Approximations for the Distribution of Perpetuities with Small Discount Rates

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Abstract

General perpetuities (i.e. random variables of the form $D = \int_0^\infty e^{-\Gamma(t-)} d\Lambda(t)$, also known as infinite horizon discounted rewards) play an important role in several application settings (e.g. insurance, finance and time series analysis). Our focus is on developing approximations for the distribution of D that are asymptotically valid when the "accumulated short rate process" or "accumulated force of interest" (represented by Γ) is small. In this paper, we emphasize approximations that are good around the "center" of the distribution of D. We provide: 1) characterizations, in terms of solutions to certain linear equations, for the distribution of D when Γ and Λ are driven by Markov processes; 2) General sufficient conditions under which weak convergence results can be derived for D; 3) Edgeworth expansions for the distribution of D in the iid case and the case in which Λ is a Levy process and $\dot{\Gamma}(t)$ is a function of a Markov process, this last setting is of particular interest in applications to life and non-life insurance problems.

1 Introduction

Perpetuities arise in the insurance and mathematical finance context when modeling long term guaranteed payments. A very basic model that is used considers benefits that are paid in perpetuity at a fixed rate, say λ, under the assumption that the interest rates are fixed at level γ (in the insurance context, γ is often called the force of interest). The present value of such benefit plan is $\lambda/\gamma = \int_0^\infty e^{-\gamma s} \lambda ds$. In more realistic situations, it is of interest to consider a pension system as a whole (which could be a private pension fund designed for a particular group) in a stochastic economic environment (i.e. under stochastic compounding) then a natural quantity to consider is the perpetuity

$$\int_0^\infty \exp\left(-\int_0^t \gamma(s)\, ds\right) \lambda(s)\, ds, \tag{1}$$

where $\lambda(t)$ is the aggregated rate paid by the insurance company at time t and $\gamma(t)$ represents the short rate (or the instantaneous force of interest) at time t. It is natural to assess how well balanced is the fund or how likely is for the pension fund to collapse by studying the distribution of the difference between the present value of the aggregated benefits and the present value of the aggregated premiums received. We refer the interested reader to the paper by Dufresne (1990) who (under stationary and ergodic assumptions) proposes a detailed model, based on perpetuities, for computing the value of a pension fund. The processes $\lambda(t)$ and $\gamma(t)$ depend on the parameters that serve to characterize the pension fund (i.e. benefit payments, actuarial liability, net premium, and rate of return). As explained in Dufresne (1990), the distribution of the value process plays an important role in risk management, as it serves to compute critical rates ensuring that the fund is being managed in a balanced manner with respect to its actuarial liabilities; see Dufresne (1990) and Bédard and Dufresne (2001) for additional detail on pension funding.

Perpetuities also arises in non-pension fund insurance settings. Consider a company that receives premiums at a rate of p dollars per unit time, and pays out claims according to the random process $A(\cdot)$. If $\gamma(t)$ represents the rate of return on the invested risk reserve at time t, the risk reserve $R(t)$ evolves according to the equation

$$dR(t) = \gamma(t) R(t) dt + (pdt - dA(t)),$$

subject to the initial condition $R(0) = r_0$. Harrison (1977) shows that the ruin probability $P(\inf_{t\geq 0} R(t) < 0)$ can be computed in terms of the distribution of a more general perpetuity that takes the form

$$D = \int_{[0,\infty)} \exp(-\Gamma(t-)) d\Lambda(t),$$

(for Λ and Γ suitably defined) when γ is deterministic. Paulsen (1998) extends this result to the case of stochastic $\gamma(\cdot)$; see also Nyrhinen (2001). Thus, the key to calculating such ruin probabilities is computing the distribution of D.

General perpetuities, such as D, are also called infinite horizon discounted rewards. One can immediately appreciate that random variables that take the form of D arise in many other contexts in addition to insurance / finance settings discussed above.

It turns out that D also plays a major role in the theory of ARCH processes. This class of time series is widely used within the statistics and econometrics communities, and has been employed to model log-returns, exchange rates, inflation, and many other financial and economic time series; see Campbell, Lo and Mackinlay (1999), Shephard (1996), Mills (1993) and Wilkie (1986). An ARCH(1) model satisfies the stochastic recursion

$$Y_{n+1} = A_n + B_{n+1} Y_n, \tag{2}$$

where the sequence $((A_i, B_i) : i \geq 1)$ is iid (independent and identically distributed.) Under mild stability conditions (see, for example, Kesten (1973), Verbaat (1979), Goldie (1991), Embrechts and Goldie (1994)), this Markov chain has a stationary distribution. This stationary distribution is a special case of D.

