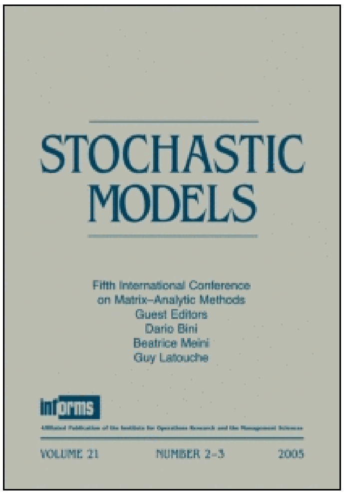

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LAWS OF LARGE NUMBERS AND FUNCTIONAL CENTRAL LIMIT THEOREMS FOR GENERALIZED SEMI-MARKOV PROCESSES

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□ *Because of the fundamental role played by generalized semi-Markov processes (GSMPs) in the modeling and analysis of complex discrete-event stochastic systems, it is important to understand the conditions under which a GSMP exhibits stable long-run behavior. To this end, we review existing work on strong laws of large numbers (SLLNs) and functional central limit theorems (FCLTs) for GSMPs; our discussion highlights the role played by the theory of both martingales and regenerative processes. We also sharpen previous limit theorems for finite-state irreducible GSMPs by establishing a SLLN and FCLT under the "natural" requirements of finite first (resp., second) moments on the clock-setting distribution functions. These moment conditions are comparable to the minimal conditions required in the setting of ordinary semi-Markov processes (SMPs). Corresponding discrete-time results for the underlying Markov chain of a GSMP are also provided. In contrast to the SMP setting, limit theorems for finite-state GSMPs require additional structural assumptions beyond irreducibility, due to the presence of multiple clocks. In our new limit theorems, the structural assumption takes the form of a "positive density" condition for specified clock-setting distributions. As part of our analysis, we show that finite moments for new clock readings imply finite moments for the od-regenerative cycles of both the GSMP and its underlying chain.*

Keywords Central limit theorem; Discrete-event stochastic systems; Generalized semi-Markov processes; Law of large numbers; Markov chains; Stability; Stochastic simulation.

1. INTRODUCTION

A wide variety of manufacturing, computer, transportation, telecommunication, and work-flow systems can usefully be viewed as *discrete-event stochastic systems*. Such systems evolve over continuous time and make

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stochastic state transitions when events associated with the occupied state occur; the state transitions occur only at an increasing sequence of random times. The underlying stochastic process of a discrete-event system records the state as it evolves over continuous time and has piecewise-constant sample paths.

The usual model for the underlying process of a complex discrete-event stochastic system is the *generalized semi-Markov process* (GSMP); see, for example, Refs.[7,19,23,26,27,30]. In a GSMP, events associated with a state compete to trigger the next state transition and each set of trigger events has its own probability distribution for determining the new state. At each state transition, new events may be scheduled. For each of these new events, a clock indicating the time until the event is scheduled to occur is set according to a probability distribution that depends on the current state, the new state, and the set of events that trigger the state transition. These clocks, along with the speeds at which the clocks run down, determine when the next state transition occurs and which of the scheduled events actually trigger this state transition. A GSMP $\{X(t) : t \geq 0\}$—here $X(t)$ denotes the state of the system at (real-valued) time t—is formally defined in terms of a general state space Markov chain $\{(S_n, C_n) : n \geq 0\}$ that records the state of the system, together with the clock readings, at successive state transitions. The GSMP model either subsumes or is closely related to a number of important applied probability models such as continuous time Markov chains, semi-Markov processes, Markovian and non-Markovian multiclass networks of queues (Ref.[27]), and stochastic Petri nets (Ref.[18]).

Given the central role played by the GSMP model in both theory and applications, it is fundamentally important to understand the conditions under which a GSMP $\{X(t) : t \geq 0\}$ exhibits stable long-run behavior. Strong laws of large numbers (SLLNs) and central limit theorems (CLTs) formalize this notion of stability. These limit theorems also provide approximations for cumulative-reward distributions, confidence intervals for statistical estimators, and efficiency criteria for simulation algorithms.

In more detail, an SLLN asserts the existence of time-average limits of the form $r(f) = \lim_{t\to\infty}(1/t)\int_0^t f(X(u))\,du$, where f is a real-valued function. If such an SLLN holds, then the quantity $\hat{r}(t) = (1/t)\int_0^t f(X(u))\,du$ is a strongly consistent estimator for $r(f)$. Viewing $R(t) = \int_0^t f(X(u))\,du$ as the cumulative "reward" earned by the system in the interval $[0, t]$, the SLLN also asserts that $R(t)$ can be approximated by the quantity $r(f)t$ when t is large. Central limit theorems (CLTs) serve to illuminate the rate of convergence in the SLLN, to quantify the precision of $\hat{r}(t)$ as an estimator of $r(f)$, and to provide approximations for the distribution of the cumulative reward $R(t)$ at large values of t. The ordinary form of the CLT asserts that under appropriate regularity conditions, the quantity $\hat{r}(t)$—suitably normalized—converges in distribution to a standard normal random variable. An ordinary CLT often can be strengthened to a *functional central limit theorem* (FCLT); see, for example, Refs.[3,6]. Roughly speaking, a stochastic process with time-average limit r obeys a FCLT if the associated cumulative (i.e., time-integrated) process—centered about the deterministic function $g(t) = rt$ and suitably compressed in space and time—converges in distribution to a standard Brownian motion as the degree of compression increases. A variety of estimation methods such as the method of batch means (with a fixed number of batches) are known to yield asymptotically valid confidence intervals for $r(f)$ when a FCLT holds (Ref.[11]). Moreover, FCLT's can be used to approximate pathwise properties of the reward process $\{R(t) : t \geq 0\}$ over finite time intervals via those of Brownian motion; see Ref.[3]. Also of interest are "discrete time" SLLNs and FCLTs for processes of the form $\{\tilde{f}(S_n, C_n) : n \geq 0\}$.

In this paper we review existing results on SLLNs and FCLTs for GSMPs. Our discussion highlights the role played by the theory of both martingales and regenerative processes. We also sharpen previous results by establishing a SLLN and FCLT for finite-state GSMPs under the "natural" conditions of irreducibility and finite first (resp., second) moments on the clock-setting distribution functions. These conditions are comparable to the minimal conditions under which SLLNs and FCLTs hold for ordinary semi-Markov processes (SMPs), namely, irreducibility and finite first (resp., second) moments for the holding-time distribution; see Ref.[9]. (Such conditions are "minimal" in that if we allow them to be violated, then we can easily find SMP models for which the conclusion of the SLLN or FCLT fails to hold.) In contrast to the case of ordinary SMPs, our new limit theorems for GSMPs impose a positive-density condition on the clock-setting distributions. Although this particular condition is by no means necessary, some such condition is needed in the face of the additional complexity caused by the presence of multiple clocks; indeed, we show that in the absence of such a condition the SLLN and FCLT can fail to hold.

2. GENERALIZED SEMI-MARKOV PROCESSES

We briefly review the notation for, and definition of, a GSMP. Following Ref.[27], let $E = \{e_1, e_2, \ldots, e_M\}$ be a finite set of *events* and S be a finite set of *states*. For $s \in S$, let $s \mapsto E(s)$ be a mapping from S to the nonempty subsets of E; here $E(s)$ denotes the set of all events that can potentially occur when the process is in state s. An event $e \in E(s)$ is said to be *active* in state s. When the process is in state s, the occurrence of one or more active events triggers a state transition. Denote by $p(s'; s, E^*)$ the probability that the new state is s' given that the events in the set E^* $(\subseteq E(s))$ occur simultaneously in state s. A "clock" is associated with each event. The clock reading for an active event indicates the remaining time until the event is scheduled to occur. These clocks, along with the speeds at which the clocks run down, determine which of the active events actually trigger the next state transition. Denote by $r(s, e)$ (≥ 0) the *speed* (finite, deterministic rate) at which the clock associated with event e runs down when the state is s; we assume that, for each $s \in S$, we have $r(s, e) > 0$ for some $e \in E(s)$. Typically in applications, all speeds for active events are equal to 1; zero speeds can be used to model preemptive-resume behavior. Let $C(s)$ be the set of possible *clock-reading vectors* when the state is s:

$$C(s) = \big\{c = (c_1, \ldots, c_M) : c_i \in [0, \infty) \text{ and } c_i > 0 \\ \text{if and only if } e_i \in E(s)\big\}.$$

Here the ith component of a clock-reading vector $c = (c_1, \ldots, c_M)$ is the clock reading associated with event e_i. Beginning in state s with clock-reading vector $c = (c_1, \ldots, c_M) \in C(s)$, the time $t^*(s, c)$ to the next state transition is given by

$$t^*(s, c) = \min_{\{i : e_i \in E(s)\}} c_i / r(s, e_i), \tag{1}$$

where $c_i/r(s, e_i)$ is taken to be $+\infty$ when $r(s, e_i) = 0$. The set of events $E^*(s, c)$ that trigger the next state transition is given by

$$E^*(s, c) = \{e_i \in E(s) : c_i - t^*(s, c) r(s, e_i) = 0\}.$$

At a transition from state s to state s' triggered by the simultaneous occurrence of the events in the set E^*, a finite clock reading is generated for each *new event* $e' \in N(s'; s, E^*) = E(s') - (E(s) - E^*)$. Denote the *clock-setting distribution function* (that is, the distribution function of such a new clock reading) by $F(\cdot; s', e', s, E^*)$. We assume that $F(0; s', e', s, E^*) = 0$, so that new clock readings are a.s. positive, and that $\lim_{x \to \infty} F(x; s', e', s, E^*) = 1$, so that each new clock reading is a.s. finite. For each *old event* $e' \in O(s'; s, E^*) = E(s') \cap \big(E(s) - E^*\big)$, the old clock reading is kept after the state transition. For $e' \in (E(s) - E^*) - E(s')$, event e' is cancelled and the clock reading is discarded. When E^* is a singleton set of the form $E^* = \{e^*\}$, we write $p(s'; s, e^*) = p(s'; s, \{e^*\})$, $O(s'; s, e^*) = O(s'; s, \{e^*\})$, and so forth. The GSMP is a continuous-time stochastic process $\{X(t) : t \geq 0\}$ that records the state of the system as it evolves.

