not attempt here to perform similar comparisons for the various other low-bias estimators present in the literature, we find no a priori reason to believe that any of them would be universally superior to the time average; see the discussion at the end of Section 4.

In contrast with the above, a first order variance expansion may suffice if the estimator is to be used in a multiple replications/parallel simulation setting. In the parallel simulation setting, use of the Meketon–Heidelberger estimator (and other low-bias estimators) can be advantageous from a completion time viewpoint. We discuss this issue in Section 5.

The key contributions of this article are:

(1) A theoretical framework for estimator comparison.
(2) Second-order variance and MSE expansions for the time-average in both the Markov and regenerative settings; see Theorem 2.8 and Theorem 3.1.
(3) A second-order asymptotic analysis for the Meketon–Heidelberger estimator and its closely related variant (involving averaging over the first N(t) cycles); see Theorems 4.1, 4.2 and 4.5.
(4) Analysis in the multiple replication/parallel simulation setting; see Section 5.

2. ASYMPTOTIC EXPRESSIONS FOR THE TIME-AVERAGE ESTIMATOR IN THE MARKOV SETTING

As noted in the Introduction, the most fundamental of all steady-state estimators is the time-average estimator. We therefore focus, in the next two sections, on providing expressions for the bias, variance, and mean square error of this estimator. Our particular emphasis, in this section, will be on deriving the appropriate expressions in the Markov process setting. As has been argued elsewhere (see, e.g., Glynn [1989] and Henderson and Glynn [2001]), the typical discrete-event simulation can be viewed as a Markov process (by appending suitable supplementary variables to the state descriptor). Consequently, the Markov assumption can be viewed as holding quite generally from an applications standpoint.

We start by obtaining a bias expansion for $\alpha(t)$ that holds without any Markov hypothesis whatsoever.

PROPOSITION 2.1. *Suppose that* $X = (X(t) : t \geq 0)$ *satisfies* $\mathrm{E}\, X(t) = \alpha + o(t^{-p})$ *as* $t \to \infty$, *where* $p > 1$. *If* $\sup_{t \geq 0} \mathrm{E}\, |X(t)| < \infty$, *then*

$$\mathrm{E}\, \alpha(t) = \alpha + \frac{b}{t} + o(t^{-p})$$

as $t \to \infty$, *where* $b = \int_0^\infty [\mathrm{E}\, X(t) - \alpha] dt$. *More generally, if* $\int_0^\infty |\mathrm{E}\, X(t) - \alpha| dt < \infty$, *then*

$$\mathrm{E}\, \alpha(t) = \alpha + \frac{b}{t} + o(t^{-1})$$

as $t \to \infty$.

PROOF. For the first part, note that because $\sup_{t\geq 0} \mathrm{E}\,|X(t)| < \infty$, Fubini's theorem applies, yielding

$$\begin{aligned} t(\mathrm{E}\,\alpha(t) - \alpha) &= \mathrm{E} \int_0^t [X(s) - \alpha] ds \\ &= \int_0^t \mathrm{E}\,[X(s) - \alpha] ds. \end{aligned}$$

But

$$\begin{aligned} \int_0^t \mathrm{E}\,[X(s) - \alpha] ds &= b - \int_t^\infty \mathrm{E}\,[X(s) - \alpha] ds \\ &= b + o(t^{1-p}) \end{aligned}$$

as $t \to \infty$, proving the result. The proof of the second part is identical, except that $\int_t^\infty \mathrm{E}\,[X(s) - \alpha] ds$ is only known to be $o(1)$ as $t \to \infty$. □

Many Markov processes converge to their steady-state exponentially fast; see for example, Brémaud [1999, Theorem 4.2.6], Stroock [2005, Section 5.3], Meyn and Tweedie [1993, Chapter 15] and Down et al. [1995] for sufficient conditions in various settings. In that case, $\mathrm{E}\,X(t) = \alpha + o(t^{-p})$ as $t \to \infty$, for every $p \geq 1$. Proposition 2.1 therefore implies that in any bias expansion of the form

$$\mathrm{E}\,\alpha(t) = \alpha + \sum_{j=1}^{l} b_j t^{-j} + o(t^{-l})$$

as $t \to \infty$, all the high-order bias coefficients must therefore vanish (i.e., $0 = b_2 = b_3 = \cdots$).

We now show that the variance of a typical stationary process exhibits a similar expansion.

PROPOSITION 2.2. *Suppose that* $X = (X(t) : t \geq 0)$ *is a square-integrable stationary process for which* $\mathrm{cov}\,(X(0), X(t)) = o(t^{-p})$ *as* $t \to \infty$*, where* $p > 2$*. Then,*

$$\mathrm{var}\ \alpha(t) = \frac{\sigma^2}{t} + \frac{\nu}{t^2} + o(t^{-p})$$

as $t \to \infty$*, where*

$$\begin{aligned} \sigma^2 &= 2 \int_0^\infty \mathrm{cov}\,(X(0), X(t)) dt, \\ \nu &= -2 \int_0^\infty t\, \mathrm{cov}\,(X(0), X(t)) dt. \end{aligned}$$

PROOF. Observe that

$$\begin{aligned} t^2 \mathrm{var}\ \alpha(t) &= 2 \int_0^t \int_s^t \mathrm{cov}\,(X(s), X(u))\, du\, ds \\ &= 2 \int_0^t \int_s^t \mathrm{cov}\,(X(0), X(u-s)) du\, ds \end{aligned}$$

$$
\begin{aligned}
&= 2\int_0^t \int_0^{t-s} \mathrm{cov}\,(X(0), X(r))dr\,ds \\
&= 2\int_0^t (t-r)\mathrm{cov}\,(X(0), X(r))dr \\
&= t\left[\sigma^2 - 2\int_t^\infty \mathrm{cov}\,(X(0), X(r))dr\right] \\
&\quad + \left[\nu + 2\int_t^\infty r\,\mathrm{cov}\,(X(0), X(r))dr\right] \\
&= t\Big[\sigma^2 + o(t^{1-p})\Big] + \Big[\nu + o(t^{2-p})\Big]
\end{aligned}
$$

as $t \to \infty$. □

Thus, the second order term νt^{-2} appears in the expansion of the variance of $\alpha(t)$ even in the setting of a stationary stochastic process (in which no initial transient exists). As for the bias expansion, note that if $\mathrm{cov}\,(X(0), X(t)) = o(t^{-p})$ as $t \to \infty$ for each $p \geq 2$ (as would typically occur for a Markov process with exponentially rapid "mixing"), any variance expansion of the form

$$
\mathrm{var}\ \alpha(t) = \frac{\sigma^2}{t} + \sum_{j=1}^{\ell} \nu_j t^{-1-j} + o(t^{-1-\ell})
$$

as $t \to \infty$ must satisfy $0 = \nu_2 = \nu_3 = \cdots$.

To explore the form of these expressions in the presence of an initial transient, we next derive their form in the Markov process setting. We assume that the output process $X = (X(t) : t \geq 0)$ can be represented as a real-valued functional of a Markov process i.e. $X(t) = f(W(t))$, where $W = (W(t) : t \geq 0)$ is an S-valued (continuous-time) Markov process and $f : S \to \mathbb{R}$ is the performance measure of interest.

We wish to provide a sufficient condition for the validity of our asymptotic expression that can be verified directly in terms of the "building blocks" of the stochastic model being simulated. As is standard in the recurrence literature for Markov processes, we state the condition in terms of a Lyapunov criterion. In continuous time, the criterion is expressed in terms of the infinitesimal transition structure of the process, as specified through the so-called "extended generator".[1]

We follow Meyn and Kontoyiannis [2003] in defining the generator as follows:

Definition 2.3. We say that $\psi : S \to \mathbb{R}$ is in the domain of the extended generator of W if there exists a function $\varphi : S \to \mathbb{R}$ such that for $x \in S$ and $t \geq 0$,

$$
E_x \left| \int_0^t \varphi(W(s))ds \right| < \infty
$$

[1]Readers unfamiliar with the theory of Markov processes in general state spaces may want, on first reading, to restrict attention to the continuous time Markov chain (CTMC) case; this can be done by skipping directly to Proposition 2.5, which can be used as a surrogate for hypothesis A1, and interpreting $\widetilde{A}$ in Theorem 2.8 as the rate matrix of the CTMC.

(where $\mathrm{E}_x(\cdot) \triangleq \mathrm{E}(\cdot | W(0) = x)$) and

$$\psi(W(t)) - \int_0^t \varphi(W(s)) ds$$

is a martingale (adapted to W), conditional on $W(0) = x$. In this case, we write $\varphi = \widetilde{A}\psi$ and call $\widetilde{A}$ the extended generator of W.

See Breiman [1968] for a discussion of martingales. To make the above definition more concrete, consider the case in which W is a continuous-time Markov chain (CTMC).

PROPOSITION 2.4. *Suppose that W is a CTMC living on discrete state space S and having rate matrix $A = (A(x, y) : x, y \in S)$. Assume that there exists $\widetilde{V} : S \to [1, \infty)$ such that*

$$\begin{aligned} ||A||_{\widetilde{V}} &\triangleq \sup_{x \in S} \sum_y |A(x, y)| \widetilde{V}(y) / \widetilde{V}(x) \\ &< \infty. \end{aligned}$$

Then W is nonexplosive and the domain of the extended generator $\widetilde{A}$ includes $\{\psi : ||\psi||_{\widetilde{V}} < \infty\}$, where $||\psi||_{\widetilde{V}} \triangleq \sup\{|\psi(x)|/\widetilde{V}(x) : x \in S\}$. Furthermore, $\widetilde{A} = A$ when restricted to such functions.

PROOF. Note that

$$|A(x, x)| \leq \sum_y |A(x, y)| \frac{\widetilde{V}(y)}{\widetilde{V}(x)} \leq ||A||_{\widetilde{V}},$$

so A is a uniformizable rate matrix. As a consequence, W is nonexplosive (see, e.g., Resnick [1992]). Furthermore, the transition probabilities for such rate matrices can be represented as

$$\mathrm{P}(W(t) = y | W(0) = x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n(x, y).$$

Hence,

$$\begin{aligned} \mathrm{E}_x |\psi(W(t))| &\leq \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_y |A^n(x, y)| \, |\psi(y)| \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_y |A^n(x, y)| \frac{\widetilde{V}(y)}{\widetilde{V}(x)} \frac{|\psi(y)|}{\widetilde{V}(y)} \widetilde{V}(x) \\ &\leq \sum_{n=0}^{\infty} \frac{t^n}{n!} ||A^n||_{\widetilde{V}} ||\psi||_{\widetilde{V}} \widetilde{V}(x). \end{aligned}$$

But $||A||_{\widetilde{V}}$ is a matrix (operator) norm, so $||A^n||_{\widetilde{V}} \leq (||A||_{\widetilde{V}})^n$. It follows that

$$\mathrm{E}_x |\psi(W(t))| \leq \widetilde{V}(x) ||\psi||_{\widetilde{V}} \exp(t ||A||_{\widetilde{V}}).$$

Similarly,

$$\mathrm{E}_x |A\psi(W(t))| \leq \widetilde{V}(x) ||A||_{\widetilde{V}} ||\psi||_{\widetilde{V}} \exp(t ||A||_{\widetilde{V}}).$$

So,

$$\psi(W(t)) - \int_0^t (A\psi)(W(s))ds$$

is integrable for all $t \geq 0$. It follows that

$$\psi(W(t)) - \int_0^t (A\psi)(W(s))ds$$

is not only a local martingale (see, e.g., Rogers and Williams [1994]) but is actually a martingale, proving the result. □

Returning to the general Markov process setting, we assume (for technical reasons) that S is a complete separable metric space (so that discrete state spaces and open and closed subsets of $\mathbb{R}^d$ are special cases), and that W has right-continuous paths with left limits. The key role played by the following hypothesis in Markov process theory is described in detail by Down et al. [1995].