We also note several other applications settings in which the distribution of D arises as a central object. Goldie and Grübel (1996) describe its relevance to complexity theory (in the context of sorting algorithms related to "Quicksort") and analytic number theory. Carmona, Petit, and Yor (2001) describe several other applications arising in mathematical physics and finance.

Study of approximations for the distribution of D can be traced back (at least) to the early seventies. Gerber (1971) established a Central Limit Theorem (CLT), as well as its Berry-Esséen companion, for

$$D = \sum_{k=0}^{\infty} \exp\left(-\alpha k\right) X_k,$$

in the case of a (small) deterministic discount rate α and iid rewards $(X_k)_{k\geq 0}$. Whitt (1972) obtained more general central limit theorems for D, also under the assumption of deterministic interest rates. The aim of Whitt's paper was to establish discounted stochastic limit theorems based on postulating a functional limit theorem for the (undiscounted) reward process (in our notation, Λ).

The setting of stochastic discount rates has also been studied in the literature. Pollack and Siegmund (1985) computed the distribution of D in the case in which Γ follows a Brownian motion with negative drift and $\Lambda(t) = t$; see also Dufresne (1990). The distribution of D has also been computed explicitly by Gjessing and Paulsen (1997) in some other particular cases in which both Γ and Λ follow particular types of Levy processes. Computing the distribution of D in complete generality is clearly unfeasible. Even in Markovian settings, such as those previously described, the type of integro-differential equations that arise (see Gjessing and Paulsen (1997), Yor (2001) and Section 3 below) are challenging to solve both analytically and numerically. Therefore, part of our goal is to provide approximations to D that hold in great generality and require relatively "easy-to-obtain" information for their implementation.

As main contributions of the paper we provide:

1) Characterizations, in terms of solutions to certain linear equations, of the distribution of D and its Laplace transform when Γ and Λ are driven by an underlying Markov process.

2) General sufficient conditions are given under which D properly centered and scaled converges weakly to some random variable Z, which typically is Gaussian, as the "interest rate" tends to zero.

3) Edgeworth expansions for the distribution of D in the iid case and the case in which Λ is a Levy process and $\dot{\Gamma}(t)$ is a function of a Markov process. This case is of particular interest in the insurance context under investments since it is common to model pure claim processes as Levy motions and interest rates as mean reverting

processes. Formal Edgeworth expansions are also given for more general Markovian settings.

The rest of the paper is organized as follows. As we indicated before Section 2 discusses some motivating examples in which the distribution of D plays an important role. In Section 3 we deal with item 2) of the previous list, namely, sufficient conditions that guarantee weak convergence of D as the "interest rate" goes to zero. The development of the Edgeworth expansions is given in Section 4.

2 Exact Computations

Our main goals in this section are to provide means to compute either numerically or analytically the exact distribution of D and to provide a brief discussion of the computational issues that would arise in implementing a numerical scheme to evaluate the distribution of D "exactly". We argue that approximations can often be a convenient alternative to exact computation.

In the applications described in previous sections, it is often the case that the discount and reward rates are modeled as functions of some underlying Markov process. To be more precise, let $Y = (Y(s) : s \geq 0)$ be a homogeneous Markov process taking values in a compact Polish space Ξ and let $\mathcal{B}(\Xi)$ be the Borel sigma-field in Ξ. Let $P(t, y, B)$ ($t \in \mathbb{R}_+$, $y \in \Xi$ and $B \in \mathcal{B}(\Xi)$) be the corresponding transition probability function. Assume that Y satisfies the Feller condition (i.e. $P(t, y, B_\delta(y)) \to 1$ as $t \searrow 0$, for all $\delta > 0$, where $B_\delta(y)$ is a ball of radius δ around y) and that the mapping $y \to E_y f(Y(t))$ is continuous for all $f(\cdot) \in C(\Xi)$ (the space of continuous function taking values on Ξ). Let A be the associated infinitesimal generator of the process Y, defined via the relation

$$Af(y) = \lim_{t \downarrow 0} \frac{E_y f(Y(t)) - f(y)}{t},$$

where $f \in C(\Xi)$. The domain $D(A)$ of A is composed by those functions $f \in C(\Xi)$ for which the previous limit exists (uniformly in Ξ) (See Skorohod, Hoppensteadt and Salehi (2002)). Finally, suppose that $Y(\cdot)$ has right continuous with left limits sample paths. The discount rate at time t, satisfies $\gamma(t) \triangleq g(Y(t))$, while the reward rate at time t is defined as $\lambda(t) \triangleq f(Y(t))$. Where both f and g are continuous functions and g is positive. We are interested in computing the distribution of

$$D = \int_0^\infty \exp(-\Gamma(t)) f(Y(t))\, dt,$$

where $\Gamma(t) = \int_0^t g(Y(s))\, ds$. In this setting, one can develop linear equations that are satisfied for the distribution of D or its Laplace transform, thereby providing the necessary means to compute either numerically or analytically the exact distribution of D. Our first result concerns the moment generating function of D.