Formal definition of the process $\{X(t) : t \geq 0\}$ is in terms of a general state space Markov chain $\{(S_n, C_n) : n \geq 0\}$ that describes the process at successive state-transition times. Heuristically, S_n represents the state and $C_n = (C_{n,1}, \ldots, C_{n,M})$ represents the clock-reading vector just after the nth state transition; see Ref.[27] for a formal definition of the chain. The chain takes values in the set $\Sigma = \bigcup_{s\in S}(\{s\} \times C(s))$. Denote by μ the *initial distribution* of the chain; for a (measurable) subset $B \subseteq \Sigma$, the quantity $\mu(B)$ represents the probability that $(S_0, C_0) \in B$. We use the notations P_μ and E_μ to denote probabilities and expected values associated with the chain, the idea being to emphasize the dependence on the initial distribution μ; when the initial state of the underlying chain is equal to some $(s, c) \in \Sigma$ with probability 1, we write $P_{(s,c)}$ and $E_{(s,c)}$. The symbol P^n denotes the n-step *transition kernel* of the chain: $P^n((s, c), A) = P_{(s,c)}\{(S_n, C_n) \in A\}$ for $(s, c) \in \Sigma$ and $A \subseteq \Sigma$; when $n = 1$ we simply write P to denote the 1-step transition kernel.

We construct a continuous time process $\{X(t) : t \geq 0\}$ from the chain $\{(S_n, C_n) : n \geq 0\}$ in the following manner. Let ζ_n $(n \geq 0)$ be the (nonnegative, real-valued) time of the nth state transition: $\zeta_0 = 0$ and

$$\zeta_n = \sum_{j=0}^{n-1} t^*(S_j, C_j)$$

for $n \geq 1$. Because S is finite, an argument as in Theorem 3.13 of Ref.[18], Ch. 3, shows that $P_\mu\{\sup_{n\geq 0} \zeta_n = \infty\} = 1$. Set

$$X(t) = S_{N(t)}, \tag{2}$$

where

$$N(t) = \sup\{n \geq 0 : \zeta_n \leq t\}. \tag{3}$$

The stochastic process $\{X(t) : t \geq 0\}$ defined by (2) is the GSMP. By construction, the GSMP takes values in the set S and has piecewise constant, right-continuous sample paths.

Example 2.1 (Patrolling Repairman). Following Ref.[27], consider a group of N (≥ 2) machines under the care of a single patrolling repairman who walks round the group of machines in a strictly defined order: $1, 2, \ldots, N, 1, 2, \ldots$. The repairman repairs and restarts a machine that is stopped and passes a machine that is running. For machine j, the successive times between completion of repair and the next stoppage are i.i.d. as a positive random variable L_j with finite second moment and a continuous distribution function. The time for the repairman to walk from machine j to the next machine and inspect it (before effecting repair or proceeding) is a positive constant W_j. The successive times to repair and restart machine j are i.i.d. as a positive random variable R_j with finite second moment. Set

$$X(t) = \big(Z_1(t), \ldots, Z_N(t), M(t), N(t)\big),$$

where

$$Z_j(t) = \begin{cases} 1 & \text{if machine } j \text{ is awaiting repair at time } t; \\ 0 & \text{otherwise,} \end{cases}$$

$$M(t) = \begin{cases} j & \text{if machine } j \text{ is under repair at time } t; \\ 0 & \text{if no machine is under repair at time } t, \end{cases}$$

and $N(t) = j$ if, at time t, machine j is the next machine to be visited by the repairman. The process $\{X(t) : t \geq 0\}$ can be specified as a GSMP with unit speeds, finite state space $S \subset \{0,1\}^N \times \{0, 1, \ldots, N\} \times \{1, 2, \ldots, N\}$, and event set $E = \{e_1, e_2, \ldots, e_{N+2}\}$, where e_j = "stoppage of machine j" for $1 \leq j \leq N$, e_{N+1} = "completion of repair," and e_{N+2} = "arrival of repairman at a machine." The set $E(s)$ of active events is defined as follows. For $s = (z_1, z_2, \ldots, z_N, m, n) \in S$, event $e_j \in E(s)$ (where $1 \leq j \leq N$) if and only if $z_j = 0$ and $m \neq j$, event $e_{N+1} \in E(s)$ if and only if $m > 0$, and event $e_{N+2} \in E(s)$ if and only if $m = 0$. Each state-transition probability is equal to 0 or 1. For example, if $e^* = e_j$ (with $1 \leq j \leq N$), then $p(s'; s, e^*) = 1$ when $s = (z_1, \ldots, z_{j-1}, 0, z_{j+1}, \ldots, z_N, m, n)$ and $s' = (z_1, \ldots, z_{j-1}, 1, z_{j+1}, \ldots, z_N, m, n)$, and $p(s'; s, e^*) = 0$ otherwise. The clock-setting distribution function $F(x; s', e', s, e^*)$ is defined as follows. If $e' = e_j$ $(1 \leq j \leq N)$, then $F(x; s', e', s, e^*) = P\{L_j \leq x\}$; if $e' = e_{N+1}$ and $s' = (z_1, \ldots, z_N, m, n)$, then $F(x; s', e', s, e^*) = P\{R_m \leq x\}$; if $e' = e_{N+2}$ and $s' = (z_1, \ldots, z_N, 0, n)$, then $F(x; s', e', s, e^*) = 1_{[0,x]}(W_{n-1})$. (Here 1_A denotes the indicator function for the set A and W_{n-1} is taken as W_N when $n = 1$.) See Ref.[27], pp. 29–31, for further details.

When $E(s)$ is a singleton set for each $s \in S$, so that there is exactly one event active at any time point, the GSMP reduces to an ordinary SMP as defined, for example, in Ref.[4]. The limit theory considered in this paper simplifies considerably under this restriction. Alternatively, when each clock-setting distribution is of the form $F(x; s', e', s, E^*) \equiv 1 - e^{-\lambda(e')x}$, then the GSMP coincides (Ref.[18], Sec. 3.4) with a continuous-time Markov chain, and the well known limit theory for such chains applies.

3. A SURVEY OF LIMIT THEORY FOR GSMPs

In this section we give an overview of previous work on SLLNs and FCLTs for GSMPs and their underlying Markov chains.

3.1. Strong Laws of Large Numbers

A SLLN for a GSMP gives conditions under which there exists a finite constant $r(f)$, independent of the initial distribution μ, such that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(X(u))\,du = r(f) \quad \text{a.s.} \tag{4}$$

for a specified function f. Similarly, a discrete-time SLLN for the underlying chain of a GSMP gives conditions under which

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \tilde{f}(S_j, C_j) = \tilde{r}(\tilde{f}) \quad \text{a.s.} \tag{5}$$

SLLNs have been obtained for GSMPs either by directly exploiting limit theorems for Harris recurrent Markov chains or by appealing to the theory of regenerative processes. We describe these two approaches below.

3.1.1. *Direct Approach via Harris-Chain Theory*

One approach to obtaining an SLLN in discrete time is to apply results for Harris recurrent Markov chains to the underlying chain of a GSMP. To this end, we review some pertinent terminology for a general Markov chain $\{Z_n : n \geq 0\}$ taking values in a (possibly uncountably infinite) state space Γ; see Ref.[24] for details. Such a chain is *ϕ-irreducible* if ϕ is a nontrivial measure on subsets of Γ and, for each $z \in \Gamma$ and subset $A \subseteq \Gamma$ with $\phi(A) > 0$, there exists $n \geq 1$—possibly depending on both z and A—such that $P^n(z, A) > 0$. (Here P^n is the n-step transition kernel for the chain.) A ϕ-irreducible chain is *Harris recurrent* if $P_z\{Z_n \in A \text{ i.o.}\} = 1$ for all $z \in \Gamma$ and $A \subseteq \Gamma$ with $\phi(A) > 0$. Recall that a probability distribution π_0 is *invariant* with respect to $\{Z_n : n \geq 0\}$ if and only if $\int P(z, A)\,\pi_0(dz) = \pi_0(A)$ for each $A \subseteq \Gamma$. A Harris recurrent chain admits an invariant distribution π_0 that is unique up to constant multiples. If $\pi_0(\Gamma) < \infty$, then $\pi(\cdot) = \pi_0(\cdot)/\pi_0(\Gamma)$ is the unique invariant probability distribution for the chain. A Harris recurrent chain that admits such a probability distribution is called *positive Harris recurrent*. Given an invariant probability distribution π together with a real-valued function f defined on Γ, we often write $\pi(f) = \int f(z)\,\pi(dz) = E_\pi[f(Z_0)]$. Observe that $\pi(f)$ is well defined and finite if and only if $\pi(|f|) < \infty$, where $|f|(z) = |f(z)|$ for $z \in \Gamma$.

If $\{Z_n : n \geq 0\}$ is positive Harris recurrent with invariant distribution π, then $\lim_{n\to\infty}(1/n)\sum_{i=0}^{n-1} f(Z_n) = \pi(f)$ a.s. for any $f : \Gamma \mapsto \Re$ such that $\pi(|f|) < \infty$. The idea behind the proof of this assertion—see Ref.[24], Sec. 17.1, for details—is to apply the SLLN for stationary sequences (Ref.[5]) to establish the desired result when the initial distribution is π.

The result can then be extended to arbitrary initial conditions by showing that (i) any bounded "harmonic" function—that is any bounded function h satisfying $\int P(x, dy)h(y) = h(x)$ for all $x \in \Gamma$—must be constant when $\{Z_n : n \geq 0\}$ is Harris recurrent, and (ii) the function $h(x) = P_x\{\lim_{n\to\infty}(1/n) \sum_{i=0}^{n-1} f(Z_n) = \pi(f)\}$ is harmonic.

Thus the main challenge in establishing an SLLN is to show that $\{Z_n : n \geq 0\}$ is positive Harris recurrent. Stochastic Lyapunov functions provide an effective tool for this purpose. Following Ref.[24], we say that a subset $B \subseteq \Gamma$ is *petite* with respect to the chain if there exists a probability distribution q on the nonnegative integers and a nontrivial measure ψ on subsets of Γ such that

$$\inf_{z\in B} \sum_{n=0}^{\infty} q(n)P^n(z, A) \geq \psi(A)$$

for $A \subseteq \Gamma$. For real-valued functions f and g defined on Γ, write $f = O(g)$ if $\sup_{z\in\Gamma}|f(z)|/|g(z)| < \infty$. (Here we take $0/0 = 0$.) The following result is given in Ref.[24].

Proposition 3.1.1.1. *Let $\{Z_n : n \geq 0\}$ be a ϕ-irreducible Markov chain. Suppose that there exists a petite set B, an integer $m \geq 1$, a function $v : \Gamma \mapsto [0, \infty)$ and a function $g : \Gamma \mapsto [1, \infty)$ such that*

$$E_z[v(Z_m) - v(Z_0)] \leq -g(z) \tag{6}$$

for all $z \in \Gamma - B$, and

$$\sup_{z\in B} E_z[v(Z_m) - v(Z_0)] < \infty. \tag{7}$$

Then

(i) $\{Z_n : n \geq 0\}$ *is positive Harris recurrent with recurrence measure ϕ and hence admits an invariant probability measure π; and*
(ii) $\pi(|f|) < \infty$ *for any function $f : \Gamma \mapsto \Re$ such that $f = O(g)$.*

The function v is the stochastic Lyapunov function, and $v(z)$ can be viewed as the "distance" between state z and the set B. The quantity $E_z[v(Z_m) - v(Z_0)]$ in (6) and (7) is called the m-step expected *drift* of the chain. Thus, the condition in (6) asserts that the m-step expected drift is strictly negative whenever the chain lies outside of B; the exact "rate of drift" is specified by the function g.