A1. Suppose that W is a nonexplosive Markov process for which there exists a nonempty subset $K \subseteq S$, a probability distribution ϕ on S, positive constants λ, β, b, and c and $V : S \to [1, \infty)$ in the domain of the extended generator $\widetilde{A}$ such that

$$(\widetilde{A}V)(x) \leq -\beta V(x) + cI(x \in K)$$

for $x \in S$ (where $I(\cdot \in K)$ is the indicator function of the set K),

$$\mathrm{P}(W(b) \in \cdot \,|W(0) = x) \geq \lambda\phi(\cdot)$$

for $x \in K$, and

$$\mathrm{P}(W(h) \in K \,|W(0) = x) > 0$$

for $h \geq b$ and $x \in K$.

(An introductory discussion of Foster–Lyapunov criteria for discrete-time Markov chains on discrete state spaces, and its connection to martingale theory for Markov randon walks, can be found in Brémaud [1999]; most of the ideas carry over to the continuous time, general state space case, albeit with more technicalities.)

In order to render the assumption more concrete from an applications standpoint, we specialize the hypothesis to the CTMC setting.

Proposition 2.5. *Suppose that W is a CTMC living on discrete state space S and having irreducible rate matrix $A = (A(x, y) : x, y \in S)$. Assume that A is uniformizable (i.e., $\inf_{x\in S} A(x, x) > -\infty$) and that there exists a finite subset $K \subseteq S$, positive constants β, and c, and $V : S \to [1, \infty)$ for which*

$$(AV)(x) \leq -\beta V(x) + cI(x \in K)$$

for $x \in S$. Then W, β, c, and V satisfy A1.

The hypothesis A1 guarantees that W is a Markov process that converges to its steady-state exponentially rapidly.

PROPOSITION 2.6. *Suppose that W satisfies A1 and that there exists a positive constant a for which $|f(x)| \le aV(x)$ for $x \in S$. Then, W has a unique stationary distribution π and there exists $r > 0$ such that*

$$\mathrm{E}_x f(W(t)) = \int_S \pi(dx) f(x) + o(\exp(-rt))$$

as $t \to \infty$.

For the proof, see Down et al. [1995]. To get a sense of how this result can be applied, we consider the $M/M/1$ queue.

Example 2.7. Suppose that W is the queue-length process in an $M/M/1$ queue model with arrival rate λ and service rate μ for which $0 < \lambda < \mu$. For $0 < \theta < \log(\mu/\lambda)$, put $V(x) = \exp(\theta x)$. Then, W satisfies A1 with $K = \{0\}$, and V as given. So, for each $k \ge 1$, there exists $r > 0$ such that

$$\mathrm{E}_x W(t)^k = \sum_{\ell=0}^{\infty} \ell^k (1-\rho)\rho^\ell + o(\exp(-rt))$$

as $t \to \infty$, where $\rho \triangleq \lambda/\mu$.

We are now ready to state our main theorem of this section. Let $W^* = (W^*(t) : t \ge 0)$ be a stationary version of W, put $f_c(x) = f(x) - \int_S \pi(dx) f(x)$ and set

$$\begin{aligned}
u(x) &\triangleq \int_0^\infty \mathrm{E}_x f_c(W(t))dt,\\
v(x) &\triangleq \int_0^\infty \mathrm{E}_x u(W(t))dt,\\
\sigma^2 &\triangleq 2\mathrm{E}\, f_c(W^*(0))u(W^*(0)),\\
w(x) &\triangleq \int_0^\infty (\mathrm{E}_x f_c(W(t))u(W(t)) - \sigma^2/2)dt.
\end{aligned}$$

THEOREM 2.8. *Suppose that W satisfies A1 and that there exists $a > 0$ such that $|f(x)| \le aV(x)^{1/2}$ for $x \in S$. Then, there exists a unique stationary distribution π. Furthermore,*

(a)

$$\begin{aligned}
\int_0^\infty |\mathrm{E}_x f_c(W(t))|dt &< \infty,\\
\int_0^\infty |\mathrm{E}_x u(W(t))|dt &< \infty,\\
\mathrm{E}\,|f_c(W^*(0))u(W^*(0))| &< \infty,\\
\int_0^\infty |\mathrm{E}_x f_c(W(t))u(W(t)) - \sigma^2/2|dt &< \infty.
\end{aligned}$$

(b) *Each of the functions u, v and w lies in the domain of the extended generator $\widetilde{A}$ of W, and*

$$\widetilde{A}u = -f_c,$$

$$\widetilde{A}v = -u,$$

$$\widetilde{A}w = -(f_c u - \sigma^2/2).$$

(c) *There exists* $r > 0$ *such that*

$$\begin{aligned} \mathrm{E}_x \alpha(t) &= \alpha + \frac{1}{t} u(x) + o(\exp(-rt)), \\ \mathrm{var}\,[\alpha(t) | W(0) = x] &= \frac{\sigma^2}{t} + \frac{\nu}{t^2} + \frac{1}{t^2}(-u^2(x) + 2w(x)) + o(\exp(-rt)), \\ \mathrm{E}_x(\alpha(t) - \alpha)^2 &= \frac{\sigma^2}{t} + \frac{\nu}{t^2} + \frac{2}{t^2} w(x) + o(\exp(-rt)) \end{aligned}$$

as $t \to \infty$, *where* $\nu = -2\mathrm{E}\, f_c(W^*(0)) v(W^*(0))$.

(d) *In addition,*

$$\begin{aligned} \sigma^2 &= 2 \int_0^\infty \mathrm{E}\, f_c(W^*(0)) f_c(W^*(t)) dt, \\ \nu &= -2 \int_0^\infty t \mathrm{E}\, f_c(W^*(0)) f_c(W^*(t)) dt. \end{aligned}$$

This result extends to nonstationary Markov processes the theory developed earlier in this section in the stationary process setting. The constant ν appears in the mean square expansion in both the Markov and stationary settings; the impact of the initial transient manifests itself in the $w(x)$ and $u(x)$ coefficients.

3. ASYMPTOTIC EXPRESSIONS FOR THE TIME-AVERAGE ESTIMATOR IN THE REGENERATIVE SETTING

In this section, we develop expressions for the bias, variance, and mean square error of the time-average estimator in the regenerative setting. In the context of steady-state simulation, the regenerative assumption holds in significant generality (see, e.g., Glynn [1994]). In particular, discrete-event simulations typically possess a regenerative structure [Glynn 1989]—although this structure can only be exploited in practice within the smaller class of processes for which the regeneration epochs can be (easily) identified from the simulation output (see Henderson and Glynn [2001, 2003] for a discussion).

Suppose $X = (X(t) : t \geq 0)$ is a real-valued nondelayed classically regenerative process, with regeneration times $T(0) = 0 < T(1) < T(2) < \ldots$. Let $(\tau_i : i \geq 1)$ be the cycle lengths, $\tau_i \triangleq T(i) - T(i-1)$, and $N = \big(N(t) : t \geq 0\big)$ the associated counting process, $N(t) = \sup\{i : T(i) \leq t\}$.

If $\mathrm{E}\,\tau_1 < \infty$ and $\mathrm{E} \int_0^{\tau_1} |X(s)| ds < \infty$, then X has a steady state, and

$$\alpha(t) = \frac{1}{t} \int_0^t X(s) ds \longrightarrow \alpha = \lambda \mathrm{E} \int_0^{\tau_1} X(s) ds < \infty,$$

where $\lambda \triangleq 1/\mathrm{E}\,\tau_1$.

For technical reasons, we assume the cycle length distribution is *spread-out*, that is, that there exists $m > 0$ and a nonnegative function g satisfying

$\int g > 0$ and $\mathrm{P}\,(\tau_1 + \tau_2 + \cdots + \tau_m \in A) \geq \int_A g$ (see, e.g., Thorisson [2000, p. 98]). In particular, any distribution that has a density is spread out.

To state the main result of this section, it is convenient to introduce some notation. Put $X_c(t) \triangleq X(t) - \alpha$, $t \geq 0$, and let

$$\beta(t) \triangleq \int_0^t X_c(s)ds, \qquad Z_i \triangleq \int_{T(i-1)}^{T(i)} X_c(s)ds \qquad \text{and } \overline{Z}_i \triangleq \int_{T(i-1)}^{T(i)} |X_c(s)|ds,$$

$i \geq 1$.

THEOREM 3.1. *Assume τ_1 spread out, $\mathrm{E}\,\tau_1^{p+2} < \infty$, $\mathrm{E}\,Z_1^2 < \infty$ and $\mathrm{E}\,\tau_1^{p+2}\overline{Z}_1^2 < \infty$, where $p > 0$. Then*

$$\begin{aligned} \mathrm{E}\,\alpha(t) &= \alpha + \frac{\gamma}{t} + o\big(t^{-(p+2)}\big), \\ \operatorname{var}\alpha(t) &= \frac{\sigma^2}{t} + \frac{\eta - \gamma^2}{t^2} + o\big(t^{-(p+2)}\big), \\ \mathrm{E}\,(\alpha(t) - \alpha)^2 &= \frac{\sigma^2}{t} + \frac{\eta}{t^2} + o\big(t^{-(p+2)}\big), \end{aligned} \tag{4}$$

where $\sigma^2 \triangleq \lambda \mathrm{E}\,Z_1^2$, $\gamma \triangleq \lambda \mathrm{E}\,[\int_0^{\tau_1} \beta(s)ds - Z_1\tau_1]$ and $\eta = (\lambda^2/2)\mathrm{E}\,Z_1^2 \mathrm{E}\,\tau_1^2 - \lambda \mathrm{E}\,Z_1^2\tau_1 + 2\lambda^2(\mathrm{E}\,Z_1\tau_1)^2 + \lambda \mathrm{E}\,\int_0^{\tau_1} \beta^2(s)ds - 2\lambda^2 \mathrm{E}\,Z_1\tau_1 \mathrm{E}\,\int_0^{\tau_1} \beta(s)ds$.

For many regenerative processes of interest τ_1 and $\overline{Z}_1$ have finite exponential moments, that is, $\mathrm{E}\,\exp(\theta(\tau_1 + \overline{Z}_1)) < \infty$ for some $\theta > 0$. In that case, Theorem 3.1 holds for all $p > 2$, so any bias expansion for $\alpha(t)$ of the form

$$\mathrm{E}\,\alpha(t) = \alpha + \sum_{\ell=1}^{k} \gamma_\ell t^{-\ell} + o(t^{-k})$$

satisfies $0 = \gamma_2 = \gamma_3 = \cdots$ (just as was seen in Section 2 for Markov processes that converge to steady state exponentially fast). Similarly, in the presence of exponential moments, the variance and MSE expansions for $\alpha(t)$ have zero coefficients for all terms except those in t^{-1} and t^{-2}.