Theorem 1 *Suppose that there exists a solution to the linear equation*

$$(A\phi)(y,\theta) = (\partial_\theta \phi(y,\theta)\, g(y)\,\theta - \phi(y,\theta)\, f(y)\,\theta), \quad (y,\theta) \in \Xi \times \mathbb{R}, \tag{3}$$

with $\sup_{(y,\theta)\in\Xi\times[-a,a]} \phi(y,\theta) < \infty$ *for every* $a > 0$ *(i.e. locally bounded),* ϕ *non-negative and satisfying that* $\phi(y,0) = 1$*. Then,* $\phi(y,\theta) = E_y \exp(\theta D)$*.*

Remarks

a) It should not be any confusion when evaluating $(A\phi)(y,\theta)$ (naturally, the operator A acts on ϕ as a function of the first argument, y, only).

b) Note that the domain of A introduces hidden boundary conditions in the previous linear equation. For example, if Y is a one dimensional reflected Brownian motion in $[0,1]$, the domain of A will comprise functions $h \in C^2[0,1]$ such that $h'(0) = 0 = h'(1)$ (otherwise, we cannot guarantee uniform convergence in the definition of Ah).

c) In many applications (for example in the insurance context) it is important to study the distribution of $D = \int_0^\infty e^{-\Gamma(t)} d\Lambda(t)$, where $\Gamma(t) = \int_0^t g(Y(s))\, ds$ and Λ is a Levy process, typically independent of Y. Assuming that $E\exp(\theta\Lambda(1)) = \exp(\psi_\Lambda(\theta)) < \infty$ for $\theta \in \mathbb{R}$, the linear system that must be solved takes the form

$$(A\phi)(y,\theta) = (\partial_\theta \phi(y,\theta)\, g(y) - \phi(y,\theta)\, \psi_\Lambda(\theta)), \quad (y,\theta) \in \Xi \times \mathbb{R}.$$

d) Note that sufficient conditions for existence/uniqueness to (3) are well studied at least in the case in which Y is a diffusion with uniformly elliptic generator A. In this case, we can transform equation (3) into a parabolic equation by introducing the change of variable $\theta = e^t$ (which imposes the restriction $\theta > 0$). We then let $\phi(y, e^t) = \widetilde{\phi}(y,t)$ and obtain that (3) is transformed into

$$\left(A\widetilde{\phi}\right)(y,t) = \left(\partial_t \widetilde{\phi}(y,t)\, g(y) - \widetilde{\phi}(y,t)\, f(y)\, e^t\right), \quad (y,t) \in \Xi \times \mathbb{R},$$

which is a non-homogeneous parabolic equation.

Proof. Consider the process,

$$\begin{aligned} & M(t) \\ = \; & \phi\left(Y(t), \theta e^{-\Gamma(t)}\right) \exp\left(-\int_0^t \left(\frac{A\phi + \partial_\theta \phi \partial_s e^{-\Gamma(s)}}{\phi}\right)\left(Y(s), \theta e^{-\Gamma(s)}\right) ds\right). \end{aligned}$$

As in Lemma 2, p. 82 of Skorohod, Hoppensteadt and Salehi (2001) we see that $M(t)$ is a local Martingale. However, note that

$$\left(\frac{A\phi + \partial_\theta \phi \partial_s e^{-\Gamma(s)}}{\phi}\right)\left(Y(s), \theta e^{-\Gamma(s)}\right) = -\theta e^{-\Gamma(s)} f(Y(s)),$$

therefore

$$M(t) = \phi\left(Y(t), \theta e^{-\Gamma(t)}\right) \exp\left(\theta \int_0^t e^{-\Gamma(s)} f(Y(s))\, ds\right).$$