Proposition 3.1.1.2 below is proved in Ref.[17] and can be used to apply the foregoing results in the GSMP setting. To prepare for the proposition, we introduce some notation and terminology. For a GSMP with state space S and event set E and for $s, s' \in S$ and $e \in E$, write $s \xrightarrow{e} s'$ if $p(s'; s, e)r(s, e) > 0$ and write $s \to s'$ if $s \xrightarrow{e} s'$ for some $e \in E(s)$. Also write $s \rightsquigarrow s'$ if either $s \to s'$ or there exist states $s_1, s_2, \ldots, s_n \in S$ $(n \geq 1)$ such that $s \to s_1 \to \cdots \to s_n \to s'$.

Definition 3.1.1.1. A GSMP is *irreducible* if $s \rightsquigarrow s'$ for each $s, s' \in S$.

Recall that a nonnegative function G is a *component* of a distribution function F if G is not identically equal to 0 and $G \leq F$. If G is a component of F and G is absolutely continuous, so that G has a density function g, then we say that g is a *density component* of F.

Assumption PD(q), defined below, encapsulates the key conditions used in Proposition 3.1.1.2 and elsewhere.

Definition 3.1.1.2. *Assumption* PD(q) holds for a specified GSMP and real number $q \geq 0$ if

(i) the state space S of the GSMP is finite;
(ii) the GSMP is irreducible;
(iii) all speeds of the GSMP are positive; and
(iv) there exists $\bar{x} \in (0, \infty)$ such that each clock-setting distribution function $F(\cdot; s', e', s, E^*)$ of the GSMP has finite qth moment and a density component that is positive and continuous on $(0, \bar{x})$.

Observe that when Assumption PD(q) holds for some $q \geq 0$, there can be at most a finite number of state transitions at which two or more events occur simultaneously. Also observe that if Assumption PD(q) holds for some $q \geq 0$, then Assumption PD(r) holds for $r \in [0, q)$.

For $0 < u \leq \infty$, Let ϕ_u be the unique measure on Borel subsets of Σ such that

$$\phi_u\big(\{s\} \times [0, a_1] \times [0, a_2] \times \cdots \times [0, a_M]\big) = \prod_{\{i: e_i \in E(s)\}} \min(a_i, u) \tag{8}$$

for all $s \in S$ and $a_1, a_2, \ldots, a_M \geq 0$. If, for example, a set $B \subseteq \Sigma$ is of the form $B = \{s\} \times A$ with $E(s) = E$, then $\phi_u(B)$ is equal to the Lebesgue measure of the set $A \cap [0, u)^M$. For $b > 0$, denote by H_b the set of all states $(s, c) \in \Sigma$ such that each clock reading is bounded above by b:

$$H_b = \big(S \times [0, b]^M\big) \cap \Sigma. \tag{9}$$

Finally, for $s \in S$, $c = (c_1, c_2, \ldots, c_M) \in C(s)$, and $q \geq 0$, set

$$h_q(s, c) = 1 + \max_{1 \leq i \leq M} c_i^q. \tag{10}$$

Proposition 3.1.1.2. *Suppose that Assumption* PD(0) *holds. Then*

(i) *the underlying chain* $\{(S_n, C_n) : n \geq 0\}$ *is* ϕ_u*-irreducible for some* $0 < u \leq \infty$*; and*
(ii) *the set* H_b *defined by* (9) *is petite with respect to* $\{(S_n, C_n) : n \geq 0\}$ *for each* $b > 0$.

If, moreover, Assumption PD(q) *holds for some* $q \geq 1$*, then for all sufficiently large values of* b

(iii) *the function* h_q *defined by* (10) *satisfies*

$$\sup_{(s,c)\in H_b} E_{(s,c)}[h_q(S_M, C_M) - h_q(S_0, C_0)] < \infty$$

and
(iv) *there exists* $\beta > 0$ *such that*

$$E_{(s,c)}[h_q(S_M, C_M) - h_q(S_0, C_0)] \leq -\beta h_{q-1}(s, c) \tag{11}$$

for $(s, c) \in \Sigma - H_b$.

Henderson and Glynn[21] have extended the result in Proposition 3.1.1.2(i) to GSMPs with infinite state space. Following Ref.[17], we can combine the foregoing results to obtain a discrete-time SLLN.

Theorem 3.1.1.1. *Suppose that Assumption* PD($u + 1$) *holds for some* $u \geq 0$. *Then the underlying chain* $\{(S_n, C_n) : n \geq 0\}$ *is positive Harris recurrent and hence admits an invariant probability measure* π. *Moreover, if* $\tilde{f} : \Sigma \mapsto \Re$ *satisfies* $\tilde{f} = O(h_u)$*, then* $\pi(|\tilde{f}|) < \infty$ *and*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \tilde{f}(S_j, C_j) = \pi(\tilde{f}) \quad a.s.$$

To obtain a SLLN in continuous time for a specified function $f : S \mapsto \Re$, recall the definition of the function t^* from (1) and set $\tilde{f}(s, c) = f(s)t^*(s, c)$ for $(s, c) \in \Sigma$ or, more concisely, $\tilde{f} = ft^*$. Observe that

$$\frac{1}{t}\int_0^t f\big(X(u)\big)\,du = \frac{\big(1/N(t)\big)\sum_{i=0}^{N(t)-1} \tilde{f}(S_n, C_n) + R_1(t)}{\big(1/N(t)\big)\sum_{i=0}^{N(t)-1} t^*(S_n, C_n) + R_2(t)} \tag{12}$$

for $t \geq 0$, where $R_1(t)$ and $R_2(t)$ are remainder terms and $N(t)$ is defined as in (3). Observe that $\tilde{f} = O(h_1)$ and $t^* = O(h_1)$, so that, with probability 1, $\lim_{n\to\infty}(1/n)\sum_{j=0}^{n-1} \tilde{f}(S_j, C_j) = \pi(\tilde{f})$ and $\lim_{n\to\infty}(1/n)\sum_{j=0}^{n-1} t^*(S_j, C_j) =$

$\pi(t^*)$ when the conditions of Theorem 3.1.1.1 hold. Under these conditions, it can be shown (Refs.[10,18]) that $N(t) \to \infty$ a.s. and that $R_1(t)$ and $R_2(t)$ become negligible relative to the other terms, thereby establishing the following result (Ref.[17]).

Theorem 3.1.1.2. *Suppose that Assumption* PD(2) *holds. Then the chain* $\{(S_n, C_n) : n \geq 0\}$ *is positive Harris recurrent and hence admits an invariant probability measure* π. *Moreover,*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f\big(X(u)\big)\,du = \frac{\pi(ft^*)}{\pi(t^*)} \quad a.s.$$

for any function $f : S \mapsto \Re$.

3.1.2. *Approach via Regenerative-Process Theory*

An alternative approach to establishing SLLNs in discrete and continuous time rests on properties of regenerative processes and their extensions. Informally, a continuous-time process $\{X(t) : t \geq 0\}$ is regenerative if it admits a sequence of random time points, called *regeneration points*, at which the process "probabilistically restarts." The regeneration points serve to decompose sample paths of the process into i.i.d. cycles. It is convenient to work with a slightly more general class of processes, defined below.

Definition 3.1.2.1. The stochastic process $\{X(t) : t \geq 0\}$ with state space S is an *od-regenerative process* in continuous time if there exists an increasing sequence $0 \leq T_0 < T_1 < T_2 < \cdots$ of a.s. finite random times such that, for $k \geq 1$, the post-T_k process $\{X(T_k + t) : t \geq 0;\ \tau_{k+l} : l \geq 1\}$

(i) is distributed as the post-T_0 process $\{X(T_0 + t) : t \geq 0;\ \tau_l : l \geq 1\}$, and
(ii) is independent of the pre-T_{k-1} process $\{X(t) : 0 \leq t < T_{k-1};\ \tau_1, \ldots, \tau_{k-1}\}$,

where $\tau_j = T_j - T_{j-1}$ for $j \geq 1$.

The od-regeneration points serve to decompose sample paths of $\{X(t) : t \geq 0\}$ into one-dependent stationary cycles. The random variable τ_k defined above is the length of the kth cycle. A classical regenerative process is a special case of an od-regenerative process in which the cycles are not only identically distributed, but are also mutually independent. Thorough discussions of od-regenerative and related processes can be found, for example, in Refs.[1,8,18,28,29].

When $T_0 = 0$ the process $\{X(t) : t \geq 0\}$ is *nondelayed*; otherwise, it is called *delayed*. For a delayed process $\{X(t) : t \geq 0\}$, the "0th cycle" $\{X(t) :$

$0 \le t < T_0\}$ need not have the same distribution as the other cycles. Similarly, the length of this cycle—denoted by τ_0—need not have the same distribution as τ_1, τ_2, and so forth.

The usefulness of od-regenerative structure stems from the fact that it permits application of well known results for m-dependent random variables. Specifically, given an od-regenerative process $\{X(t) : t \ge 0\}$ with state space S and od-regeneration points $\{T_k : k \ge 0\}$, along with a real-valued function f defined on S, set

$$Y_k(f) = \int_{T_{k-1}}^{T_k} f(X(u))\,du$$

for $k \ge 0$. (Take $T_{-1} = 0$.) It follows from the definition of an od-regenerative process that the sequence $\{(Y_k(f), \tau_k) : k \ge 1\}$ consists of one-dependent identically distributed random pairs. Set

$$r(f) = \frac{E[Y_1(f)]}{E[\tau_1]}$$

and observe that $r(f)$ is well defined and finite if and only if $E[Y_1(|f|)]$, and hence $r(|f|)$, is finite. Denoting by $M(t)$ the number of regeneration points in $[0, t]$, we have $\int_0^t f\big(X(u)\big)du = \sum_{k=0}^{M(t)-1} Y_k(f) + R_1(t)$ and $t = \sum_{k=0}^{M(t)-1} \tau_k + R_2(t)$ for appropriate remainder terms $R_1(t)$ and $R_2(t)$, and an argument similar to the proof of Theorem 3.1.1.2 using the classical SLLN for m-dependent random variables (Ref.[2], p. 86), yields the following result.

Proposition 3.1.2.1. *Suppose that* $E[\tau_1] < \infty$. *Then* $r(|f|) < \infty$ *and*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(X(u))\,du = r(f) \quad a.s.$$

for any real-valued function f *such that* $Y_0(|f|) < \infty$ *a.s. and* $E[Y_1(|f|)] < \infty$.