4. MEAN-SQUARE ERROR ANALYSIS OF THE MEKETON–HEIDELBERGER ESTIMATOR

In this section, we turn our attention to the low-bias estimator introduced by Meketon and Heidelberger [1982] for regenerative steady-state simulation. The setup and notation are the same as in Section 3. The Meketon–Heidelberger estimator, $\alpha_{MH}(t)$, is defined as the time average after completing the cycle in progress at time t,

$$\alpha_{MH}(t) \triangleq \frac{1}{T(N(t)+1)} \int_0^{T(N(t)+1)} X(s)ds.$$

It was shown by Meketon and Heidelberger [1982] that $\alpha_{MH}(t)$ satisfies

$$\mathrm{E}\,\alpha_{MH}(t) = \alpha + o(1/t).$$

Our first result extends this to a full asymptotic expansion in powers of $1/t$.

THEOREM 4.1. *Assume τ_1 is spread out,* $\mathrm{E}\,\tau_1^{p+q} < \infty$ *and* $\mathrm{E}\,Z_1^2\tau_1^{p+q-1} < \infty$ *for some integer $p \geq 1$ and real $q > 4$. Then*

$$\mathrm{E}\,\alpha_{MH}(t) = \alpha + \sum_{k=1}^{p}(-1)^k \frac{\bar{a}_k}{t^{k+1}} + o\big(t^{-(p+1)}\big),$$

where $\bar{a}_k \triangleq \big(\lambda \mathrm{E}\,Z_1\tau_1^{k+1} - \lambda^2 \mathrm{E}\,Z_1\tau_1\,\mathrm{E}\,\tau_1^{k+1}\big)/(k+1)$.

In previous sections, we saw that the time-average estimator has a bias expansion in powers of $1/t$ in which typically only the coefficient of $1/t$ is non-zero. In contrast, in Theorem 4.1 the coefficient of $1/t$ is zero, but all those of higher-order terms are generally non-zero.

We point out that the $\bar{a}_k$'s can be estimated from the simulation output. In principle, one can use such estimates together with Theorem 4.1 to further reduce the bias of $\alpha_{MH}(t)$, which is potentially useful in the context of parallel simulation (see Section 5), but we do not pursue this here.

Our next result provides second-order variance and MSE expansions for $\alpha_{MH}(t)$.

THEOREM 4.2. *Assume τ_1 is spread out,* $\mathrm{E}\,Z_1^2\,\tau_1^p < \infty$ *and* $\mathrm{E}\,\tau_1^p < \infty$*, for some $p > 5$. Then*

$$\mathrm{var}\,\alpha_{MH}(t) = \frac{\sigma^2}{t} + \frac{\kappa}{t^2} + o(t^{-2}), \qquad \textit{and}$$

$$\mathrm{E}\,(\alpha_{MH}(t) - \alpha)^2 = \frac{\sigma^2}{t} + \frac{\kappa}{t^2} + o(t^{-2}),$$

where $\sigma^2 = \lambda \mathrm{E}\,Z_1^2$ *and* $\kappa = -\lambda^2 \mathrm{E}\,Z_1^2 \mathrm{E}\,\tau_1^2/.2$.

The $o(t^{-2})$ term in these asymptotic expansions is typically of order t^{-3}, even if τ_1 has moments of all order. This is in contrast with the variance and MSE expansions for the time-average obtained in Sections 2 and 3, where the $o(t^{-2})$ term typically decays faster than any power (under appropriate mixing or moment conditions).

It follows from Theorems 4.2 and 3.1 that the coefficient of $1/t^2$ in the MSE expansion for $\alpha_{MH}(t)$ is smaller than the corresponding coefficient in the MSE expansion for $\alpha(t)$ if and only if the following condition holds:

$$\mathrm{E}\,Z_1^2 \mathrm{E}\,\tau_1^2 > -\mathrm{E}\,\tau_1 \mathrm{E} \int_0^{\tau_1} \big(\beta(s)^2 - Z_1^2\big)ds - 2\mathrm{E}\,Z_1\tau_1 \mathrm{E} \int_0^{\tau_1} (Z_1 - \beta(s))ds. \quad (5)$$

Thus, if (5) holds, then $\alpha_{MH}(t)$ has smaller mean squared error than $\alpha(t)$ for all large enough t, and the opposite is true if the inequality is reversed.

Next we provide an example for which the inequality in (5) can hold in either direction, depending on the distribution of the cycle length τ_1. Hence, neither of $\alpha_{MH}(t)$ and $\alpha(t)$ performs universally better than the other.

Example 4.3. Let $V = (V_i : i \geq 1)$ be an i.i.d. sequence of random variables such that $\mathrm{E}\,V_1^2 < \infty$ and $\mathrm{E}\,V_1 = 0$. Suppose that $\mathrm{E}\,|\tau_1|^7 < \infty$ and that V is independent of $\tau = (\tau_i : i \geq 1)$. For $i \geq 1$ and $0 \leq s < \tau_i$, let $X(T(i-1)+s) = V_i$. Then $Z_i = V_i \tau_i$ and $\beta(s) = sV_1$, $0 \leq s \leq \tau_1$, and it is easy to verify that (5) holds if and only if

$$\frac{\mathrm{E}\,\tau_1^2}{\mathrm{E}\,\tau_1} > \frac{2}{3} \cdot \frac{\mathrm{E}\,\tau_1^3}{\mathrm{E}\,\tau_1^2}. \tag{6}$$

By choosing the distribution of τ_1 appropriately, one can make the above inequality hold in either direction. To illustrate, suppose τ_1 can take only two values, a and b, with probabilities $1-\varepsilon$ and ε respectively. If $a = 1$, $b > 2$ and $\varepsilon = 1/2$, then (6) holds, whereas if $a = 1$, $b = 3n$ and $\varepsilon = 1/n^3$, then for large enough n the opposite inequality holds.

Another simple example is $(Z_i : i \geq 1)$ independent of the cycle lengths. In this case, the MSE of $\alpha_{MH}(t)$ is asymptotically smaller than that of $\alpha(t)$:

Example 4.4. Let $V = (V_i : i \geq 1)$ be an i.i.d. sequence of random variables such that $0 < \mathrm{E}\,V_1^2 < \infty$ and $\mathrm{E}\,V_1 = 0$. Assume that $\mathrm{E}\,|\tau_1|^5 < \infty$ and that V is independent of $\tau = (\tau_i : i \geq 1)$. Suppose $Z_i = V_i$ for $i \geq 1$ (e.g., set $X(T(i-1)+s) = V_i/\tau_i$ for $0 \leq s < \tau_i$). Then, $Z_i = V_i$, and it is easy to verify that (5) is equivalent to

$$\mathrm{E}\,V_1^2\,\mathrm{E}\,\tau_1^2 > \mathrm{E}\,V_1^2(\mathrm{E}\,\tau_1)^2 - \mathrm{E}\,\tau_1\,\mathrm{E}\int_0^{\tau_1} \beta(s)^2\,ds$$

which holds as long as $\mathrm{var}(\tau_1) > 0$.

In general, there appears to be no obvious easily verifiable sufficient condition guaranteeing (5). In principle, all the terms in (5) can be estimated via simulation (as they involve expectations of cycle-type random variables), and this may provide a practical way to choose between $\alpha_{MH}(t)$ and $\alpha(t)$ in some situations: for example, if one is simulating a parametric model and wants to estimate the steady-state for many different values of the parameters, then it may be worthwhile to perform one simulation run to estimate the terms in (5), and thus gain insight on whether (5) holds for the class of models considered. In most situations, however, the simulationist will have no simple a priori guarantee as to whether MSE will be reduced by using $\alpha_{MH}(t)$ relative to the time average. This suggests that, to the extent that one wishes to make a theoretical argument for practical use of an estimator such as $\alpha_{MH}(t)$ in the single replication context, such an argument is likely to rest upon a theoretical criterion/analysis other than MSE. The fact that (5) appears difficult to interpret at a practical level further suggests that the argument for practical use of such estimators in the single replication setting is likely to hinge more upon a large-scale empirical comparison of the performance of such estimators across many test problems representative of real examples. Empirical work in this spirit has been performed by Hsieh et al. [2004].

Note that computing $\alpha_{MH}(t)$ requires simulating X to the end of the cycle in progress at time t. This cycle can be (much) longer than a typical cycle, due to length-biasing effects. A natural alternative is to average only over the cycles completed by time t, which is in fact a commonly used variant of the above estimator, and that we denote $\alpha_{CC}(t)$:

$$\alpha_{CC}(t) \triangleq \frac{\int_0^{T(N(t))} X(s)ds}{T(N(t))}.$$

Our next result provides asymptotic expansions for the bias, variance and MSE for this estimator.

THEOREM 4.5. *Assume τ_1 is spread out, $\mathrm{E}\, Z_1^2 \tau_1^p < \infty$ and $\mathrm{E}\, \tau_1^{p+2} < \infty$, for some $p > 4$. Also, assume that τ_1 is bounded away from zero, that is, there exists $\varepsilon_0 > 0$ such that $\mathrm{P}(\tau_1 > \varepsilon_0) = 1$. Then*

$$\begin{aligned} \mathrm{E}\, \alpha_{CC}(t) &= \alpha + \frac{\zeta}{t} + o(t^{-1}), \\ \mathrm{var}\, \alpha_{CC}(t) &= \frac{\sigma^2}{t} + \frac{\xi}{t^2} + o(t^{-2}), \\ \mathrm{E}\, (\alpha_{CC}(t) - \alpha)^2 &= \frac{\sigma^2}{t} + \frac{\xi + \zeta^2}{t^2} + o(t^{-2}), \end{aligned}$$

where $\sigma^2 = \lambda \mathrm{E}\, Z_1^2$, $\zeta = -\lambda \mathrm{E}\, Z_1 \tau_1$ and $\xi = 3\lambda^2 \mathrm{E}\, Z_1^2 \mathrm{E}\, \tau_1^2/2 - \lambda \mathrm{E}\, Z_1^2 \tau_1 + \lambda^2(\mathrm{E}\, Z_1 \tau_1)^2$.

The proof is similar to that of Theorem 4.2, and is omitted for brevity. There is only one minor additional complication, related to the fact that in computing $\alpha_{CC}(t)$ one is dividing by $T(N(t))$, which can take arbitrarily small values; the assumption that τ_1 be bounded below is added to deal with this issue without introducing additional difficulty in the argument. Note that this assumption is immediately satisfied by discrete time regenerative processes, like discrete time Markov chains.

Theorem 4.5 can be used to compare the MSE of α_{CC}(t) to that of $\alpha_{MH}(t)$ or $\alpha(t)$. The same scenarios described in Examples 4.3 and 4.4 can be used to show that its MSE can be asymptotically larger or smaller than that of the previous estimators; it does not perform universally better or worse than either of them.