Since ϕ is locally bounded it follows that M is a bounded Martingale. On the other hand, we are assuming that ϕ is in the domain of A and differentiable in θ, thus ϕ is continuous in both arguments. It is not hard to verify that $u_1(\cdot) \triangleq \sup_{y\in\Xi} \phi(y,\cdot)$ and $u_2(\cdot) \triangleq \inf_{y\in\Xi} \phi(y,\cdot)$ are both continuous functions, therefore we have that

$$\phi(y,\theta) = E_y M(t) \leq E_y u_1\left(\theta e^{-\Gamma(t)}\right) \exp\left(\theta \int_0^t e^{-\Gamma(s)} f(Y(s))\, ds\right),$$

and

$$\phi(y,\theta) = E_y M(t) \geq E_y u_2\left(\theta e^{-\Gamma(t)}\right) \exp\left(\theta \int_0^t e^{-\Gamma(s)} f(Y(s))\, ds\right).$$

Using that ϕ is locally bounded and the continuity of u_1 and u_2 we obtain, after applying the bounded convergence theorem

$$\phi(y,\theta) = E_y \exp\left(\theta \int_0^t e^{-\Gamma(s)} f(Y(s))\, ds\right),$$

which is the statement of the theorem. ■

Our next result provides a more direct way of computing the distribution of D, the inconvenient is that the types of boundary conditions involved are harder to deal with numerically.

Theorem 2 *Suppose that for every $y \in \Xi$, $P_y(D \leq \cdot)$ is continuous and that there exists a solution to the linear equation*

$$(Ah)(y,z) = (f(y) - zg(y))(\partial_z h)(y,z), \quad (y,z) \in \Xi \times \mathbb{R}, \tag{4}$$

where $0 \leq h(y,z) \leq 1$ and such that $\lim_{z\to\infty} h(y,z) = 1$ and $\lim_{z\to-\infty} h(y,z) = 0$. Then,

$$h(y,z) = P_y(D \leq z).$$

Proof. Let

$$D(t) = \int_0^t e^{-\Gamma(s)} f(Y(s))\, ds,$$

and define $M = (M(t) : t \geq 0)$ as

$$M(t) = h\left(Y(t), (z - D(t))\, e^{\Gamma(t)}\right),$$

it follows from Lemma 2, p. 82 of Skorohod, Hoppensteadt and Salehi (2001), that M is a local Martingale. Since h is bounded we therefore must have that M is indeed a bounded Martingale. Hence, we obtain that

$$h(y,z) = E_y h\left(Y(t), (z - D(t))\, e^{\Gamma(t)}\right).$$

An argument similar to that given in the proof of Theorem 1 (using the continuity of h and the continuity of the distribution of D) implies, letting $t \to \infty$, that

$$h(y, z) = P_y (D \leq z)$$

as we claimed. ■

The previous results provide means for numerical evaluation of the distribution of D. In order to get a sense of the computational complexity implied in equation (3) consider the case in which Y follows a diffusion process. In this case, equation (3) becomes a PDE that has the level of complexity of a parabolic equation. In addition, there is a transform inversion that is required to recover the distribution from the solution to (3). This inversion has the level of complexity of a numerical integration, but it has to be carried out for each value of y and z in $P_y (D \leq z)$. Equation (4), on the other hand seems more convenient. One problem, however, is that equation (4) requires handling more complicated boundary conditions and the corresponding PDE does not correspond to a parabolic equation, thus the existence/uniqueness issues in this case are more subtle. In the next sections we shall propose several approximations that can be implemented at a lower computational cost. Some of these approximations (for example, the case of the Edgeworth expansions and the exact large deviations) require solving linear systems such as (3) and (4), but there is no inversion involved in those procedures and the existence/uniqueness issues are easier to deal with as we shall see.

3 Weak Convergence

The computational analysis described in the previous section suggests that developing approximations for the distribution of D can be a convenient alternative in many of the applications previously discussed. In this section, we are interested in developing distributional approximations for D that are asymptotically valid in the presence of low interest rates. In order to accomplish this, we shall introduce a parameter $\alpha > 0$ that will be used to control the size of the "discount rate" (essentially what we have denoted as Γ). Later we shall offer convenient interpretations of the results that can be applied to practical problems, in which typically there is no scaling parameter $\alpha > 0$. To be more precise, let us define

$$D(\alpha) = \int_0^\infty \exp(-\alpha \Gamma(t-)) \, d\Lambda(t).$$

Our goal is then to derive distributional approximation for $D(\alpha)$ that are asymptotically valid as $\alpha \searrow 0$. It is important to recognize that in the particular case in which $(\Gamma, \Lambda)_{t \geq 0}$ have stationary increments, $D(\alpha)$ will typically be equal in distribution to $\widetilde{D}(\infty)$, where $\widetilde{D}(t)$ satisfies

$$d\widetilde{D}(t) = -\alpha \widetilde{D}(t) \, d\Gamma(t-) + d\Lambda(t). \tag{5}$$