See Ref.[1] for a detailed proof in the setting of classical regenerative processes. The foregoing development has an obvious analog for a discrete-time process $\{X_n : n \ge 0\}$; the discrete-time results can be obtained by applying the continuous-time theory to the process $\{X_{\lfloor t \rfloor} : t \ge 0\}$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

To apply Proposition 3.1.2.1 in the GSMP setting, we must show that the GSMP of interest (or its underlying chain) is od-regenerative and that quantities such as τ_1 and $Y_1(|f|)$ have finite mean. A number of authors (Refs.[18,19,22,27]) have identified simple conditions on the building blocks of a GSMP that ensure probabilistic restart, and hence classical regenerative structure. For example, suppose that state $\bar{s} \in S$ is a "single state" in that $E(\bar{s}) = \{\bar{e}\}$ for some $\bar{e} \in E$. Then the successive times at which event $\bar{e}$ occurs in state $\bar{s}$ form a sequence of regeneration points. Indeed, at each such regeneration point the new state s' is always chosen according to the fixed distribution function $p(\cdot; \bar{s}, \bar{e})$, the clock for each new event e' is set according to $F(\cdot; s', e', \bar{s}, \bar{e})$ regardless of the past history of the GSMP, and there are no old events.

The other sufficient conditions for probabilistic restart discussed in Refs.[18,19,22,27] are variations on this theme. For example, let $\bar{s} \in S$ and suppose that each event $e' \in E(\bar{s})$ has a clock-setting distribution function of the form $F(x; s', e', s, E^*) \equiv F(x; e') = 1 - \exp\big(-\lambda(e')x\big)$. [An event e' such that $F(x; s', e', s, E^*) \equiv F(x; e')$ is called *simple*.] It follows (nontrivially) from the memoryless property of the exponential distribution that the successive state-transition times at which the new state is $\bar{s}$ form a sequence of regeneration points.

Glynn[7] provides a different set of conditions on an irreducible finite-state GSMP that ensure probabilistic restart. Specifically, each event is assumed to be simple, and each clock-setting distribution function $F(\cdot; e)$ is assumed to be absolutely continuous with density function $f(\cdot; e)$ and "exponentially bounded" in that the hazard function $h(x; e) = f(x; e)/\big(1 - F(x; e)\big)$ is bounded both above and below by positive constants. It then follows that $f(x; e) = \delta_e \beta_e \exp(-\beta_e x) + (1 - \delta_e) q_e(x)$ for some density function q_e and constants $\beta_e > 0$ and $\delta_e \in (0, 1)$. Conceptually, each new clock reading for event e is selected according to either an exponential distribution or according to q_e, depending on the outcome of a Bernoulli trial having success probability δ_e. For any fixed state $\bar{s} \in S$, a geometric trials argument then shows that, with probability 1, the GSMP makes infinitely many transitions to $\bar{s}$ such that the clock for each event $e \in E(\bar{s})$ has most recently been set according to an exponential distribution. As discussed previously, these state-transition times form a sequence of regeneration points.

Once probabilistic restart has been established, it remains to show that the regenerative cycle length τ_1 has finite mean, which in turn implies that $Y_1(|f|)$ has finite mean when S is finite. One approach (Refs.[18,19,22,27]) to establishing finite moments combines geometric trials arguments with "new better than used" (NBU) distributional assumptions. A distribution F with support on $[0, \infty)$ is NBU if $\overline{F}(x + y) \leq \overline{F}(x)\overline{F}(y)$ for all $x, y \geq 0$, where $\overline{F} = 1 - F$. Equivalently, a random variable L is NBU if $P\{L > x + y \mid L > x\} \leq P\{L > y\}$; viewing L as the lifetime of a component, the NBU condition asserts that a new component is more likely than a used component (which has been running for x time units) to survive for at least the next y time units.

The following example illustrates the basic ideas. Suppose that each event is simple and there exist infinite sequences of random times $\{\zeta_{\alpha(n)} : n \geq 0\}$ and $\{\zeta_{\beta(n)} : n \geq 0\}$ with $\zeta_{\alpha(n)} < \zeta_{\beta(n)} \leq \zeta_{\alpha(n+1)}$ for $n \geq 0$ and such that

(i) A probabilistic restart occurs at time $\zeta_{\beta(n)}$ if every event that belongs to a specified set $\widetilde{E}$ and is active at time $\zeta_{\alpha(n)}$ occurs "soon enough," i.e., before the occurrence of another specified new event $e_{i^*} \notin \widetilde{E}$;
(ii) The clock-setting distribution $F(\cdot; e_i)$ is NBU for each $e_i \in \widetilde{E}$;
(iii) For $e_i \in \widetilde{E}$, there is a positive probability that an independent sample from $F(\cdot; e_i)$ is smaller than an independent sample from $F(\cdot; e_{i^*})$; and
(iv) $\liminf_{n\geq 0} E_\mu[\zeta_{\beta(n+1)} - \zeta_{\beta(n)}] < \infty$.

We claim that $E_\mu[\tau_1] < \infty$, where $\tau_1, \tau_2, \ldots$ are the lengths of the i.i.d. cycles delineated by the successive points in $\{\zeta_{\beta(n)} : n \geq 0\}$ at which a probabilistic restart occurs. The idea is that a "trial" occurs at each time $\zeta_{\alpha(n)}$, where a "success" corresponds to a probabilistic restart at time $\zeta_{\beta(n)}$. The NBU and structural assumptions can be used to uniformly bound the success probability away from zero: letting R_n be the event in which a probabilistic restart occurs at time $\zeta_{\beta(n)}$, $\mathscr{G}_n = \{X(t) : 0 \leq t \leq \zeta_{\alpha(n)}\}$, and $\widetilde{E}_n = \widetilde{E} \cap E(S_{\alpha(n)})$, we have

$$\begin{aligned} P_\mu\{R_n \mid \mathscr{G}_n\} &\geq P_\mu\{C_{\alpha(n),i} \leq C_{\alpha(n),i^*} \text{ for } e_i \in \widetilde{E}_n \mid \mathscr{G}_n\} \\ &\geq P_\mu\{A_i \leq C_{\alpha(n),i^*} \text{ for } e_i \in \widetilde{E}_n \mid \mathscr{G}_n\} \\ &\geq P\{A_i \leq A_{i^*} \text{ for } e_i \in \widetilde{E}\} \stackrel{\text{def}}{=} \delta \end{aligned}$$

for $n \geq 0$, where A_i denotes an independent sample from $F(\cdot; e_i)$. Here the first inequality follows from the assumption in (i). The second inequality follows (nontrivially) from the assumption in (ii); intuitively, the clock reading for event e_i is more likely to be "small" (i.e., less than the new clock reading $C_{\alpha(n),i^*}$) than is a fresh sample from the clock-setting distribution for e_i. The assumption in (iii) ensures that $\delta > 0$. A "conditional" geometric trials argument (Ref.[18], pp. 88–89), now shows that the number of trials γ between regeneration points is stochastically dominated by a geometrically-distributed random variable (having success probability δ) and hence has finite moments of all orders. The quantity τ_1 can be represented as a random sum containing γ terms, and the assumption in (iv), which is often easy to verify in practice, controls the expected size of these terms.

Example 3.1.2.1 (Patrolling Repairman). For the model of Example 2.1, it can be seen that the system probabilistically restarts whenever the repairman arrives at machine 1 and all machines are stopped. Indeed, just prior to this event the state of the system is $s = (1, 1, \ldots, 1, 0, 1)$, which is a single state. We can therefore take $\zeta_{\beta(n)}$ (resp., $\zeta_{\alpha(n)}$) to be the nth time at which the repairman arrives at machine 1 (resp., leaves machine N), event e_{i^*} to be e_{N+2} = "arrival of repairman at a machine," and the set $\widetilde{E}$ to be $\{e_1, e_2, \ldots, e_N\}$. Thus $\{X(t) : t \geq 0\}$ is a regenerative process with finite expected cycle length if each machine lifetime L_j is NBU with $P\{L_j < W_N\} > 0$. (Recall that W_N is the deterministic walking time from machine N to machine 1.) Note that the expected time between successive arrivals to machine 1 is bounded above by $\sum_{j=1}^{N}\big(W_j + E[R_j]\big)$, so that the condition in (iv) above is satisfied.

Refs.[18,19,22,27] give refinements and extensions of the foregoing approach. For example, the NBU requirements can be relaxed to require that specified distributions have a "generalized NBU" (GNBU) property; a distribution F is GNBU if $\sup_{y\geq 0} \overline{F}(x+y)/\overline{F}(y) < 1$ for some $x \geq 0$.

The geometric trials approach avoids imposition of positive density assumptions on the clock-setting distributions, but establishing the conditions in (i)–(iv) typically requires detailed knowledge of the GSMP under study. A more generic regenerative approach to the SLLN is discussed in Ref.[17] for GSMPs having a single state $\bar{s}$, and hence classical regenerative structure. The set of states $\boldsymbol{\alpha} = \{(\bar{s}, c) : c \in C(\bar{s})\}$ is called an "atom" of the underlying chain $\{(S_n, C_n) : n \geq 0\}$ and has the defining property that $P\big((s, c), A\big) = P\big((s', c'), A\big)$ for any set A whenever $(s, c), (s', c') \in \boldsymbol{\alpha}$. For any Markov chain $\{Z_n : n \geq 0\}$ having an atom $\boldsymbol{\alpha}$ and satisfying the conditions of Proposition 3.1.1.1 it can be shown (Ref.[24], p. 334) that $E_z\big[\sum_{n=1}^{T_{\boldsymbol{\alpha}}} u(Z_n)\big] < \infty$ for any $z \in \boldsymbol{\alpha}$ and $u = O(g)$, where $T_{\boldsymbol{\alpha}}$ is the first hitting time of $\boldsymbol{\alpha}$. Thus, if Assumption PD(2) holds for a GSMP with a single state, then Proposition 3.1.1.2 implies that $E[\tau_1] < \infty$ and $E[Y_1(|f|)] < \infty$ for any function $f : S \mapsto \Re$, and an application of Proposition 3.1.2.1 establishes the desired SLLN for $\{X(t) : t \geq 0\}$. This result differs from Theorem 3.1.1.2 in that there is an additional assumption (the presence of a single state) and a corresponding representation of the time-average limit as a ratio of quantities defined in terms of a regenerative cycle. Analogous arguments in discrete time yield a "regenerative version" of Theorem 3.1.1.1.

Finally, we note that in the setting of Ref.[7], the assumption that each clock-setting distribution function is exponentially bounded ensures that each distribution has finite moments of all orders. The finiteness of the mean cycle length then follows by an argument similar to the proof of Wald's identity.