Various other low-bias estimators have been proposed in the context of regenerative steady-state simulation. Iglehart [1975] provides an early empirical study of four estimators that are known to be low-bias; his work is set, however, in a different time scale (the process is simulated for a fixed number of cycles, rather than up to a fixed simulated time). The estimators he considers are the Fieller estimator [Fieller 1940], Beale estimator [Beale 1962], Tin estimator [Tin 1965] and jackknife estimator [Quenouille 1956; Durbin 1959]. He concludes recommending the jackknife estimator. In the time scale of fixed simulated time (as we have been using here), a jackknife estimator takes the form

$$\alpha_J(t) \triangleq \begin{cases} \alpha(t), & N(t) \le 1; \\ N(t) \frac{\sum_{i=1}^{N(t)} Y_i}{\sum_{i=1}^{N(t)} \tau_i} - \frac{N(t)-1}{N(t)} \sum_{i=1}^{N(t)} \frac{\sum_{j \ne i} Y_i}{\sum_{j \ne i} \tau_i}, & N(t) \ge 2, \end{cases}$$

where $Y_i = \int_{T(i-1)}^{T(i)} X(s)ds$; see Glynn and Heidelberger [1992b]. A version of the Tin estimator in the time scale of simulated time is given by

$$\alpha_2(t) \triangleq \begin{cases} \alpha(t), & N(t) = 0; \\ \alpha(t) + \frac{1}{t^2}\left[\sum_{i=1}^{N(t)} Y_i \tau_i - \alpha(t) \sum_{i=1}^{N(t)} \tau_i^2\right], & N(t) \geq 1; \end{cases}$$

see Glynn and Heidelberger [1990, Theorem 8 and subsequent discussion]. A low-bias estimator inspired on the bias expansion of the time average was introduced by Glynn [1994], and is given by

$$\alpha_3(t) \triangleq \begin{cases} \alpha(t), & N(t) = 0; \\ \alpha(t) + \frac{1}{t^2} \sum_{i=1}^{N(t)} \left(\int_0^{\tau_i} sX(T(i-i)+s)ds \; - \; \alpha(t)\tau_i^2/2\right), & N(t) \geq 1. \end{cases}$$

To compare these (and other) estimators against the time-average and each other, one can develop second order MSE expansions, as done here for α_{MH}. While we do not perform such expansions here, we find no a priori reason to believe any one of them would be universally superior to the others. This belief based on the one hand on our earlier conclusion that α_{MH} is not universally better than the time average, and also on the findings of Hsieh et al. [2004]: they provide an empirical comparison of $\alpha_{MH}(t)$, $\alpha_J(t)$, $\alpha_2(t)$ and $\alpha_3(t)$ against $\alpha(t)$ and other benchmarks, and find that $\alpha_{MH}(t)$ performs as well as the best of the others (though they recommend $\alpha_2(t)$ for computer time considerations).

5. ASYMPTOTIC ANALYSIS IN THE MULTIPLE REPLICATE SETTING

Throughout the previous sections, we have argued that a first-order variance analysis is not sufficient to assess whether a low-bias estimator is preferable to the time-average (or, more generally, to compare two such estimators). So far, we have always assumed a "single-run" approach. Here, we show that the situation changes drastically in the multiple replicate/parallel steady-state simulation setting.

In the context of multiple replications, the variance will be reduced by averaging the estimates from many runs, so that the bias plays a critical role in improving the quality of the (aggregate) estimator—see Heidelberger [1988] and Glynn and Heidelberger [1990, 1992a].

To be specific, suppose $Z = (Z(t) : t \geq 0)$ is an S-valued stochastic process that has a steady-state, in the sense that

$$Z(t) \Longrightarrow \alpha$$

as $t \to \infty$, for some constant α, which we are interested in estimating. We assume the availability of a (single run) estimator $\alpha_1(t, Z)$ that satisfies the following conditions:

H1 (CLT) : $t^{1/2}(\alpha_1(t, Z) - \alpha) \Longrightarrow \sigma N(0, 1)$, where $\sigma > 0$,
H2 : $(t(\alpha_1(t, Z) - \alpha)^2 : t \geq 0)$ is uniformly integrable (U.I.),
H3 : $\mathrm{E}\, \alpha_1(t, Z) = \alpha + O(t^{-p})$, $p \geq 1$.

Sometimes, we will replace H3 by the stronger version H4:

H4 : $\mathrm{E}\, \alpha_1(t, Z) = \alpha + b/t^p + o(t^{-p})$, $p \geq 1$, $b \neq 0$.

If $\{Z^i : i \geq 1\}$ are i.i.d. replicates of Z, one can construct a multiple replications estimator, $\alpha_A(t)$, by averaging the estimates of $k(t)$ independent runs, that is, setting

$$\alpha_A(t) = \frac{1}{k(t)} \sum_{i=1}^{k(t)} \alpha_1(m(t),\ Z^i),$$

where $m(t) \triangleq t/k(t)$.

For $\alpha_A(t)$ to be consistent, it is necessary that the length of each run, $m(t)$, increases fast enough compared to $k(t)$. When $\alpha_1(t, Z)$ is the time-average, and under assumptions that ensure that H1, H2 and H3 hold with $p = 1$, it is known that it is necessary that $m(t)/k(t) \to \infty$ as $t \to \infty$ [Glynn 1987; Glynn and Heidelberger 1991]. In contrast, if $p > 1$, it is enough that $m(t)^{2p-1}/k(t) \to \infty$, as our next result shows.

THEOREM 5.1

(i) *Assume H1, H2, H3. If* $m(t)^{2p-1}/k(t) \to \infty$ *and* $k(t) \to \infty$ *as* $t \to \infty$, *then*

$$t^{1/2}(\alpha_A(t) - \alpha) \Longrightarrow \sigma\, N(0, 1) \quad \text{as } t \to \infty.$$

(ii) *Assume H1, H2, H4. If* $m(t)^{2p-1}/k(t) \to 0$ *as* $t \to \infty$ *then*

$$t^{1/2} \left|\alpha_A(t) - \alpha\right| \Longrightarrow \infty \quad \text{as } t \to \infty.$$

(iii) *Assume H1, H2, H4. If* $m(t)^{2p-1}/k(t) \to c > 0$ *as* $t \to \infty$ *then*

$$t^{1/2} \left(\alpha_A(t) - \alpha\right) \Longrightarrow \sigma\, N(0, 1) \;+\; b/c^{1/2} \quad \text{as } t \to \infty.$$

(iv) *Assume H1, H2, H3. If* $m(t) \sim t^{\frac{1}{2p}} g(t)$ *as* $t \to \infty$, *where* $g(t) \nearrow \infty$ *and* $g(t) = o(t^{1-\frac{1}{2p}})$ *as* $t \to \infty$, *then*

$$\text{MSE}\, \alpha_A(t) = \frac{\sigma^2}{t} + o(t^{-1}) \quad \text{as } t \to \infty.$$

The proof uses the same arguments as that in Theorem 3 of Glynn [1987]. For the sake of completeness, we include it in Section 6.

Note that as long as $m(t)^{2p-1}/k(t) \to \infty$, $\alpha_A(t)$ satisfies the same CLT as $\alpha_1(t, Z)$ (the single-run estimator), and which doesn't depend on p. If we are interested in the quality of the estimator as a function of the *total computational effort*, t, then to first order, there is no advantage in using a low-bias estimator (i.e., one that satisfies H1–H3 with the same σ but greater p), and no first-order advantage in using a multiple replication scheme instead of a single run. The estimator comparison would then require a second-order analysis, as discussed in previous sections.

However, if one takes advantage of (massive) parallel processing capability, running each replicate of Z in a different processor, then there is a clear benefit in terms of *completion time* in using a multiple replications scheme, and there are also significant additional gains from using a low-bias estimator. Indeed, for $\alpha_A(t)$ to have the same first-order MSE of the single-run estimator, $m(t)$ (which we identify with the completion time) needs to increase only slightly faster than $t^{\frac{1}{2p}}$. Moreover, if $\alpha_1'(t, Z)$ satisfies H1–H3 with the same σ but a greater p, then the resulting estimator $\alpha_A'(t)$ will yet again have the same first-order MSE , but would require the completion time to increase at a smaller rate; this can be viewed as a first-order efficiency improvement, provided completion time (rather than total computational effort) is what one cares about. In the framework of Glynn and Whitt [1992], if the *cost* of interest is the completion time and the *loss function* is quadratic, then, in the parallel simulation context, a low-bias estimator produces a multiple replications estimator which is *more asymptotically efficient* than the multiple replications estimator obtained with the time average, which in turn is more asymptotically efficient than the single run estimator.

6. PROOFS OF KEY RESULTS

PROOF (PROPOSITION 2.5). Note that

$$\sum_y |A(x, y)| \frac{V(y)}{V(x)} = -A(x,x) + \sum_{y \neq x} A(x, y) \frac{V(y)}{V(x)}$$
$$= -2A(x,x) + \sum_y A(x, y) \frac{V(y)}{V(x)}.$$

Hence, for $x \in K^c$,

$$\sum_y |A(x, y)| \frac{V(y)}{V(x)} \leq -2A(x,x) - \beta$$

whereas for $x \in K$,

$$\sum_y |A(x, y)| \frac{V(y)}{V(x)} \leq -2A(x,x) + c.$$

So,

$$||A||_V \leq 2 \sup_{x \in S} -A(x,x) + c.$$

Proposition 2.4 implies that W is nonexplosive and that V is in the domain of the extended generator. Also, the irreducibility ensures that $\mathrm{P}(W(t) = y | W(0) = x) > 0$ for $x, y \in S$ and $t > 0$. Setting $\phi(y) = I(y = z)$ for some $z \in S$ completes the proof. □

PROOF (THEOREM 2.8). Theorem 5.1, part (d), of Down et al. [1995] establishes the existence of a petite set $\widetilde{K}$ for $(W(n) : n \geq 0)$, $\beta < 1$, and $\widetilde{c} > 0$ for which

$$\mathrm{E}_x V(W(1)) \leq \beta V(x) + \widetilde{c} I(x \in \widetilde{K}).$$

We next apply Lemma 15.2.9 of Meyn and Tweedie [1993], thereby obtaining the existence of a petite set $\widetilde{\widetilde{K}}$, $\widetilde{\beta} < 1$, and $\widetilde{\widetilde{c}}$ for which

$$\mathrm{E}_x V^{1/2}(W(1)) \leq \widetilde{\beta} V^{1/2}(x) + \widetilde{\widetilde{c}} I(x \in \widetilde{\widetilde{K}}).$$

Theorem 15.0.1 of Meyn and Tweedie [1993] proves that there exists r_1 and R positive such that

$$\sup\{|\mathrm{E}_x g_c(W(n))| : |g(\cdot)| \le V^{1/2}(\cdot)\} \le R \exp(-r_1 n) V(x)^{1/2}, \tag{7}$$

where $g_c(\cdot) = g(\cdot) - \mathrm{E}\, g(W^*(0))$.