Indeed, note that the solution to the previous stochastic differential equation (SDE) is (assuming that Γ and Λ do not jump at the same time) given by

$$\widetilde{D}(t) = e^{-\alpha\Gamma(t-)} \left(\widetilde{D}(0) + \int_0^t e^{-\alpha\Gamma(s-)} d\Lambda(s) \right).$$

Assuming that $\Gamma(t)/t \to \gamma > 0$ as $t \to \infty$ and noting that because of stationarity (using a time reverse argument)

$$e^{-\alpha\Gamma(t-)} \int_0^t e^{-\alpha\Gamma(s-)} d\Lambda(s) \stackrel{D}{=} \int_0^t e^{-\alpha\Gamma(s-)} d\Lambda(s),$$

we conclude (using convergence together results) that $\widetilde{D}(t) \Rightarrow D(\alpha)$.

Of course, not every infinite horizon discounted reward can be analyzed assuming stationary input. For instance, note that Theorem 2 indicates that the distribution of $D(\alpha)$ depends on the initial state of the underlying Markov process, and this will actually be the typical situation in applications settings such as insurance. However, stationarity is reasonable in some applied contexts such as time series analysis. In fact, in this context, by properly scaling certain types of auto-regressive processes (which satisfy the discrete analog of (5), see equation (2) above), Nelson (1990) obtained sample-path weak convergence results a Gaussian Ornstein-Uhlenbech process as the sample frequency increases. More recently Forniari and Mele (1997) extended Nelson's results to cover more general type of non-linear ARCH and GARCH time series models. From the time series analysis perspective, a central limit theorem approximation for the distribution of $D(\alpha)$ is related to the convergence of the stationary distributions of auto-regressive type models to that of Ornstein-Uhlenbeck (namely a Gaussian law).

One of the contributions of Whitt (1972) is to show that weak convergence of properly scaled processes Γ and Λ in the standard Skorohod topology is not enough to guarantee weak convergence of a suitably scaled and centered version of $D(\alpha)$. Thus, the general weak convergence analysis at the level of stationary distributions in auto-regressive processes does not follow directly from previous results in the literature (such as those by Nelson (1990), and Forniari and Mele (1997)). Our results here complement previous analysis on the structure of auto-regressive processes by providing additional conditions that guarantee weak convergence at the level of stationary distributions.

In the context of stochastic approximation algorithms, Bucklew, Kurtz, and Sethares (1993) analyzed weak convergence (on compact sets) of processes following certain stochastic recursive equations that give rise to stationary distributions related to $D(\alpha)$. As in the previous discussion regarding the time series setting, this type of analysis does not directly imply weak convergence of stationary distributions.

For the case in which the increments of (Γ, Λ) are not stationary, there is also a SDE related to $D(\alpha)$, which is probably more natural to consider from the perspective of insurance applications. If we let $R = (R(t) : t \geq 0)$ satisfy

$$dR(t) = \alpha R(t)\, d\Gamma(t-) + d\Lambda(t), \quad R(0) = 0$$

and define $D(t) = \exp(-\alpha\Gamma(t-))\, R(t)$, then, assuming that Γ and Λ do not jump have common jumps, we have that

$$D(t) = \int_0^t \exp(-\alpha\Gamma(s-))\, d\Lambda(s).$$

One can take advantage of a number of results that are available in the literature to develop functional weak convergence results in $D[0,\infty)$. In particular, for example, the results by Kurtz and Protter (1991) can be applied by considering the sequence of processes $(\Gamma_\alpha, \Lambda_\alpha) = (\Gamma(\cdot/\alpha), \Lambda(\cdot/\alpha))$ and developing the corresponding weak convergence theory for

$$dR_\alpha(t) = \alpha R_\alpha(t)\, d\Gamma_\alpha(t-) + d\Lambda_\alpha(t), \tag{6}$$

$$D_\alpha(t) = \exp(-\alpha\Gamma_\alpha(t-))\, R_\alpha(t). \tag{7}$$

Typically, after centering and scaling (6) and (7) standard results such as those developed by Kurtz and Protter (1991) would yield weak convergence (in $D[0,\infty)$ under the standard Skorohod topology, also denoted as the J_1 topology) to an Ornstein-Uhlenbeck type process driven by a stable process. If the stable process is Brownian motion, then one obtains convergence to a Gaussian process which suggests a central limit theorem for $D(\alpha)$. Again, the problem is that in the analysis of $D(\alpha)$ one first sends $t \to \infty$ and then $\alpha \to 0$, while the weak convergence results in $D[0,\infty)$ are developed on compact sets (i.e for fixed t) as $\alpha \to \infty$. In other words, there is an interchange of limits that must be justified. The main result of this section justifies this interchange of limits under general assumptions that we state next.