3.2. Functional Central Limit Theorems

Given a GSMP satisfying the SLLN in (4) for a specified function f, set

$$U_\nu(f)(t) = \frac{1}{\sqrt{\nu}} \int_0^{\nu t} \big(f(X(u)) - r(f)\big)\,du \tag{13}$$

for $t, v \geq 0$; each random function $U_v(f)$ is an element of $C[0,\infty)$, the space of continuous real-valued functions on $[0,\infty)$. A FCLT gives conditions under which there exists a finite constant $\sigma(f) \geq 0$ such that $U_v(f) \Rightarrow \sigma(f)W$ as $v \to \infty$ for any initial distribution μ. Here $W = \{W(t) : t \geq 0\}$ denotes a standard Brownian motion on $[0,\infty)$ and $\Rightarrow$ denotes weak convergence on $C[0,\infty)$; see Refs.[3,6]. Weak convergence on $C[0,\infty)$ generalizes to a sequence of random functions—i.e., a sequence of stochastic processes—the usual notion of convergence in distribution of a sequence of random variables.

Similarly, supposing that (5) holds, a discrete-time FCLT give conditions under which there exists a finite constant $\tilde{\sigma}(\tilde{f}) \geq 0$ such that $U_n(\tilde{f}) \Rightarrow \tilde{\sigma}(\tilde{f})W$ as $n \to \infty$ for any initial distribution μ, where

$$U_n(\tilde{f})(t) = \frac{1}{\sqrt{n}} \int_0^{nt} \big(\tilde{f}(S_{\lfloor u \rfloor}, C_{\lfloor u \rfloor}) - \tilde{r}(\tilde{f})\big)\,du. \tag{14}$$

The two main techniques for establishing FCLTs rest on the theory of martingales and regenerative processes, respectively.

3.2.1. *Approach via Martingale Theory*

One approach[17] to establishing a discrete-time FCLT for the underlying chain of a GSMP is to apply results in Refs.[13,24], which are in turn obtained by combining Lyapunov-function arguments with martingale methods. Limit theorems for the continuous-time process $\{X(t) : t \geq 0\}$ can then be obtained using random-time-change arguments.

Specifically, suppose that the underlying chain of a GSMP is positive Harris recurrent with invariant distribution π, and suppose that $\pi(|\tilde{f}|) < \infty$ for the function $\tilde{f}$ of interest. Then the key idea is to establish the existence of a solution $\tilde{h}$ to *Poisson's equation*:

$$\tilde{f}(s,c) - \pi(\tilde{f}) = \tilde{h}(s,c) - P\tilde{h}(s,c), \quad (s,c) \in \Sigma,$$

where $P\tilde{h}(s,c) = E_{(s,c)}[\tilde{h}(S_1, C_1)]$. Setting $L_n = \sum_{j=0}^{n-1} \tilde{f}(S_j, C_j)$ and $M_n = L_n + \tilde{h}(S_n, C_n) - \tilde{h}(S_0, C_0)$ for $n \geq 0$, it follows from Poisson's equation that

$$M_n = \sum_{j=0}^{n-1} \big\{\tilde{h}(S_{j+1}, C_{j+1}) - E_\mu\big[\tilde{h}(S_{j+1}, C_{j+1}) \mid (S_j, C_j)\big]\big\}.$$

Thus $\{M_n : n \geq 0\}$ is a martingale, and hence the partial sum process of interest $\{L_n : n \geq 0\}$ is "almost" a martingale. Provided that $\pi(\tilde{h}^2) < \infty$, an application of a FCLT for martingales as in Ref.[20] establishes a FCLT for $\{M_n : n \geq 0\}$ under initial distribution π, and a corresponding FCLT for $\{\tilde{f}(S_n, C_n) : n \geq 0\}$ follows directly.

To show that the solution to Poisson's equation has finite second moment with respect to π, we can use the following result from Ref.[13] for a general Markov chain $\{Z_n : n \geq 0\}$. Suppose that the drift conditions in (6) and (7) hold. Then, for any $f = O(g)$, Poisson's equation $f - \pi(f) = h - Ph$ admits a solution h satisfying the bound $|h| \leq c_0(v + 1)$ for some constant c_0. Hence if $\pi(v^2) < \infty$, then $\pi(h^2) < \infty$ and the chain satisfies a FCLT when the initial distribution is π. It can then be shown[13] that the FCLT in fact holds for any initial distribution of the chain.

We can now obtain an FCLT in the GSMP setting in a manner analogous to our derivation of the SLLN in Theorem 3.1.1.1. Suppose that Assumption $PD(2u + 3)$ holds for some $u \geq 0$. First apply Proposition 3.1.1.2 with $q = u + 1$ to show that the conditions in (6) and (7) hold with $v = h_{u+1}$ and $g = h_u$. Apply Proposition 3.1.1.2 again with $q = 2u + 3$ followed by Proposition 3.1.1.1(ii) to show that $\pi(h_{2u+2}) < \infty$, and hence $\pi(v^2) < \infty$. The discussion in the preceding paragraph now implies the following result, where we define $U_n(\tilde{f})$ as in (14), but with $\tilde{r}(\tilde{f})$ replaced by $\pi(\tilde{f})$.

Theorem 3.2.1.1. *Suppose that Assumption* PD$(2u + 3)$ *holds for some* $u \geq 0$. *Let* $\tilde{f} : \Sigma \mapsto \Re$ *be a specified function such that* $\tilde{f} = O(h_u)$. *Then there exists* $\tilde{\sigma}(\tilde{f}) \geq 0$ *such that* $U_n(\tilde{f}) \Rightarrow \tilde{\sigma}(\tilde{f})W$ *as* $n \to \infty$ *for any initial distribution* μ.

Ref.[18] gives variants of Theorems 3.1.1.1 and 3.2.1.1 in which the assumption of finite qth moments for the clock-setting distributions is strengthened to an assumption of convergent Laplace–Stieltjes transforms in a neighborhood of the origin. Then the SLLN and FCLT can be shown to hold, e.g., for any function $\tilde{f}(s, c)$ such that $|\tilde{f}|$ is bounded above by some polynomial function of the clock readings.

To derive a continuous-time FCLT from Theorem 3.2.1.1, we can proceed almost as in the derivation of Theorem 3.1.1.2 and apply the discrete time result (with $u = 1$) to the function $\tilde{f}(s, c) = (ft^*)(s, c) = f(s)t^*(s, c)$. Instead of using (12), however, we use a random-time-change argument; see Ref.[17] for details. The resulting continuous-time FCLT is as follows, where we define $U_v(\tilde{f})$ as in (13), but with $r(f)$ replaced by $\pi(ft^*)/\pi(t^*)$.

Theorem 3.2.1.2. *Suppose that Assumption* $PD(5)$ *holds and let* f *be an arbitrary real-valued function defined on* S. *Then there exists* $\sigma(f) \geq 0$ *such that* $U_v(f) \Rightarrow \sigma(f)W$ *as* $v \to \infty$ *for any initial distribution* μ.

Glynn and Haas[9] apply the foregoing approach in the setting of finite-state irreducible SMPs, establishing a FCLT under essentially the assumption that the holding time in each state has finite second moment. In this simpler stochastic process setting, the martingale approach leads to a closed-form expression for the variance constant $\sigma^2(f)$ in the FCLT.

3.2.2. *Approach via Regenerative-Process Theory*

The regenerative approach to establishing an FCLT parallels the development of the SLLN in Section 3.1.2, using the following FCLT for od-regenerative processes. Let $\{X(t) : t \geq 0\}$ be an od-regenerative process with state space S and od-regeneration points $\{T_k : k \geq 0\}$, and let f be a real-valued function defined on S. Define the quantities τ_k, $Y_k(f)$, and $r(f)$ as in Section 3.1.2. Suppose that $r(|f|) < \infty$ and define $U_v(f)$ as in (13). Set

$$\sigma^2(f) = \frac{\mathrm{Var}_\mu[Z_1(f)] + 2\mathrm{Cov}_\mu[Z_1(f), Z_2(f)]}{E_\mu^2[\tau_1]}, \tag{15}$$

where $Z_k(f) = Y_k(f) - r(f)\tau_k$ for $k \geq 1$. Note that the value of $\sigma^2(f)$ is actually independent of the initial distribution μ by virtue of the od-regenerative property.

Proposition 3.2.2.1. *Suppose that $Y_0(|f|) < \infty$ a.s. and $E[Y_1^2(|f|) + \tau_1^2] < \infty$. Then $U_v(f) \Rightarrow \sigma(f)W$ as $v \to \infty$.*

The proof of Proposition 3.2.2.1 rests on the FCLT for "mixing" stationary random variables (Ref.[3], Th. 19.2) together with a random-time-change result (Ref.[3], Sec. 14); see Ref.[17] for further details. When applying Proposition 3.2.2.1 to a classical regenerative process, the covariance term in (15) vanishes. Analogous results are available for both od-regenerative and classical regenerative processes in discrete time.

As with the SLLN, previous work has focused on establishing classical regenerative structure and finite cycle-length moments in order to obtain an FCLT for GSMPs via Proposition 3.2.2.1. The approach to establishing probabilistic restart is identical to that in Section 3.1.2. As in the latter section, the geometric-trials approach can be used to show that the cycle length τ_1, and hence the cycle integral $Y_1(f)$, has finite second moment. The condition that $\liminf_{n\geq 0} E_\mu[\zeta_{\beta(n+1)} - \zeta_{\beta(n)}] < \infty$ is now strengthened to require that $\liminf_{n\geq 0} E_\mu[(\zeta_{\beta(n+1)} - \zeta_{\beta(n)})^{2+\epsilon}] < \infty$ for some $\epsilon > 0$. Similarly to the situation described in Section 3.1.2, the more generic approach in Ref.[17] leads to "regenerative versions" of Theorems 3.2.1.1 and 3.2.1.2.

4. IMPROVED LIMIT THEOREMS

In this section we provide new SLLNs and FCLTs that weaken the moment conditions of Theorems 3.1.1.1, 3.1.1.2, 3.2.1.1, and 3.2.1.2. Our approach exploits the fact that every positive Harris chain is an od-regenerative process, and applies Propositions 3.1.2.1 and 3.2.2.1 in their full generality. An argument reminiscent of the proof of Wald's identities establishes the required finite-moment properties of quantities such as the od-regenerative cycle length.

4.1. Statement of Results

Consider a GSMP $\{X(t) : t \geq 0\}$ with finite state space S and underlying chain $\{(S_n, C_n) : n \geq 0\}$ having initial distribution μ and state space Σ. Recall the definition of the holding-time function t^* in (1) and denote by $\mathscr{H}$ the set of real-valued functions defined on Σ. For $u \geq 0$, set

$$\mathscr{H}_u = \Big\{ h \in \mathscr{H} : |h(s,c)| \leq a + b\left(t^*(s,c)\right)^u \text{ for some } a, b \geq 0 \text{ and all } (s,c) \in \Sigma \Big\}.$$

Also write $x \vee y = \max(x, y)$.