Hence,

$$|\mathrm{E}_x f_c(W(n))| \le aR \exp(-r_1 n) V(x)^{1/2}.$$

We now wish to prove that there exists $\widetilde{a} > 0$ for which

$$|\mathrm{E}_x f_c(W(t))| \le \widetilde{a} \exp(-r_1 t) V(x)^{1/2}. \tag{8}$$

Note that for $0 \le t \le 1$,

$$\mathrm{E}_x f_c(W(n+t)) = \mathrm{E}_x \widetilde{g}(W(n)),$$

where

$$\widetilde{g}(x) = \mathrm{E}_x f_c(W(t)).$$

But

$$\begin{aligned} |\mathrm{E}_x f_c(W(t))| &\le |\mathrm{E}\, f(W^*(0))| + \mathrm{E}_x f(W(t)) \\ &\le |\mathrm{E}\, f(W^*(0))| + a\mathrm{E}_x V^{1/2}(W(t)) \\ &\le |\mathrm{E}\, f(W^*(0))| + a\sqrt{\mathrm{E}_x V(W(t))}. \end{aligned}$$

Because V is in the domain of the extended generator $\widetilde{A}$,

$$\begin{aligned} \mathrm{E}_x V(W(t)) &= V(x) + \mathrm{E}_x \int_0^t (AV)(W(s))ds \\ &\le V(x) + c\mathrm{E}_x \int_0^t I(W(s) \in K)ds \\ &\le V(x) + ct \\ &\le V(x) + c \end{aligned}$$

for $0 \le t \le 1$. Hence

$$|\widetilde{g}(x)| \le |\mathrm{E}\, f(W^*(0))| V(x)^{1/2} + a\sqrt{1+c}\, V(x)^{1/2}.$$

Put $g(x) = \widetilde{g}(x)(|\mathrm{E}\, f(W^*(0))| + a\sqrt{1+c})^{-1}$, so that $|g(\cdot)| \le V^{1/2}(\cdot)$. It follows from (7) that

$$|\mathrm{E}_x f_c(W(t))| \le \exp(r_1)(|\mathrm{E}\, f(W^*(0))| + a\sqrt{1+c})R \exp(-r_1 t) V(x)^{1/2}$$

for $t \ge 0$, proving (8).

Consequently,

$$|u(x)| \le \frac{\widetilde{a}}{r_1} V(x)^{1/2}.$$

Applying the same argument as for f_c to u, we find that

$$|\mathrm{E}_x u(W(t))| \le \widetilde{\widetilde{a}} \exp(-r_1 t) V(x)^{1/2}, \tag{9}$$

so that

$$|v(x)| \leq \frac{\widetilde{\widetilde{a}}}{r_1} V(x)^{1/2}.$$

Thus, $|f_c(x)u(x)| \leq a(\widetilde{a}/r_1)V(x)$. We now apply Theorem 5.2 of Down et al. [1995] to conclude that there exists r_2 such that

$$\mathrm{E}_x[f_c(W(t))u(W(t)) - \sigma^2/2] = o(\exp(-r_2 t))$$

as $t \to \infty$, and hence

$$\int_0^\infty |\mathrm{E}_x f_c(W(t))u(W(t)) - \sigma^2/2| dt < \infty.$$

Also, because $|f_c(W^*(t))u(W^*(t))| \leq a(\widetilde{a}/r_1)V(W^*(t))$, it follows from the same result that $f_c(W^*(t))u(W^*(t))$ is integrable. This proves that all the quantities of part (a) are finite and well defined.

As for (b), observe that

$$\begin{aligned} \mathrm{E}_x u(W(t)) &= \int_t^\infty \mathrm{E}_x f_c(W(s)) ds \\ &= u(x) - \int_0^t \mathrm{E}_x f_c(W(s)) ds, \end{aligned}$$

proving the martingale property for

$$u(W(t)) + \int_0^t f_c(W(s)) ds.$$

This shows that u is in the domain of $\widetilde{A}$ with $\widetilde{A}u = -f_c$. The arguments for v and w are identical.

For (c), note that

$$\begin{aligned} t\mathrm{E}_x(\alpha(t) - \alpha) &= \mathrm{E}_x \int_0^t f_c(W(s)) ds \\ &= u(x) - \mathrm{E}_x u(W(t)) \\ &= u(x) + o(\exp(-r_1 t/2)) \end{aligned}$$

as $t \to \infty$. On the other hand,

$$\begin{aligned} t^2\mathrm{E}_x(\alpha(t) - \alpha)^2 &= 2\int_0^t \int_s^t \mathrm{E}_x f_c(W(s)) f_c(W(u)) du\, ds \\ &= 2\int_0^t \mathrm{E}_x f_c(W(s))[u(W(s)) - u(W(t))] ds \\ &= \sigma^2 t + 2(w(x) - \mathrm{E}_x w(X(t))) - 2\int_0^t \mathrm{E}_x f_c(W(s))u(W(t)) ds \\ &= \sigma^2 t + 2w(x) + o(\exp(-r_2 t)) - 2\int_0^t \mathrm{E}_x f_c(W(s))u(W(t)) ds. \end{aligned}$$

To deal with this last term, we first split the integral into two pieces:

$$\begin{aligned}&\int_0^t \mathrm{E}_x f_c(W(s))u(W(t))ds\\ &= \int_0^{t/2} \mathrm{E}_x f_c(W(s))u(W(t))ds + \int_{t/2}^t \mathrm{E}_x f_c(W(s))u(W(t))ds\end{aligned}$$

It follows from (9) that

$$\begin{aligned}&|\mathrm{E}_x f_c(W(s))u(W(t))|\\ &\le \tilde{\tilde{a}} e^{-r_1(t-s)}\mathrm{E}_x |f_c(W(s))V(W(s))^{1/2}|\\ &\le a\tilde{\tilde{a}} e^{-r_1(t-s)}\mathrm{E}_x V(W(s)).\end{aligned}$$

Consequently,

$$\left|\int_0^{t/2} \mathrm{E}_x f_c(W(s))u(W(t))ds\right| \le \exp(-r_1 t/2) a\tilde{\tilde{a}}(t/2)\sup_{r\ge 0}\mathrm{E}_x V(W(r)).$$

For the other term, observe that

$$\mathrm{E}_x f_c(W(s))u(W(t)) = \mathrm{E}_x \tilde{g}(W(s)),$$

where

$$\tilde{g}(x) = f_c(x)\mathrm{E}_x u(W(t-s)).$$

Because

$$|\tilde{g}(x)| \le a\tilde{\tilde{a}}\exp(-r_1(t-s))V(x) \le a\tilde{\tilde{a}}V(x),$$

Theorem 5.2 of Down et al. [1995] proves that there exists $\underline{a} > 0$ for which

$$\sup_{t/2\le s\le t} |\mathrm{E}_x f_c(W(s))u(W(t)) - \mathrm{E}\, f_c(W^*(0))u(W^*(t-s))| \le \underline{a}\exp(-r_2 t/2)V(x).$$

Hence,

$$\begin{aligned}\int_{t/2}^t \mathrm{E}_x f_c(W(s))u(W(t))ds &= o(\exp(-r_2 t/4)) + \int_{t/2}^t \mathrm{E}\, f_c(W^*(0))u(W^*(t-s))ds\\ &= o(\exp(-r_2 t/4)) + \int_0^{t/2} \mathrm{E}\, f_c(W^*(0))u(W^*(s))ds\\ &= o(\exp(-r_2 t/4)) + \mathrm{E}\, f_c(W^*(0))v(W^*(0))\\ &\quad + o(\exp(-r_1 t/4)),\end{aligned}$$

completing the proof of c.).

The identity for σ^2 is obvious. For ν, note that

$$\begin{aligned}\mathrm{E}\, f_c(W^*(0))v(W^*(0)) &= \int_0^\infty \mathrm{E}\, f_c(W^*(0))u(W^*(t))dt\\ &= \int_0^\infty \mathrm{E}\, f_c(W^*(0))\int_t^\infty f_c(W^*(s))ds\,dt\\ &= \int_0^\infty \int_0^s dt\, \mathrm{E}\, f_c(W^*(0))f_c(W^*(s))ds,\end{aligned}$$

proving the result. □

PROOF (THEOREM 3.1). We denote $F(x) = \mathrm{P}(\tau_1 \le x)$ and $U(t) \triangleq \sum_{k=0}^{\infty} F^{*k}(t)$ (the renewal kernel), where F^{*k} denotes the k-fold convolution of F, that is, $F^{*k+1}(t) = (F^{*k} * F)(t) \triangleq \int_{-\infty}^{t} F^{*k}(t-s)F(ds)$, and $F^{*0}(t) = I(t \ge 0)$.

For the bias expansion, let $h(t) \triangleq t\mathrm{E}(\alpha(t)-\alpha) = \mathrm{E}\,\beta(t)$. Note that h satisfies the renewal equation

$$h(t) = q(t) + (F * h)(t),$$

$t \ge 0$, where

$$q(t) \triangleq \mathrm{E}(\beta(t) - Z_1;\ \tau_1 > t).$$

Note that

$$\begin{aligned} |q(t)| &\le t^{-(p+2)}\mathrm{E}\left(\overline{Z}_1\tau_1^{p+2};\ \tau_1 > t\right) \\ &= o\big(t^{-(p+2)}\big), \end{aligned}$$

since $\mathrm{E}(\overline{Z}_1\tau_1^{p+2};\ \tau_1 > t) \to 0$ by dominated convergence. Also, $|h(t)| \le \mathrm{E}\sum_{j=1}^{(N(t)+1)}\overline{Z}_j = \mathrm{E}(N(t)+1)\mathrm{E}\,\overline{Z} < \infty$, so h is bounded in compact sets, whence $h(t) = (U * q)(t)$. It then follows from Theorem 4.2(ii) in Nummelin and Tuominen [1983] that

$$\begin{aligned} h(t) &= \lambda\int_0^{\infty} q(s)ds + o\big(t^{-(p+1)}\big) \\ &= \gamma + o\big(t^{-(p+1)}\big). \end{aligned} \tag{10}$$

Dividing by t gives the desired bias expansion.

For the MSE expansion, let $g(t) \triangleq t^2\,\mathrm{E}(\alpha(t)-\alpha)^2$. Note that

$$\begin{aligned} g(t) &= \mathrm{E}(\beta(t)^2;\ \tau_1 > t) + \mathrm{E}\left[Z_1^2 + 2Z_1\int_{T(1)}^{t} X_c(s)ds \right. \\ &\quad \left. + \left(\int_{T(1)}^{t} X_c(s)ds\right)^2;\ \tau_1 \le t\right] \\ &= \mathrm{E}(\beta(t)^2;\ \tau_1 > t) + \mathrm{E}\,Z_1^2 - \mathrm{E}\left(Z_1^2;\ \tau_1 > t\right) + 2\mathrm{E}(Z_1 h(t-\tau_1);\ \tau_1 \le t) \\ &\quad + (F * g)(t), \end{aligned}$$

so g satisfies the renewal equation

$$g = \mathrm{E}\,Z_1^2 + v + F * g,$$

where

$$v(t) \triangleq \mathrm{E}\left(\beta(t)^2 - Z_1^2;\ \tau_1 > t\right) + 2\mathrm{E}(Z_1 h(t-\tau_1);\ \tau_1 \le t).$$