Throughout the rest of the present section we adopt the setting of Kurtz and Protter (1991), namely, we shall assume that Λ is a semi-martingale (with respect to some filtration) with decomposition

$$\Lambda = \Lambda_b + \Lambda_M,$$

where Λ_M is a local Martingale and Λ_b is a process with local bounded variation (i.e. bounded variation on compact sets). Also, we shall assume that both Λ and Γ are cadlag (right continuous with left limits) processes and Γ is adaptedWe shall discuss this set of assumptions after we state our main result of the section. We will also explain later how this result can be applied in practice.

Theorem 3 *Let* $(\Gamma_\alpha, \Lambda_\alpha) = (\Gamma(\cdot/\alpha), \Lambda(\cdot/\alpha))$ *and assume that*

W1 $\alpha^{-\beta}(\alpha(\Gamma_\alpha(\cdot), \Lambda_\alpha(\cdot)) - (\gamma\cdot, \lambda\cdot)) \Longrightarrow (Z_\Gamma(\cdot), Z_\Lambda(\cdot))$ *in* $D[0,\infty) \times D[0,\infty)$ *(under the standard Skorohod* J_1 *topology), for* $\beta \in (0,1)$ and $\gamma > 0$.

W2

$$\overline{\lim}_{t\to\infty}\frac{|\Lambda_b|(t)}{t}+E\frac{[\Lambda_M](t)}{t}<\infty.$$

W3

$$\begin{aligned}\overline{\lim_{\alpha\searrow 0}} E\log\left(1+\sup_{u\in[0,1]}\alpha^{-\beta}\left|\alpha\Lambda(u/\alpha)-\lambda u\right|\right) &< \infty,\\ \overline{\lim_{\alpha\searrow 0}} E\left(\sup_{u\in[0,1]}\alpha^{-\beta}\left|\alpha\Gamma(u/\alpha)-\gamma u\right|\right) &< \infty.\end{aligned}$$

Then,

$$\alpha^{-\beta}\left(D(\alpha)-\lambda/\gamma\right)\Longrightarrow\int_0^\infty e^{-\gamma s}dZ_1^\Lambda(s)-\lambda\int_0^\infty e^{-\gamma s}dZ_2^\Gamma(s),$$

where

$$D(\alpha)=\int_0^\infty \exp\left(-\alpha\Gamma(t-)\right)d\Lambda(t).$$

Remark: condition **W2** can be relaxed by means of localization as in Kurtz and Protter (1991), condition C2.2(i). The two assumptions given in **W3** resamble classical conditions that are required to define a perpetuity such as $D(\alpha)$, see Kesten (1973).

Before providing the proof of the previous theorem, we shall present the following lemma that makes clear the connection of condition **W3** to a more direct technical condition that can be expressed in terms of the tail behavior of the processes Λ and Γ. The proof of this lemma is given at the end of the present section.

Lemma 4 *If condition* ***W3*** *holds, then for each* $\delta,\delta_0>0$ *we have that*

$$\overline{\lim_{t_0\nearrow\infty}}\,\overline{\lim_{\alpha\searrow 0}} P\left(\sup_{t\geq t_0}e^{-\delta t}\alpha^{-\beta}\left|\alpha\Lambda(t/\alpha)-\lambda t\right|>\delta_0\right)=0.$$

Moreover,

$$\overline{\lim_{t_0\nearrow\infty}}\,\overline{\lim_{\alpha\searrow 0}} P\left(\sup_{t\geq t_0}\alpha^{-\beta}\left|\alpha\Gamma(t/\alpha)-\gamma t\right|>\delta_0\right)=0.$$

With the aid of the previous lemma we can proceed to the proof of the main result of this section.