We first state a discrete-time SLLN and FCLT for the underlying chain of a GSMP; see Section 5 below for proofs. Recall from Theorem 3.1.1.1 that if Assumption PD(1) holds for a GSMP, then the underlying chain is positive Harris recurrent and admits a unique invariant distribution.

Theorem 4.1.1. *Suppose that Assumption* PD$(u \vee 1)$ *holds for some* $u \geq 0$, *so that there exists a unique invariant distribution* π *for the underlying chain* $\{(S_n, C_n) : n \geq 0\}$. *Then* $\pi(|\tilde{f}|) < \infty$ *and*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \tilde{f}(S_n, C_n) = \pi(\tilde{f}) \quad a.s.$$

for any $\tilde{f} \in \mathscr{H}_u$ *and initial distribution* μ.

Theorem 4.1.2. *Let* $u \geq 0$ *and* $\tilde{f} \in \mathscr{H}_u$. *If Assumption* PD$\big(2(u \vee 1)\big)$ *holds, then there exists a finite constant* $\tilde{\sigma}(\tilde{f}) \geq 0$ *such that* $U_n(\tilde{f}) \Rightarrow \tilde{\sigma}(\tilde{f})W$ *as* $n \to \infty$ *for any initial distribution* μ, *where* $U_n(\tilde{f})$ *is given by* (14) *with* $\tilde{r}(\tilde{f}) = \pi(\tilde{f})$.

A variant of the foregoing result asserts weak convergence to a limiting Brownian motion on $D[0, \infty)$, the space of real-valued functions on $[0, \infty)$ that are right-continuous and have limits from the left. The statement of this theorem is identical to that of Theorem 4.1.2, except that the sequence $U_1(\tilde{f}), U_2(\tilde{f}), \ldots$ is defined by setting

$$U_n(\tilde{f})(t) = \frac{1}{\sqrt{n}} \sum_{j=0}^{\lfloor nt \rfloor} \big(\tilde{f}(S_j, C_j) - \pi(\tilde{f})\big)$$

for $n, t \geq 0$. The proof is essentially identical to that of Theorem 4.1.2, and we omit the details.

We now give limit theorems in continuous time. Given an invariant distribution π for the underlying chain of a GSMP together with a function

$f : S \mapsto \Re$, set

$$r(f) = \frac{\pi(ft^*)}{\pi(t^*)},$$

where the functions t^* and ft^* are defined as before.

Theorem 4.1.3. *Suppose that Assumption* PD(1) *holds. Then* $r(|f|) < \infty$ *and*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f\big(X(u)\big)\,du = r(f) \quad a.s.$$

for any real-valued function f *defined on* S *and any initial distribution* μ.

Theorem 4.1.4. *Suppose that Assumption* PD(2) *holds, and let* f *be a real-valued function defined on* S. *Then there exists a finite constant* $\sigma(f) \geq 0$ *such that* $U_v(f) \Rightarrow \sigma(f)W$ *as* $v \to \infty$ *for any initial distribution* μ, *where* $U_v(f)$ *is given by* (13).

4.2. Discussion

The conclusions of Theorems 4.1.1 and 4.1.2 hold for a function $f \in \mathcal{H}_u$ $(u \geq 1)$ under the respective assumptions PD(u) and PD($2u$), and the conclusions of Theorems 4.1.3 and 4.1.4 hold under the respective assumptions PD(1) and PD(2). The moment conditions in Theorems 3.1.1.1, 3.1.1.2, 3.2.1.1, and 3.2.1.2 are substantially stronger: the SLLN and FCLT for the underlying chain hold for a function $f \in \mathcal{H}_u$ under the respective assumptions PD($u+1$) and PD($2u+3$), and the corresponding limit theorems for the process $\{X(t) : t \geq 0\}$ hold under the respective assumptions PD(2) and PD(5).

The moment conditions in Theorems 4.1.3 and 4.1.4 are natural in light of known conditions for semi-Markov processes and continuous-time Markov chains. E.g., it is shown in Ref.[9] that, for a finite-state irreducible semi-Markov process, finite second moments on the holding time distributions are necessary and sufficient for the conclusion of the FCLT to hold; thus the moment condition in Theorem 4.1.4 is the weakest general condition possible. The appropriateness of the moment conditions in Theorems 4.1.1 and 4.1.2 may not be quite as apparent. For example, it may not be clear why Theorem 4.1.2 requires finite second moments on the clock-setting distributions even when $\tilde{f}(s, c) \equiv g(s)$ for some function g, so that the constant u in the theorem can be taken as 0. The following example shows that the conclusion of Theorem 4.1.2 can fail when clock-setting distribution functions are allowed to have infinite second moments.

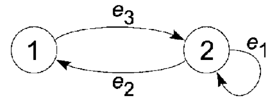

FIGURE 1 State-transition diagram for GSMP of Example 4.2.1.

Example 4.2.1. Consider a GSMP with unit speeds, state space $S = \{1, 2\}$, event set $E = \{e_1, e_2, e_3\}$, and active event sets given by $E(1) = \{e_3\}$ and $E(2) = \{e_1, e_2\}$. The state-transition probabilities are

$$p(2; 1, e_3) = p(2; 2, e_1) = p(1; 2, e_2) = 1$$

(see Figure 1). The clock-setting distribution functions have the simple form $F(\cdot; e_i)$ for $i = 1, 2, 3$. Denote by α_i and β_i the first and second moment of $F(\cdot; e_i)$. We assume that $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_3 < \infty$ and $\beta_2 = \infty$. We also assume that each $F(\cdot; e_i)$ has a density function that is positive on $[0, \infty)$.

Set $\theta(-1) = -1$ and $\theta(n) = \inf\{k > \theta(n-1) : S_{\theta(n)} = 1\}$ for $n \geq 0$. Because there are never any old events just after a transition from state 2 to state 1, the underlying chain probabilistically restarts whenever it hits the set $\{1\} \times C(1)$. Because Assumption PD(1) holds, it follows from Theorem 5.2.1 below that the random indexes $\{\theta(n) : n \geq 0\}$ form a sequence of classical regeneration points for the underlying chain and that the cycle length $\eta_1 = \theta(1) - \theta(0)$ has finite mean. It can then be shown[14] that a necessary condition for the conclusion of Theorem 4.1.2 to hold with $\tilde{f}(s, c) = s$ is that η_1 have finite second moment. Observe that η_1 is distributed as $N(T) + 1$, where $\{N(t) : t \geq 0\}$ is a renewal counting process with inter-renewal distribution function $F(\cdot; e_1)$ and T is an independent sample from $F(\cdot; e_2)$. Using the Cauchy-Schwartz inequality together with a standard result for renewal counting processes (Ref.[1], p. 158), we have $E[N^2(t)] \geq E^2[N(t)] \geq t^2/\alpha_1^2$ for $t \geq 0$. Thus

$$E[\eta_1^2] \geq E[N^2(T)] = E[E[N^2(T) \mid T]] \geq E[T^2/\alpha_1^2] = \infty,$$

so that the conclusion of Theorem 4.1.2 fails to hold.

A slight modification of the foregoing example shows that conclusion of Theorem 4.1.1 can fail to hold if we allow clock-setting distributions to have infinite mean.

Our assumption in Theorems 4.1.1, 4.1.2, 4.1.3, and 4.1.4 of positive density components for the clock-setting distributions is by no means necessary—it is easy to construct GSMPs that violate this assumption but still satisfy SLLNs and FCLTs. The following example, however, shows that some such assumption is needed in order to ensure that value of a time-average limit does not depend upon the initial distribution.

Example 4.2.2. (An irreducible GSMP with no unique time-average limit) Consider a GSMP with unit speeds, state space $S = \{1, 2, 3, 4\}$, event set $E = \{e_1, e_2\}$ and active event sets given by $E(1) = E(3) = \{e_1, e_2\}$ and $E(2) = E(4) = \{e_2\}$. The state-transition probabilities are

$$p(1; 3, e_1) = p(3; 1, e_1) = 1$$

and

$$p(1; 2, e_2) = p(2; 1, e_2) = p(3; 4, e_2) = p(4; 3, e_2) = 1$$

(see Figure 2). Observe that this GSMP is irreducible in the sense of Definition 3.1.1.1. Suppose that each successive new clock reading for event e_i $(i = 1, 2)$ is uniformly distributed on a specified interval $[a_i, b_i]$, and that $0 \leq a_2 < b_2 < a_1 < b_1$. Then with probability 1 event e_2 always occurs before event e_1 whenever both events simultaneously become active. It follows that if the initial state is equal to 1 or 2, then the GSMP never hits state 3 or 4; if the initial state is equal to 3 or 4, then the GSMP never hits state 1 or 2. Thus, in general, the value of a limit of the form $\lim_{t\to\infty}(1/t)\int_0^t f\big(X(u)\big)\,du$ depends on the initial distribution. Similar observations hold for the underlying chain. Of course, this GSMP does not satisfy Assumption PD(q) for any $q \geq 0$ since the clock-setting distribution function for event e_1 does not have a density component that is positive on an interval of the form $(0, \bar{x})$.

The positive-density condition in Theorems 4.1.1–4.1.4 can actually be weakened slightly. Denote by $\mathscr{B}$ the subset of the clock-setting distribution functions such that $F(\cdot; s', e', s, E^*) \in \mathscr{B}$ if and only if $E(s') = \{e'\}$. Then, in Definition 3.1.1.2, we need only require that each clock-setting distribution function $F(\cdot; s', e', s, E^*) \notin \mathscr{B}$ have a density component that is positive and

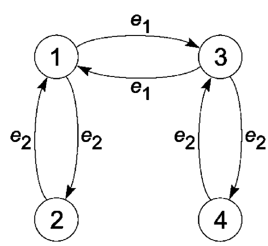

FIGURE 2 State-transition diagram for GSMP of Example 4.2.2.

continuous on $(0, \bar{x})$. The point is that the positive-density assumption is only needed when two or more events are simultaneously active, in order to ensure that the events can occur in any specified order with positive probability; see Ref.[17]. When the GSMP reduces to a semi-Markov process, then every clock-setting distribution function is an element of $\mathscr{B}$, so that there are no positive-density requirements in this case.