Note

$$\begin{aligned} \mathrm{E}\left(\beta(t)^2 - Z_1^2;\ \tau_1 > t\right) &\le 2t^{-(p+2)}\mathrm{E}\left(\overline{Z}_1\tau_1^{(p+2)};\ \tau_1 > t\right) \\ &= o\big(t^{-(p+2)}\big), \end{aligned}$$

and

$$\begin{aligned}
2\mathrm{E}\,(Z_1 h(t-\tau_1);\, \tau_1 \le t) \;&=\; 2\gamma \mathrm{E}\,(Z_1;\, \tau_1 \le t) \;+\; 2\mathrm{E}\,(Z_1[h(t-\tau_1)-\gamma];\, \tau_1 \le t)\\
&=\; -2\gamma \mathrm{E}\,(Z_1;\, \tau_1 > t) \;+\; 2\mathrm{E}\,(Z_1[h(t-\tau_1)-\gamma];\\
&\qquad 0 \le \tau_1 \le t/2)\\
&\quad + 2\mathrm{E}\,(Z_1[h(t-\tau_1)-\gamma];\, t/2 \le \tau_1 \le t)\\
&=\; o\big(t^{-(p+2)}\big) + o\big((t/2)^{-(p+1)}\big) + o\big((t/2)^{-(p+2)}\big)\\
&=\; o\big(t^{-(p+1)}\big),
\end{aligned}$$

so that $v(t) = o(t^{-(p+1)})$. Since g and v are bounded on compact sets, g is given by

$$g(t) = \big(U * \big(\mathrm{E}\, Z_1^2 + v\big)\big)(t),$$

and it follows from Lemma 6.1 below that

$$g(t) \;=\; \lambda\, t\, \mathrm{E}\left(Z_1^2\right) \;+\; \lambda^2 \mathrm{E}\, Z_1^2\, \mathrm{E}\, \tau_1^2/2 \;+\; \lambda \int_0^\infty v(s)ds + o(t^{-p}). \tag{11}$$

Note that

$$\begin{aligned}
&\int_0^\infty v(s)ds\\
&= \int_0^\infty \mathrm{E}\left(\beta(s)^2 - Z_1^2;\, \tau_1 > s\right)ds + 2\int_0^\infty \mathrm{E}\,(Z_1 h(s-\tau_1);\, \tau_1 \le s)ds\\
&= \mathrm{E}\int_0^{\tau_1} \left(\beta(s)^2 - Z_1^2\right)ds + 2\gamma \int_0^\infty \mathrm{E}\; Z_1 I(\tau_1 \le s)\, ds + 2\int_0^\infty \mathrm{E}\; Z_1[h(s-\tau_1)-\gamma]\\
&\quad \times I(\tau_1 \le s)\, ds\\
&= \mathrm{E}\int_0^{\tau_1} \left(\beta(s)^2 - Z_1^2\right) - 2\gamma \int_0^\infty \mathrm{E}\; Z_1 I(\tau_1 > s)\, ds + 2\mathrm{E}\, Z_1 \int_0^\infty [h(s)-\gamma]\, ds\\
&= \mathrm{E}\int_0^{\tau_1} \left(\beta(s)^2 - Z_1^2\right) - 2\gamma\, \mathrm{E}\,(Z_1 \tau_1).
\end{aligned}$$

The application of Fubini's theorem in the second step above is justified since $\mathrm{E}\int_0^\infty |\, Z_1[h(s-\tau_1)-\gamma] I(\tau_1 \le s)\,|ds \le \mathrm{E}\,|Z_1| \int_0^\infty |\, h(s)-\gamma|ds < \infty$ in light of (10).

Substituting into (11) gives the desired MSE expansion. The variance expansion follows immediately from the bias and MSE expansions. □

LEMMA 6.1. *Suppose that* $b : [0, \infty) \longrightarrow \mathbb{R}$ *is bounded and there exists* $\bar{b}$ *such that* $|b(t) - \bar{b}| = o(t^{-(p+1)})$ *for some* $p > 0$. *Assume also that* $\mathrm{E}\, \tau_1^{p+2} < \infty$. *Then*

$$(U * b)(2t) = 2\lambda\, \bar{b}\, t + \lambda^2 \bar{b} \mathrm{E}\, \tau_1^2/2 \;+\; \lambda \int_0^\infty [b(s) - \bar{b}]dt + o(t^{-p}).$$

PROOF

$$\begin{aligned}
&(U * b)(2t)\\
&\quad = \int_0^t b(2t-s)dU(s) + \int_t^{2t} b(2t-s)dU(s)
\end{aligned}$$

$$
\begin{aligned}
&= \bar{b}U(t) + \lambda \int_t^{2t} b(2t-s)ds + \int_t^{2t} b(2t-s)dU(s) - \lambda \int_t^{2t} b(2t-s)ds \\
&\quad + \int_0^t [b(2t-s) - \bar{b}]dU(s) \\
&= \bar{b}U(t) + \lambda \int_0^t b(s)ds + \int_t^{2t} b(2t-s)dU(s) - \lambda \int_t^{2t} b(2t-s)ds \\
&\quad + \int_0^t [b(2t-s) - \bar{b}]dU(s) \\
&= 2\,\bar{b}\,\lambda t \,+\, \lambda^2 \bar{b} \mathrm{E}\,\tau_1^2/2 \,+\, \lambda \int_0^\infty [b(s) - \bar{b}]ds \\
&\quad \lambda \int_t^\infty [b(s) - \bar{b}]ds \,+\, \bar{b}\big(U(t) - \lambda t - \lambda^2 \mathrm{E}\,\tau_1^2/2\big) \\
&\quad + \left[\int_t^{2t} b(2t-s)dU(s) - \lambda \int_t^{2t} b(2t-s)ds\right] + \int_0^t [b(2t-s) - \bar{b}]dU(s)\,.
\end{aligned}
$$

It is enough to show that the last four terms on the right-hand side are $o(t^{-p})$. For the first term,

$$
\left|\lambda \int_t^\infty [b(s) - \bar{b}]ds\right| \;\le\; \lambda \int_t^\infty |b(s) - \bar{b}| \;=\; o(t^{-p})
$$

since $|b(s) - \bar{b}| = o(t^{-(p+1)})$ by assumption.

For the second term, it is a standard result in renewal theory that $U(t) - \lambda t \longrightarrow \lambda^2 \mathrm{E}\,\tau_1^2/2$ as $t \to \infty$ (see, e.g., Karlin and Taylor [1975, p. 195]). That the rate of convergence is such that

$$
\big|\big(U(t) - \lambda t - \lambda^2 \mathrm{E}\,\tau_1^2/2\big)\big| = o(t^{-p}) \tag{12}
$$

follows from the renewal equation satisfied by $t \mapsto (U(t) - \lambda t)$, the assumption that $E\tau_1^{p+2} < \infty$, and Theorem 4.2 (ii) in Nummelin and Tuominen [1983].

For the third term,

$$
\begin{aligned}
\left|\int_t^{2t} b(2t-s)dU(s) - \lambda \int_t^{2t} b(2t-s)ds\right| \;&\le\; \|b\| \sup_{A \in \mathcal{B}([0,\,\infty))} \left|\int_{t+A} U(ds) - \int_{t+A} \lambda\, ds\right| \\
&=\; o(t^{-p}),
\end{aligned}
$$

where $\mathcal{B}([0,\,\infty))$ denotes the Borel sets in $[0,\,\infty)$ and the last follows from the assumption that $\mathrm{E}\,\tau_1^{p+2} < \infty$ and the argument in Thorisson [2000, p. 421].

Finally, for the last term, it follows from the assumption that $|b(t) - \bar{b}| = o(t^{-(p+1)})$ that

$$
\begin{aligned}
\left|\int_0^t [b(2t-s) - \bar{b}]dU(s)\right| \;&\le\; \int_0^t |b(2t-s) - \bar{b}|dU(s) \\
&\le\; o(t^{-(p+1)})U(t) = o\big(t^{-p}\big)\frac{U(t)}{t} \\
&=\; o(t^{-p})O(1)\,. \qquad \square
\end{aligned}
$$

PROOF (THEOREM 4.1). Let $R(t) \triangleq T(N(t)+1) - t$ denote the residual life at time t. Note that

$$\alpha_{MH}(t) - \alpha = \frac{\sum_{i=1}^{N(t)+1} Z_i}{T(N(t)+1)} = \frac{\frac{1}{t}\sum_{i=1}^{N(t)+1} Z_i}{1 + R(t)/t}.$$

Put

$$\varepsilon(t) \triangleq t^{p+1}\left[\alpha_{MH}(t) - \alpha - \left(\frac{1}{t}\sum_{i=1}^{N(t)+1} Z_i\right)\sum_{k=0}^{p}(-1)^k \frac{R(t)^k}{t^k}\right].$$

A Taylor expansion of $x \mapsto 1/(1+x)$ shows that, for $x > 0$,

$$\left|\frac{1}{1+x} - \sum_{k=0}^{p}(-1)^k x^k\right| \le x^{p+1},$$

and applying this with $x = R(t)/t$, we conclude

$$|\varepsilon(t)| \le R(t)^{p+1}\left|\frac{1}{t}\sum_{i=1}^{N(t)+1} Z_i\right| \le \frac{1}{\sqrt{t}} R(t)^{p+1}\left(1 + \frac{1}{t}\left(\sum_{i=1}^{N(t)+1} Z_i\right)^2\right).$$

Hence,

$$\mathrm{E}\,|\varepsilon(t)| \;\le\; t^{-1/2}\mathrm{E}\,R(t)^{p+1} + t^{-3/2}\mathrm{E}\,R(t)^{p+1}\left(\sum_{i=1}^{N(t)+1} Z_i\right)^2.$$

Note that $\mathrm{E}\,R(t)^{p+1} \to \lambda \mathrm{E}\,\tau_1^{p+2}/(p+2) < \infty$. Also, we show below that

$$\mathrm{E}\,R(t)^{p+1}\left(\sum_{i=1}^{N(t)+1} Z_i\right)^2 = O(t). \tag{13}$$

Assuming this for the moment, it follows that

$$\mathrm{E}\,|\varepsilon(t)| \;=\; O(t^{-1/2}) \longrightarrow 0$$

as $t \to \infty$, and hence

$$\mathrm{E}\,\varepsilon(t) \longrightarrow 0$$

as $t \to \infty$, which implies

$$\mathrm{E}\,\alpha_{MH}(t) - \alpha \;=\; \sum_{k=0}^{p}\frac{(-1)^k}{t^{k+1}}\mathrm{E}\left[R(t)^k \sum_{i=1}^{N(t)+1} Z_i\right] \;+\; o\big(t^{-(p+1)}\big).$$

Note that for $k = 0$, $\mathrm{E}\,[R(t)^k \sum_{i=1}^{N(t)+1} Z_i] = 0$ by Wald's first moment identity. Hence, it is enough to show that for $1 \le k \le p$

$$a_k(t) = \bar{a}_k + o\big(t^{-(p-k)}\big), \tag{14}$$

where $a_k(t) \triangleq \mathrm{E}\,(R(t)^k \sum_{i=1}^{N(t)+1} Z_i)$. Note a_k satisfies the renewal equation

$$a_k(t) \;=\; b(t) \;+\; (F * a_k)(t),$$

where

$$b(t) \; = \; \mathrm{E}\,(Z_1 R(t)^k).$$

Since both a_k and b are bounded on compact intervals, it follows that a_k is given by

$$a_k(t) = (U * b)(t)\,,$$

$t \geq 0$. We next show that $b(t) = o(t^{-(p+q-k-1)})$. For this, let $r_k(t) \triangleq \mathrm{E}\,R(t)^k$ and $\bar{r_k} \triangleq \lim_{t\to\infty} r_k(t) = \lambda \mathrm{E}\,\tau_1^{k+1}/(k+1)$. Observe that

$$\begin{aligned} b(t) \; &= \; \mathrm{E}\,(Z_1(\tau_1 - t)^k;\; \tau_1 > t) \; + \; \mathrm{E}\,(Z_1 r_k(t-\tau_1);\; \tau_1 \leq t) \\ &= \; \mathrm{E}\,(Z_1(\tau_1 - t)^k;\; \tau_1 > t) - \bar{r_k}\mathrm{E}\;(Z_1;\; \tau_1 > t) \\ &\quad + \mathrm{E}\,(Z_1[r_k(t-\tau_1) - \bar{r_k}];\; \tau_1 \leq t). \end{aligned} \tag{15}$$