Proof of Theorem. 3 Conditions **W1** and **W2** imply by virtue of Kurtz and Protter (1991) Theorem 2.2 that

$$\begin{aligned}&\alpha^{-\beta}\left(\int_0^t e^{-\alpha\Gamma_\alpha(s-)}\alpha d\Lambda_\alpha(s)-\int_0^t e^{-\gamma s}\lambda ds\right)\\ &\overset{D[0,\infty)}{\Longrightarrow}\int_0^t e^{-\gamma s}dZ_1^\Lambda(s)-\lambda\int_0^t e^{-\gamma s}Z_2^\Gamma(s)\,ds.\end{aligned}$$

Note that in our context we do not require to take $E(\cdot)$ in the total variation process $|\Lambda_b|(\cdot)$ as condition C2.2(i) of Kurtz and Protter (1991) requires. Looking at the proof of Kurtz and Protter (1991), specifically from equations (2.6) to (2.7), we see that taking expectation to the total variation process is not necessary in our context to obtain the required convergence in probability.

We then must show that for each $\delta_0, \varepsilon > 0$ there exists $t = t(\varepsilon) > 0$ large enough such that

$$\overline{\lim}_{\alpha\to\infty} P\left(\left|\alpha^{-\beta}\left(\int_t^\infty e^{-\alpha\Gamma_\alpha(s-)}\alpha d\Lambda_\alpha(s) - \int_t^\infty e^{-\gamma s}\lambda ds\right)\right| > \delta_0\right) \leq \varepsilon.$$

So we have to study

$$\begin{aligned} & \alpha^{-\beta}\left(\int_t^\infty e^{-\alpha\Gamma_\alpha(s-)}\alpha d\Lambda_\alpha(s) - \int_t^\infty e^{-\gamma s}\lambda ds\right) \\ = & \alpha^{-\beta}\int_t^\infty \left(e^{-\alpha\Gamma_\alpha(s-)} - e^{-\gamma s}\right)\alpha d\Lambda_\alpha(s) \qquad (8) \\ & +\alpha^{-\beta}\int_t^\infty e^{-\gamma s} d\left(\alpha\Lambda_\alpha(s) - \lambda s\right). \qquad (9) \end{aligned}$$

First we analyze (8), we decompose this term using the decomposition for the semi-martingale Λ, thus we obtain that the integral (8) equals

$$\begin{aligned} & \alpha^{-\beta}\int_t^\infty \left(e^{-\alpha\Gamma_\alpha(s-)} - e^{-\gamma s}\right)\alpha d\Lambda_b(s/\alpha) \qquad (10) \\ & +\alpha^{-\beta}\int_t^\infty \left(e^{-\alpha\Gamma_\alpha(s-)} - e^{-\gamma s}\right)\alpha d\Lambda_M(s/\alpha). \qquad (11) \end{aligned}$$

We shall start by showing that the contribution of the integral (11) is small for large t (uniformly in α). Define the stopping time

$$T_1 = \inf\{u \geq t : u^{-1}\left|\alpha\Gamma_\alpha(u-) - \gamma u\right| > \delta_1\alpha^\beta\},$$

Note that

$$\begin{aligned} & P\left(\left|\alpha^\beta\int_t^\infty \left(e^{-\alpha\Gamma_\alpha(s-)} - e^{-\gamma s}\right)\alpha d\Lambda_M(s/\alpha)\right| > \delta\right) \qquad (12) \\ \leq & P\left(\left|\alpha^\beta\int_t^{T_1} \left(e^{-\alpha\Gamma_\alpha(s-)} - e^{-\gamma s}\right)\alpha d\Lambda_M(s/\alpha)\right| > \delta; T = \infty\right) \\ & +P(T_1 < \infty) \\ & \left(\alpha^{-2(\beta-1/2)}/\delta\right) E\int_t^{T_1} \left(e^{-\alpha\Gamma_\alpha(s-)} - e^{-\gamma s}\right)^2 \alpha d[\Lambda_M](s/\alpha) \qquad (13) \\ & +P(T_1 < \infty). \qquad (14) \end{aligned}$$

Now, observe that (for some constant $c_1 > 0$)

$$
\begin{aligned}
& E\int_t^{T_1} e^{-2\gamma s}\left(e^{-(\alpha\Gamma_\alpha(s-)-\gamma s)}-1\right)^2 \alpha d[\Lambda_M](s/\alpha) \\
\leq\ & c_1\alpha^\beta E\int_t^{T_1} e^{-\gamma s}\alpha d[\Lambda_M](s/\alpha) = c_1\alpha^\beta E\int_t^{\infty} e^{-\gamma s}\alpha d[\Lambda_M](s/\alpha).
\end{aligned}
$$