In the continuous-time setting, the results in this paper focus on rewards that accrue continuously at rate $f(s)$ whenever the GSMP is in state $s \in S$. It is not difficult to extend our results to handle “impulse rewards,” e.g., a reward of the form $g(s'; s, E^*)$ that accrues whenever the simultaneous occurrence of the events in E^* triggers a transition from s to s'. The idea is to consider the Markov chain $\{(S_n, C_n, S_{n+1}, C_{n+1}) : n \geq 0\}$, which inherits the stability properties of the underlying chain; cf Ref.[9]. Another straightforward extension of the results in the current paper allows for the reward functions $\tilde{f}$ and f to take values in $\Re^l$ for some $l > 1$; see Ref.[18], Sec. 7.2.1.

5. PROOFS

Before proving Theorems 4.1.1–4.1.4, we first review conditions under which a GSMP has od-regenerative structure. These conditions allow the application of Propositions 3.1.2.1 and 3.2.2.1.

5.1. OD-Regenerative Structure in GSMPs

The following proposition gives some conditions under which the underlying chain of a GSMP is an od-regenerative process.

Proposition 5.1.1. *Let $\{(S_n, C_n) : n \geq 0\}$ be the underlying chain of a GSMP. If Assumption* PD(1) *holds, then there exists a sequence $\{\theta(k) : k \geq 0\}$ of od-regeneration points for $\{(S_n, C_n) : n \geq 0\}$. Moreover, the invariant distribution π of the chain has the representation*

$$\pi(A) = \frac{E_\mu\left[\sum_{n=\theta(0)}^{\theta(1)-1} 1_A(S_n, C_n)\right]}{E_\mu[\eta_1]}$$

for $A \subseteq \Sigma$, where $\eta_1 = \theta(1) - \theta(0)$ and 1_A is the indicator function of the set A.

The idea of the proof is as follows (see Ref.[17] and references therein for further details). Because Assumption PD(1) holds by hypothesis, Propositions 3.1.1.1, and 3.1.1.2 together imply that the underlying chain $\{(S_n, C_n) : n \geq 0\}$ is positive Harris recurrent with recurrence measure $\bar{\phi}$

defined by (8). By a well known result for Harris chains, there exists a set $\mathscr{C} \subseteq \Sigma$ with $\bar{\phi}(\mathscr{C}) > 0$ such that

$$P^r((s,c),\cdot) = \epsilon\lambda(\cdot) + (1-\epsilon)Q((s,c),\cdot), \quad (s,c) \in \mathscr{C} \tag{16}$$

for some $r \geq 1$, $\epsilon \in (0,1]$, probability distribution λ, and transition kernel Q; see Asmussen[1] Sec. VI.3, Glynn and L'Ecuyer[12], and Meyn and Tweedie[24], Th. 5.2.3. Indeed, any subset $A \subseteq \Sigma$ with $\bar{\phi}(A) > 0$ contains such a $\mathscr{C}$-set (Ref.[24], Th. 5.2.2). Observe that, since $\bar{\phi}(\mathscr{C}) > 0$, it follows that $P_\mu\{(S_n, C_n) \in \mathscr{C} \text{ i.o.}\} = 1$. The decomposition in (16) permits construction of a version of the underlying chain together with a sequence $\{\theta(k) : k \geq 0\}$ of random indices that serve as od-regeneration points. The construction uses a sequence $\{I_n : n \geq 0\}$ of i.i.d. Bernoulli random variables with $P_\mu\{I_n = 1\} = 1 - P_\mu\{I_n = 0\} = \epsilon$. The idea is to generate successive states of the underlying chain according to the initial distribution μ and one-step transition kernel P until the first time $M \geq 0$ such that $(S_M, C_M) \in \mathscr{C}$. If $I_M = 1$, then generate (S_{M+r}, C_{M+r}) according to λ; if $I_M = 0$, then generate (S_{M+r}, C_{M+r}) according to $Q((S_M, C_M),\cdot)$. Next, generate the intermediate states $\{(S_n, C_n) : M+1 \leq n < M+r\}$ according to an appropriate conditional distribution (conditioned on the endpoint values (S_M, C_M) and (S_{M+r}, C_{M+r})). Now iterate this procedure starting from state (S_{M+r}, C_{M+r}). The successive times $\theta(0), \theta(1), \ldots$ at which the state of the chain is distributed according to λ form a sequence of od-regeneration points. [Observe that the length of each cycle is greater than or equal to r. In general, the conditioning on (S_M, C_M) and (S_{M+r}, C_{M+r}) mentioned above results in statistical dependence between $(S_{\theta(n)}, C_{\theta(n)})$ and $(S_{\theta(n)-r}, C_{\theta(n)-r})$ for each $n \geq 0$, which is why the cycles are one-dependent.] The second assertion of the proposition follows from Theorem VI.3.2 in Ref.[1].

Although we do not use this fact in the sequel, a close inspection of the foregoing proof shows that the cycle lengths $\{\theta(k) - \theta(k-1) : k \geq 1\}$ are i.i.d., so that the od-regeneration points form a (possibly delayed) renewal process. On the other hand, the continuous-time cycle lengths $\{\zeta_{\theta(k)} - \zeta_{\theta(k-1)} : k \geq 1\}$ are, in general, one-dependent and stationary.

5.2. Proof of the SLLNs and FCLTs

Under Assumption PD(1), Proposition 5.1.1 guarantees the existence of a sequence $\{\theta(k) : k \geq 0\}$ of od-regeneration points for the underlying chain $\{(S_n, C_n) : n \geq 0\}$ and a corresponding sequence $\{\zeta_{\theta(k)} : k \geq 0\}$ of od-regeneration points for the GSMP $\{X(t) : t \geq 0\}$. For a real-valued function $\tilde{f}$ defined on Σ, set

$$\widetilde{Y}_i(\tilde{f}) = \sum_{j=\theta(i-1)}^{\theta(i)-1} \tilde{f}(S_n, C_n) \tag{17}$$

for $i \geq 0$. [Take $\theta(-1) = 0$.] Also set $\tilde{1}(s, c) \equiv 1$ for all $(s, c) \in \Sigma$. Theorem 4.1.3 follows from Proposition 3.1.2.1 provided that the cycle length $\tau_1 = \zeta_{\theta(1)} - \zeta_{\theta(0)} = \widetilde{Y}_1(t^*)$ has finite mean, and Theorem 4.1.4 follows from Proposition 3.2.2.1 provided that τ_1 has finite second moment. Similarly, Theorem 4.1.1 (resp., Theorem 4.1.2) follows from the discrete-time version of Proposition 3.1.2.1. (resp., Proposition 3.2.2.1) provided that the cycle length $\eta_1 = \theta(1) - \theta(0) = \widetilde{Y}_1(\tilde{1})$ and the cycle quantity $\widetilde{Y}_1(|\tilde{f}|)$ have finite first (resp., second) moments. [In this connection, observe that $\widetilde{Y}_0(|\tilde{f}|) < \infty$ a.s. because $\theta(0)$ is a.s. finite by Proposition 5.1.1 and each new clock reading is a.s. finite by definition.] To establish the desired limit theorems, it therefore suffices to prove the following general result on cycle moments.

Theorem 5.2.1. *Suppose that Assumption* PD$(q(u \vee 1))$ *holds for some* $q \in \{1, 2, \ldots\}$ *and* $u \geq 0$. *Then* $E_\mu \widetilde{Y}_1^q(|\tilde{f}|) < \infty$ *for any* $\tilde{f} \in \mathscr{H}_u$, *where* $\widetilde{Y}_1$ *is defined as in* (17).

We prove the assertion of Theorem 5.2.1 via a sequence of lemmas. Fix a compact set $B \subseteq \Sigma$ and denote by T_B the return time to B: $T_B = \inf\{n > 0 : (S_n, C_n) \in B\}$. Lemma 5.2.1 below gives upper bounds on the moments of T_B. To prepare for this lemma, first recall the definition of h_q from (10). By an argument that uses the drift conditions (11) in Proposition 3.1.1.2 together with Dynkin's formula, we have

$$E_{(s,c)}\left[\sum_{n=0}^{T_B-1} h_{q-1}(S_n, C_n)\right] \leq \gamma_q h_q(s, c) \tag{18}$$

for some finite positive constant $\gamma_q = \gamma_q(B)$ and all $(s, c) \in \Sigma$; see the proof of Theorem 14.2.3 in Ref.[24] for details. Next, fix finite positive constants $a_1 = 1, a_2, a_3, \ldots$ such that

$$n^{q+1} \leq a_{q+1}(1^q + 2^q + \cdots + n^q) \tag{19}$$

for $n \geq 1$ and $q \in \{0, 1, 2, \ldots\}$—it is well known that such constants exist. Finally, set $b_q = \prod_{i=1}^{q}(a_i \gamma_i)$ for $q \geq 1$.

Lemma 5.2.1. *Suppose that Assumption* PD(q) *holds for some* $q \in \{1, 2, \ldots\}$. *Then*

$$E_{(s,c)}[T_B^q] \le b_q h_q(s, c)$$

for $(s, c) \in \Sigma$.

Proof. Our proof is by induction on q. Fix $(s, c) \in \Sigma$ and observe that the desired result holds for $q = 1$ by virtue of (18). Assume for induction that the lemma holds for some $q \ge 1$ and observe that, by (19),

$$\begin{aligned} E_{(s,c)}[T_B^{q+1}] &\le a_{q+1} E_{(s,c)}\left[\sum_{n=0}^{T_B-1} (T_B - n)^q\right] \\ &= a_{q+1} \sum_{n=0}^{\infty} E_{(s,c)}[(T_B - n)^q; T_B > n], \end{aligned} \tag{20}$$

where the interchange of sum and expectation is justified by the nonnegativity of the summands. Using the Markov property together with the induction hypothesis, we find that

$$\begin{aligned} E_{(s,c)}[(T_B - n)^q; T_B > n] &= E_{(s,c)}[E_{(s,c)}[(T_B - n)^q; T_B > n \mid (S_k, C_k) : 0 \le k \le n]] \\ &= E_{(s,c)}[I(T_B > n) E_{(S_n, C_n)}[T_B^q]] \\ &\le E_{(s,c)}[I(T_B > n) b_q h_q(S_n, C_n)], \end{aligned} \tag{21}$$

where $I(A)$ is the indicator function for the event A. Substituting (21) into (20), interchanging sum and expectation, and applying (18), we find that

$$\begin{aligned} E_{(s,c)}[T_B^{q+1}] &\le a_{q+1} b_q E(s, c)\left[\sum_{n=0}^{T_B-1} h_q(S_n, C_n)\right] \\ &\le a_{q+1} \gamma_{q+1} b_q h_{q+1}(s, c) \\ &= b_{q+1} h_{q+1}(s, c), \end{aligned}$$

and the desired result follows. □

The next step in the argument is to show that the discrete-time cycle length η_1 has finite qth moment under Assumption PD(q). To this end, we use the following fact: if $X_1, X_2, \ldots, X_k$ ($k \ge 1$) are nonnegative random variables and $a_1, a_2, \ldots, a_k$ are positive integers, then

$$E[X_1^{a_1} X_2^{a_2} \cdots X_k^{a_k}] \le E^{a_1/q}[X_1^q] E^{a_2/q}[X_2^q] \cdots E^{a_k/q}[X_k^q], \tag{22}$$

where $q = a_1 + a_2 + \cdots + a_k$. The inequality in (22) follows by an easy induction argument on k that uses Hölder's inequality.