The first term on the right-hand side satisfies

$$|\mathrm{E}\,(Z_1(\tau_1 - t)^k;\; \tau_1 > t)| \leq t^{-(p+q-k-1)}\mathrm{E}\left(|Z_1|\tau_1^{p+q-1};\; \tau_1 > t\right) = o\big(t^{-(p+q-k-1)}\big)$$

by dominated convergence. The same argument applied to the second term gives

$$\big|\mathrm{E}\;(Z_1;\; \tau_1 > t)\big| = o(t^{-(p+q-1)}).$$

To deal with the third term on the right-hand side in (15), note that, from the renewal equation satisfied by $r_k(t)$ it follows that $r_k(t) = (U * v)(t)$, where $v(t) = \mathrm{E}\,((\tau_1 - t)^k;\; \tau_1 > t)$. Note $|v(t)| = o(t^{-(p+q-k)})$, so it follows from Theorem 4.2(*ii*) in Nummelin and Tuominen [1983] that $|r_k(t) - \bar{r_k}| = o(t^{-(p+q-k-1)})$. Hence,

$$\begin{aligned} &|\mathrm{E}\,(Z_1[r_k(t-\tau_1) - \bar{r_k}];\; \tau_1 \leq t)| \\ &\qquad \leq \; \sup_{s \geq t/2} |r_k(s) - \bar{r_k}| \cdot \mathrm{E}\,|Z_1| + \sup_{s \geq 0} |r_k(s) - \bar{r_k}| \cdot \mathrm{E}\,(|Z_1|;\; \tau_1 > t/2) \\ &\qquad = \; o\big(t^{-(p+q-k-1)}\big)\mathrm{E}\,|Z_1| \; + \; o\big(t^{-(p+q-1)}\big) \; = \; o\big(t^{-(p+q-k-1)}\big)\,. \end{aligned} \tag{16}$$

Thus, all three terms on the right-hand side in (15) are $o(t^{-(p+q-k-1)})$ and hence so is $b(t)$. We can use again Theorem 4.2(ii) in Nummelin and Tuominen [1983] to conclude that

$$a_k(t) \; = \; \lambda \int_0^\infty b(s)ds \; + \; o\big(t^{-(p+q-k-2)}\big)$$

as $t \to \infty$. Hence, to prove (14), it is enough to verify

$$\int_0^\infty b(s)ds = \frac{1}{\lambda}\,\bar{a}_k.$$

Indeed, from (15),

$$\begin{aligned} \int_0^\infty b(t)dt \; &= \; \int_0^\infty \{\mathrm{E}\,[Z_1(\tau_1 - t)^k I(t < \tau_1)] \; - \; \bar{r_k}\,\mathrm{E}\,[Z_1 I(\tau_1 > t)] \\ &\qquad + \mathrm{E}\,[Z_1\,[r_k(t-\tau_1) - \bar{r_k}]\,I(\tau_1 \leq t)]\}dt. \end{aligned}$$

Note that $\mathrm{E} \int_0^\infty |Z_1(\tau_1 - t)^k I(t < \tau_1)dt| = \mathrm{E}\,|Z_1|\tau_1^{k+1}(k+1)^{-1} < \infty$, $\mathrm{E} \int_0^\infty |Z_1 I(\tau_1 > t)|dt = \mathrm{E}\,|Z_1|\tau_1 < \infty$ and $\mathrm{E} \int_0^\infty |Z_1[r_k(t-\tau_1) - \bar{r_k}]I(\tau_1 \le t)|dt = \mathrm{E}\,|Z_1| \int_0^\infty |r_k(s) - \bar{r_k}|ds < \infty$, so Fubini's theorem applies and we can write

$$
\begin{aligned}
\int_0^\infty b(t)dt &= \mathrm{E}\left[Z_1 \int_0^\infty (\tau_1 - t)^k I(t < \tau_1)dt\right] - \bar{r_k}\,\mathrm{E}\left[Z_1 \int_0^\infty I(\tau_1 > t)dt\right] \\
&\quad + \mathrm{E}\left[Z_1 \int_0^\infty [r_k(t-\tau_1) - \bar{r_k}]\,I(\tau_1 \le t)dt\right] \\
&= (k+1)^{-1}\mathrm{E}\,Z_1\tau_1^{k+1} - \bar{r_k}\,\mathrm{E}\,Z_1\tau_1 + \mathrm{E}\,(Z_1)\int_0^\infty [r_k(s) - \bar{r_k}]ds \\
&= (k+1)^{-1}\mathrm{E}\,Z_1\tau_1^{k+1} - \bar{r_k}\,\mathrm{E}\,Z_1\tau_1 \\
&= \frac{1}{\lambda}\,\bar{a}_k\,.
\end{aligned}
$$

The proof of the theorem is complete if we justify our earlier claim in (13). For this purpose, let

$$
g_k(t) \triangleq \mathrm{E}\ R(t)^k \left(\sum_{i=1}^{N(t)+1} Z_i\right)^2,
$$

$k \ge 1$. (The added generality is due to g_1 and g_2 playing a role later in the proof of Theorem 4.2.)

Note that

$$
\begin{aligned}
g_k(t) &= \mathrm{E}\left(R(t)^k Z_1^2\right) + 2\mathrm{E}\,[Z_1 a_k(t-\tau_1);\ \tau_1 \le t] + (F * g_k)(t), \\
&= v + F * g_k,
\end{aligned}
$$

where $v(t) = \mathrm{E}\,R(t)^k Z_1^2 + 2\mathrm{E}\,[Z_1 a_k(t-\tau_1);\ \tau_1 \le t]$. Since g_k and v are bounded on compact intervals, $g_k(t) = (U * v)(t)$. We can rewrite v as

$$
\begin{aligned}
v(t) &= \mathrm{E}\left(Z_1^2(\tau_1 - t)^k;\ \tau_1 > t\right) + \mathrm{E}\left(Z_1^2 r_k(t-\tau_1);\ \tau_1 \le t\right) \\
&\quad + 2\mathrm{E}\,[Z_1(a_k(t-\tau_1) - \bar{a}_k;\ \tau_1 \le t] - 2\bar{a_k}\mathrm{E}\,[Z_1;\ \tau_1 > t] \\
&= \bar{r}_k\mathrm{E}\,Z_1^2 - \bar{r}_k\mathrm{E}\left(Z_1^2;\ \tau_1 > t\right) + \mathrm{E}\left(Z_1^2(\tau_1 - t)^k;\ \tau_1 > t\right) \\
&\quad - 2\bar{a_k}\mathrm{E}\,[Z_1;\ \tau_1 > t] + \mathrm{E}\left(Z_1^2(r_k(t-\tau_1) - \bar{r}_k);\ \tau_1 \le t\right) \\
&\quad + 2\mathrm{E}\,[Z_1(a_k(t-\tau_1) - \bar{a}_k);\ \tau_1 \le t].
\end{aligned}
$$

Observe that the second, third, and fourth terms on the right-hand side are $o(t^{-(p+q-k-1)})$ (by dominated convergence). The same argument that led to (16) shows that the fifth term on the right-hand side is $o(t^{-(p+q-k-1)})$, and a similar argument with $a_k(t)$ in the role of $r_k(t)$ shows that the last term is $o(t^{-(p+q-k-2)})$, (using that $a_k(t) - \bar{a}_k = o(t^{-(p+q-k-2)})$, which we obtained above). Thus, $v(t) = \bar{r}_k\mathrm{E}\,Z_1^2 + o(t^{-(p+q-k-2)})$, and it follows from Lemma 6.1 that for $1 \le k \le p+1$

$$
g_k(t) = t\,\lambda\bar{r}_k\mathrm{E}\,Z_1^2 + O(1), \tag{17}
$$

proving (13) and completing the proof. □

Remark 6.2. It is interesting to note that in computing $\int_0^\infty b(t)dt$ above, a careless (and incorrect) exchange in the order of integration yields a wrong answer:

$$\int_0^\infty b(t)dt = \int_0^\infty \mathrm{E}\,(Z_1 R(t)^k)dt \neq \sum_{i=1}^{\infty} \mathrm{E}\left[Z_1 \int_{T(i-1)}^{T(i)} R(t)^k dt\right]$$
$$= \sum_{i=1}^{\infty} \mathrm{E}\, Z_1 \tau_i^{k+1}(k+1)^{-1}$$
$$= \mathrm{E}\, Z_1 \tau_1^{k+1}(k+1)^{-1}.$$

(The second term in $\bar{a}_k/\lambda$ is missing from the right-hand side.)

PROOF (THEOREM 4.2). The argument is similar to the one in the proof of Theorem 4.1. Start by noting that

$$(\alpha_{MH}(t)-\alpha)^2 = \left(\frac{\sum_{i=1}^{N(t)+1} Z_i}{T(N(t)+1)}\right)^2 = \frac{\left(\frac{1}{t}\sum_{i=1}^{N(t)+1} Z_i\right)^2}{\left(1+R(t)/t\right)^2}.$$

Put

$$\delta(t) \triangleq t^2\left[(\alpha_{MH}(t)-\alpha)^2 - \frac{1}{t^2}\left(\sum_{i=1}^{N(t)+1} Z_i\right)^2\left(1-\frac{2}{t}R(t)\right)\right]. \tag{18}$$

A Taylor expansion of $x \mapsto 1/(1+x)$ shows that, for $x > 0$

$$\left|\frac{1}{(1+x)^2} - (1-2x)\right| \le 3x^2,$$

and applying this in (18) with $x = R(t)/t$, we conclude

$$|\delta(t)| \le \frac{3}{t^2} R(t)^2 \left(\sum_{i=1}^{N(t)+1} Z_i\right)^2$$

and hence,

$$\mathrm{E}\,|\delta(t)| \le \frac{3}{t^2}\,\mathrm{E}\; R(t)^2 \left(\sum_{i=1}^{N(t)+1} Z_i\right)^2.$$

It follows from (17) in the proof of Theorem 4.1 that $\mathrm{E}\, R(t)^2(\sum_{i=1}^{N(t)+1} Z_i)^2 = O(t)$. Hence,

$$\mathrm{E}\,|\delta(t)| = O(t^{-1}) \longrightarrow 0$$

so that

$$\mathrm{E}\,\delta(t) \longrightarrow 0$$

as $t \to \infty$, which together with (18) implies

$$\mathrm{E}\;(\alpha_{MH}(t)-\alpha)^2 = \frac{1}{t^2}\mathrm{E}\left(\sum_{i=1}^{N(t)+1} Z_i\right)^2 - \frac{2}{t^3}\mathrm{E}\,R(t)\left(\sum_{i=1}^{N(t)+1} Z_i\right)^2 + o(t^{-2}).$$