Lemma 5.2.2. *Suppose that Assumption* PD(q) *holds for some* $q \in \{1, 2, \ldots\}$. *Then* $E_\mu[\eta_1^q] < \infty$.

Proof. We give the proof under the simplifying assumption that (16) holds with $r = 1$; the extension to the general case is straightforward as in Ref.[25]. Let $\mathscr{C} \subset \Sigma$ be as in (16), and set

$$\alpha_q = \sup_{(s,c)\in\mathscr{C}} E_{(s,c)}[T_{\mathscr{C}}^q].$$

We can assume that $\mathscr{C}$ is compact, and it follows from Lemma 5.2.1 that $\alpha_q < \infty$. Assume for convenience that the initial state of the chain is an element of $\mathscr{C}$, and that the initial Bernoulli trial is successful (i.e., $I_0 = 1$), so that $\theta(0) = 1$. Denote by δ_i the number of state transitions between the $(i-1)$st and ith visit of the underlying chain to $\mathscr{C}$, where the 0th visit occurs at time 0. Also denote by N the number of returns to $\mathscr{C}$, up to and including the return that corresponds to the first successful Bernoulli trial after time 0. Observe that, by (16),

$$P_\mu\{N \geq i\} = (1-\epsilon)^{i-1} \tag{23}$$

for $i \geq 1$. Also observe that

$$E_\mu[\eta_1^q] = E_\mu\left[\left(\sum_{i=1}^{N} \delta_i\right)^q\right].$$

We can write

$$\left(\sum_{i=1}^{N} \delta_i\right)^q = b_1 V_1 + b_2 V_2 + \cdots + b_m V_m,$$

where m and $b_1, b_2, \ldots, b_m$ are finite integers and each V_j is a sum of the form

$$V_j = \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \cdots \sum_{i_k=1}^{N} \delta_{i_1}^{a_1} \delta_{i_2}^{a_2} \cdots \delta_{i_k}^{a_k}.$$

Here the integers $k, a_1, a_2, \ldots, a_m$ are such that $k = k(j) \leq q$, $a_l = a_l(j) \geq 1$ for $1 \leq l \leq k$, and $a_1 + \cdots + a_k = q$. It therefore suffices to show that each

V_j has finite mean. Consider an arbitrary fixed value of j, and observe that, using (22),

$$\begin{aligned} E_\mu[V_j] &= E_\mu\left[\sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\cdots\sum_{i_k=1}^{N}\delta_{i_1}^{a_1}\delta_{i_2}^{a_2}\cdots\delta_{i_k}^{a_k}\right] \\ &= \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty}\cdots\sum_{i_k=1}^{\infty}E_\mu\big[\delta_{i_1}^{a_1}I(N\geq i_1)\,\delta_{i_2}^{a_2}I(N\geq i_2)\cdots\delta_{i_k}^{a_k}I(N\geq i_k)\big] \\ &\leq \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty}\cdots\sum_{i_k=1}^{\infty}\left(\prod_{l=1}^{k}E_\mu^{a_l/q}\big[\delta_{i_l}^{q}I(N\geq i_l)\big]\right). \end{aligned} \tag{24}$$

Let $\mathscr{F}_0 = \sigma(S_0, C_0, I_0)$, that is, the σ-field generated by (S_0, C_0, I_0), and $\mathscr{F}_j = \sigma\big((S_n, C_n, I_n) : 0 \leq n \leq \delta_1 + \cdots + \delta_j\big)$ for $j \geq 1$. For each $i \geq 1$, observe that $I(N \geq i)$ is measurable with respect to $\mathscr{F}_{i-1}$, so that

$$E_\mu[\delta_i^q I(N\geq i)] = E_\mu[I(N\geq i)E_\mu[\delta_i^q \mid \mathscr{F}_{i-1}]] \leq \alpha_q P_\mu\{N\geq i\}.$$

Using the foregoing inequality together with (23) and (24), we find that

$$E_\mu[V_j] \leq \alpha_q\prod_{l=1}^{k}\left(\sum_{i=1}^{\infty}P_\mu^{a_l/q}\{N\geq i\}\right) = \alpha_q\prod_{l=1}^{k}\left(\sum_{i=1}^{\infty}(1-\varepsilon)^{a_l(i-1)/q}\right) < \infty$$

as desired. □

To complete the proof of Theorem 5.2.1, we need the following proposition.

Proposition 5.2.1. *Let $S_N = \sum_{n=1}^{N} X_n$, where $\{X_n : n \geq 1\}$ is a sequence of i.i.d. random variables and N is a stopping time with respect to an increasing sequence $\{\mathscr{F}_n : n \geq 1\}$ of σ-fields such that X_n is measurable with respect to $\mathscr{F}_n$ for $n \geq 1$ and independent of $\mathscr{F}_{n-1}$ for $n \geq 2$. Then for $r \geq 0$ there exists a finite constant b_r (depending only on r) such that $E[|S_N|^r] \leq b_r E[|X_1|^r]E[N^r]$.*

The proof of Proposition 5.2.1 is contained in the proof of Theorem I.5.2 in Gut[16].

Proof of Theorem 5.2.1. Fix q, u, and $\tilde{f} \in \mathscr{H}_u$. For ease of exposition, we assume that all speeds for active events are equal to 1 and that $F(\cdot; s', e', s, e^*) \equiv F(\cdot; e')$ for all s', e', s, and e^*. Denote by $A_{i,j}$ the value of the jth new clock reading generated for event e_i after time $\zeta_{\theta(0)}$, and by N_i

the number of new clock readings generated for event e_i in the interval $(\zeta_{\theta(0)}, \zeta_{\theta(1)})$. Observe that

$$\widetilde{Y}_1(|\tilde{f}|) \leq a\eta_1 + b\sum_{i=1}^{M} C^u_{\theta(0),i} + b\sum_{i=1}^{M}\sum_{j=1}^{N_i+r} A^u_{i,j} \tag{25}$$

for some $a, b \geq 0$, where r is as in (16). It therefore suffices to show that

$$E_\mu\left[\left(\sum_{j=1}^{N_i+r} A^u_{i,j}\right)^q\right] < \infty \tag{26}$$

and

$$E_\mu[C^{qu}_{\theta(0),i}] < \infty \tag{27}$$

for $1 \leq i \leq M$—this assertion follows from (25), Lemma 5.2.2, and the elementary inequality

$$E[(X_1 + X_2 + \cdots + X_k)^q] \leq k^{q-1}\big(E[X_1^q] + E[X_2^q] + \cdots + E[X_k^q]\big),$$

which holds for any $k \geq 1$ and non-negative random variables $X_1, X_2, \ldots, X_k$. To see that (26) holds, fix i and denote by $\kappa(i,j)$ the random index of the state transition at which $A_{i,j}$ is generated. Set $\mathscr{F}_j = \sigma((S_n, C_n, I_n) : 0 \leq n \leq \kappa(i,j))$ for $j \geq 1$, and observe that (i) $A_{i,j}$ is measurable with respect to $\mathscr{F}_j$ for $j \geq 1$, (ii) $A_{i,j}$ is independent of $\mathscr{F}_{j-1}$ for $j \geq 2$, and (iii) $N_i + r$ is a stopping time with respect to $\{\mathscr{F}_n : n \geq 1\}$. Moreover, since $N_i \leq \eta_1$, it follows from Lemma 5.2.2 that $E_\mu[(N_i + r)^q] < \infty$. An application of Proposition 5.2.1 now establishes (26). In light of (16), it can be seen that a sufficient condition for (27) to hold is $\sup_{(s,c)\in\mathscr{C}} \int_0^\infty x^{qu}\, dG_i(x; s, c) < \infty$, where $G_i(x; s, c) = P_{(s,c)}\{C_{r,i} \leq x\}$. Observe that if X_i is distributed according to $G_i(x; s, c)$, then X_i is stochastically dominated by $W_i(c) = W_i(c_1, c_2, \ldots, c_M) = \max(c_i, B_{i,1}, B_{i,2}, \ldots, B_{i,r})$, where $B_{i,1}, B_{i,2}, \ldots, B_{i,r}$ are i.i.d. samples from $F(\cdot; e_i)$. Because $\mathscr{C}$ is assumed compact, there is a finite constant b such that $\max_{1\leq j\leq M} c_i \leq b$ for all $c = (c_1, c_2, \ldots, c_M)$ such that $(s, c) \in \mathscr{C}$. Thus $E[X_i^{qu}] \leq E[W_i^{qu}(c)] \leq b^{qu} + rE[B_{i,1}^{qu}] < \infty$, and the desired result follows. □

As an aside, it follows from the results of this section that the limits $r(f)$ and $\pi(\tilde{f})$ in Theorems 4.1.1 and 4.1.3 can be expressed as ratios of the form

$$r(f) = \frac{E_\mu[Y_1(f)]}{E_\mu[\tau_1]} \quad \text{and} \quad \pi(\tilde{f}) = \frac{E_\mu[\widetilde{Y}_1(\tilde{f})]}{E_\mu[\eta_1]},$$

where $\widetilde{Y}_1(\tilde{f}) = \sum_{j=\theta(0)}^{\theta(1)-1} \tilde{f}(S_n, C_n)$ and $Y_1(f) = \int_{\zeta_{\theta(0)}}^{\zeta_{\theta(1)}} f(X(u))du$. Similarly, the variance constants $\sigma^2(f)$ and $\tilde{\sigma}^2(\tilde{f})$ in Theorems 4.1.2 and 4.1.4 can be expressed as

$$\sigma^2(f) = \frac{\mathrm{Var}_\mu Z_1(f) + 2\mathrm{Cov}_\mu[Z_1(f), Z_2(f)]}{E_\mu^2[\tau_1]}$$

and

$$\tilde{\sigma}^2(\tilde{f}) = \frac{\mathrm{Var}_\mu[\widetilde{Z}_1(\tilde{f})] + 2\mathrm{Cov}_\mu[\widetilde{Z}_1(\tilde{f}), \widetilde{Z}_2(\tilde{f})]}{E_\mu^2[\eta_1]},$$

where $Z_k(f) = Y_k(f) - r(f)\tau_k$ and $\widetilde{Z}_k(\tilde{f}) = \widetilde{Y}_k(\tilde{f}) - \pi(\tilde{f})\eta_k$ for $k \geq 1$. Note that none of the foregoing quantities actually depend on the initial distribution μ.

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