The first term on the right-hand side can be simplified using Wald's second moment identity,

$$\begin{aligned}\mathrm{E}\left(\sum_{i=1}^{N(t)+1} Z_i\right)^2 &= U(t)\mathrm{E}\,Z_1^2 = \lambda t\mathrm{E}\,Z_1^2 + \mathrm{E}\,Z_1^2(U(t)-\lambda t)\\ &= \lambda t\mathrm{E}\,Z_1^2 + \lambda^2\mathrm{E}\,Z_1^2\mathrm{E}\,\tau_1^2/2 + o(t^{-2}),\end{aligned}$$

where the last step is justified as in (12) above. For the second term, it follows from (17) in the proof of Theorem 4.1 that

$$\mathrm{E}\,R(t)\left(\sum_{i=1}^{N(t)+1} Z_i\right)^2 = t\,\frac{\lambda^2\mathrm{E}\,\tau_1^2\mathrm{E}\,Z_1^2}{2} + O(1).$$

Hence,

$$\begin{aligned}\mathrm{E}\,(\alpha_{MH}(t)-\alpha)^2 &= \frac{\lambda t\mathrm{E}\,Z_1^2}{t} + \frac{\lambda^2\mathrm{E}\,Z_1^2\mathrm{E}\,\tau_1^2}{2t^2} - \frac{\lambda^2\mathrm{E}\,\tau_1^2\mathrm{E}\,Z_1^2}{t^2} + o(t^{-2})\\ &= \frac{\sigma^2}{t} + \frac{\kappa}{t^2} + o(t^{-2}),\end{aligned}$$

completing the proof of the MSE expansion. The variance expansion follows immediately from the MSE expansion and Theorem 4.1. □

Proof (Theorem 5.1). For (i), let $v(t) \triangleq t\,\mathrm{E}\,(\alpha_1(t,Z)-\alpha)^2$. Note that

$$t^{1/2}(\alpha_A(t)-\alpha) = a(t)\,b(t)\sum_{i=1}^{k(t)} U_i(t) + \gamma(t), \tag{19}$$

where $a(t) \triangleq [v(m(t))]^{1/2}$, $b(t) \triangleq [m(t)\mathrm{var}\,(\alpha_1(m(t),Z))\,/\,v(m(t))]^{1/2}$, $\gamma(t) \triangleq t^{1/2}[\mathrm{E}\,(\alpha_1(m(t),Z))-\alpha]$ and

$$U_i(t) = (k(t)\,\mathrm{var}\,(\alpha_1(m(t),Z)))^{-1/2}[\alpha_1(m(t),\ Z^i) - \mathrm{E}\,(\alpha_1(m(t),Z))].$$

From H3,

$$\gamma(t) = t^{1/2}\,O(m(t)^{-p}) = k(t)^{1/2}\,O\big(m(t)^{-(p-1/2)}\big) \to 0 \tag{20}$$

as $t\to\infty$. Also, it follows from H1 and H2 that

$$v(t) \to \sigma^2 \tag{21}$$

as $t\to\infty$. Note that

$$\begin{aligned}\mathrm{var}\,(\alpha_1(m(t),Z)) &= \mathrm{E}\,[\alpha_1(m(t),Z)-\alpha]^2 - [\alpha - \mathrm{E}\,(\alpha_1(m(t),Z))]^2\\ &= v(t)/m(t) + O(m(t)^{-2p}),\end{aligned} \tag{22}$$

whence

$$b(t) = \left(1 + \frac{O(m(t)^{-2p+1})}{v(m(t))}\right)^{1/2} \to 1 \text{ as } t\to\infty. \tag{23}$$

Note that $\mathrm{E}\,U_i(t) = 0$ and $\sum_{i=1}^{k(t)}\mathrm{E}\,U_i^2(t) = 1$. Suppose, for the time being, that we verify Lindeberg's condition for the triangular array $(U_i(t) : 1 \le i \le k(t),\ t \ge 0)$, that is, that for all $\eta > 0$,

$$\sum_{i=1}^{k(t)} \mathrm{E}\left(U_i^2(t);\, U_i^2(t) > \eta\right) \to 0 \text{ as } t \to \infty\,. \tag{24}$$

Then, it follows from the Lindeberg–Feller central limit theorem (see, e.g., Durrett [1995, p. 116]) that

$$\sum_{i=1}^{k(t)} U_i(t) \Longrightarrow N(0,1)\,,$$

which together with (20), (21), (23), (19) and the converging-together lemma give the desired result.

Hence, all we need to show is that (24) holds. Fix $\eta > 0$ and note that

$$\begin{aligned}\sum_{i=1}^{k(t)} \mathrm{E}\left(U_i^2(t);\, U_i^2(t) > \eta\right) &= k(t)\,\mathrm{E}\left(U_1^2(t);\, U_1^2(t) > \eta\right)\\ &= \mathrm{E}\,(V^2(t);\, V^2(t) > \eta k(t))\,,\end{aligned} \tag{25}$$

where

$$\begin{aligned}V^2(t) &= \frac{1}{\operatorname{var}\alpha_1(m(t),\,Z)}(\alpha_1(m(t),\,Z) - \mathrm{E}\,\alpha_1(m(t),\,Z))^2\\ &\le \frac{2}{m(t)\operatorname{var}\alpha_1(m(t),\,Z)}[m(t)(\alpha_1(m(t),\,Z)-\alpha)^2 + (\alpha - \mathrm{E}\,\alpha_1(m(t),\,Z))^2].\end{aligned}$$

It follows from (22) that the factor outside the brackets is bounded. The last relation, together with H2 and H3, implies that $(V^2(t) : t \ge 0)$ is U.I., whence the right-hand side in (25) goes to zero as $t \to \infty$, which in turn implies (24), completing the proof.

The proof of (ii) and (iii) follows the same argument, only the behavior of $\gamma(t)$ is different.

For (iv), note $\operatorname{var}\alpha_A(t) = k(t)^{-1}\operatorname{var}\alpha_1(m(t), Z)$, and $\mathrm{E}\,\alpha_A(t) = \mathrm{E}\,\alpha_1(m(t), Z) = \alpha + O(m(t)^{-p})$. It follows from $H1$, H_2 and H_3 that $\operatorname{var}\alpha_1(m(t), Z) = \sigma^2 m(t)^{-1} + o(m(t)^{-1})$. Since $m(t)^{-2p} = o(t^{-1})$, the result follows. □

REFERENCES

BEALE, E. M. L. 1962. Some use of computers in operational research. *Indust. Org. 31*, 27–28.

BREIMAN, L. 1968. *Probability*. Addison-Wesley series in Statistics. Addison-Wesley, Reading, MA.

BRÉMAUD, P. 1999. *Markov Chains, Gibbs Fields, Monte Carlo Simulation and Queues*. Springer-Verlag, New York.

DOWN, D., MEYN, S., AND TWEEDIE, R. 1995. Exponential and uniform ergodicity of Markov processes. *Ann. Prob. 23*, 4, 1671–1691.

DURBIN, J. 1959. A note on the application of Quenouille's method of bias reduction to the estimation of ratios. *Biometrika 46*, 477–480.

DURRETT, R. 1995. *Probability: Theory and Examples*, 2nd ed. Duxbury Press, Belmont, CA.

FIELLER, E. C. 1940. The biological standarization of insulin. *J. Roy. Stat. Soc. Ser. B 7*, 1–64.

GLYNN, P. W. 1987. Limit theorems for the method of replications. *Commun. Statist. Stoch. Models 3*, 3, 343–355.

GLYNN, P. W. 1989. A GSMP formalism for discrete-event systems. *Proc. IEEE 77*, 1, 14–23.
GLYNN, P. W. 1994. Some topics in regenerative steady-state simulation. *Acta Appl. Math. 34*, 1-2, 225–236.
GLYNN, P. W. AND HEIDELBERGER, P. 1990. Bias properties of budget constrained simulation. *Oper. Res. 38*, 5, 801–814.
GLYNN, P. W. AND HEIDELBERGER, P. 1991. Analysis of initial transient deletion for replicated steady-state simulations. *Oper. Res. Lett. 10*, 8, 437–443.
GLYNN, P. W. AND HEIDELBERGER, P. 1992a. Analysis of initial transient deletion for parallel steady-state simulations. *SIAM J. Sci. Stat. Comput. 13*, 4, 904–922.
GLYNN, P. W. AND HEIDELBERGER, P. 1992b. Jackknifing under a budget constraint. *ORSA J. Comput. 4*, 3, 226–234.
GLYNN, P. W. AND WHITT, W. 1992. The asymptotic efficiency of simulation estimators. *Oper. Res. 40*, 3, 505–520.
HEIDELBERGER, P. 1988. Discrete event simulation and parallel processing: Statistical properties. *SIAM J. Sci. Stat. Comput. 9*, 1114–1132.
HENDERSON, S. G. AND GLYNN, P. W. 2001. Regenerative steady-state simulation of discrete-event systems. *ACM Trans. Model. Comput. Simul. 11*, 4, 313–345.
HENDERSON, S. G. AND GLYNN, P. W. 2003. Nonexistence of a class of variate generation schemes. *Oper. Res. Lett. 31*, 2, 83–89.
HSIEH, M.-H., IGLEHART, D. L., AND GLYNN, P. W. 2004. Empirical performance of bias-reducing estimators for regenerative steady-state simulations. *ACM Trans. Model. Comput. Simul. 14*, 4, 325–343.
IGLEHART, D. L. 1975. Simulating stable stochastic systems V. Comparison of ratio estimators. *Naval Res. Logist. 22*, 3, 553–565.
KARLIN, S. AND TAYLOR, H. M. 1975. *A First Course in Stochastic Processes*, 2nd ed. Academic Press, San Diego, CA.
MEKETON, M. S. AND HEIDELBERGER, P. 1982. A renewal theoretic approach to bias reduction in regenerative simulations. *Manage. Sci. 28*, 2, 173–181.
MEYN, S. AND KONTOYIANNIS, I. 2003. Spectral theory and limit theorems for geometrically ergodic Markov processes. *Ann. Appl. Prob. 13*, 1, 304–362.
MEYN, S. AND TWEEDIE, R. 1993. *Markov Chains and Stochastic Stability*. Communications and Control Engineering series. Springer-Verlag, New York.
NUMMELIN, E. AND TUOMINEN, P. 1983. The rate of convergence in Orey's theorem for Harris recurrent Markov chains with applications to renewal theory. *Stochas. Proc. Their Appl. 15*, 295–311.
QUENOUILLE, M. 1956. Notes on bias in estimation. *Biometrika 43*, 3-4, 353–360.
RESNICK, S. I. 1992. *Adventures in Stochastic Processes*. Birkhäuser, Boston, MA.
ROGERS, L. AND WILLIAMS, D. 1994. *Diffusions, Markov Processes, and Martingales*, 2nd ed. Vol. 1. Wiley, New York.
STROOCK, D. W. 2005. *An Introduction to Markov Processes*. Springer-Verlag, Berlin, Germany.
THORISSON, H. 2000. *Coupling, Stationarity, and Regeneration*. Springer-Verlag, New York.
TIN, M. 1965. Comparison of some ratio estimators. *J. ASA 60*, 309, 294–307.

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