This estimate implies that (13) is bounded by

$$
\begin{aligned}
& c_1\frac{\alpha^{-2(\beta-1/2)}}{\delta}\alpha^\beta E\int_t^\infty e^{-\gamma s}\alpha d[\Lambda_M](s/\alpha) \\
\leq\ & c_1\gamma\frac{\alpha^{1-\beta}}{\delta}E\int_t^\infty e^{-\gamma s}\alpha[\Lambda_M](s/\alpha)\, ds < c_2\alpha^{1-\beta}e^{-\gamma/2t}, \qquad (15)
\end{aligned}
$$

for some constant $c_2 > 0$. For $P(T_1 < \infty)$ note that

$$
P(T_1<\infty)\leq P\left(\sup_{u\geq t}\left|\frac{\alpha\Gamma(u/\alpha)-\gamma u}{u}\right|\geq\delta_1\alpha^\beta\right).
$$

Therefore, this estimate combined with (15) and Lemma 4 yields that for each $\varepsilon > 0$ we can find $\alpha_0 > 0$ small enough such that

$$
\sup_{0<\alpha<\alpha_0} P\left(\left|\alpha^\beta\int_t^\infty\left(e^{-\alpha\Gamma_\alpha(s-)}-e^{-\gamma s}\right)d\Lambda_M(s/\alpha)\right|>\delta\right)\leq\varepsilon, \qquad (16)
$$

this takes care of (12). The analysis of (10) is similar to that of (11). In particular, we note that

$$
\begin{aligned}
& P\left(\left|\alpha^{-\beta}\int_t^\infty\left(e^{-\alpha\Gamma_\alpha(s-)}-e^{-\gamma s}\right)\alpha d\Lambda_b(s/\alpha)\right|>\delta\right) \\
\leq\ & P\left(\left|\alpha^{-\beta}\int_t^{T_1}\left(e^{-\alpha\Gamma_\alpha(s-)}-e^{-\gamma s}\right)\alpha d\Lambda_b(s/\alpha)\right|>\delta\right)+P(T_1<\infty). \qquad (17)
\end{aligned}
$$

Note that the definition of T_1 implies that we can find a constant $c_1 > 0$ such that

$$
\begin{aligned}
& P\left(\left|\alpha^{-\beta}\int_t^{T_1}\left(e^{-\alpha\Gamma_\alpha(s-)}-e^{-\gamma s}\right)\alpha d\Lambda_b(s/\alpha)\right|>\delta\right) \\
\leq\ & P\left(c_1\int_t^{T} e^{-\gamma s/2}\alpha d\left|\Lambda_b\right|(s/\alpha)>\delta\right)\leq P\left(c_1\int_t^{\infty} e^{-\gamma s/2}\alpha d\left|\Lambda_b\right|(s/\alpha)>\delta\right) \\
\leq\ & P\left(C(\omega)\,e^{-\gamma t/2}>\delta\right), \qquad (18)
\end{aligned}
$$

where the last line above follows from integration by parts and assumption **W2**. Consequently, (18) together with (16) allows to control the behavior of (8). Finally, note we study (9). Integration by parts yields

$$\begin{aligned}
&\alpha^{-\beta}\int_t^{\infty} e^{-\gamma s} d[\alpha\Lambda_\alpha(s)-\lambda s] \\
=\ &\alpha^{-\beta}\int_t^{\infty} \gamma[\alpha\Lambda_\alpha(s)-\lambda s] e^{-\gamma s} ds - e^{-\gamma t}\alpha^{-\beta}[\alpha\Lambda_\alpha(s)-\lambda s].
\end{aligned}$$

Hence, as an immediate consequence of Lemma 4, we obtain that

$$\overline{\lim_{t_0\nearrow\infty}}\,\overline{\lim_{\alpha\searrow 0}} P\left(\alpha^{-\beta}\left|\int_t^{\infty} e^{-\gamma s} d[\alpha\Lambda_\alpha(s)-\lambda s]\right| > \delta_0\right) = 0.$$

Combining this estimate with our previous arguments concerning (8) yields the conclusion of the theorem. ■

Proof. Let t_0 be a large but fix number, $\alpha > 0$ and pick $\theta > 1$, then

$$\begin{aligned}
&P\left(\sup_{t\geq t_0} e^{-\delta t}\alpha^{-\beta}\left|\Lambda(t/\alpha)-\lambda t\right| > \delta_0\right) \\
\leq\ &\sum_{k=0}^{\infty} P\left(\sup_{t\in[t_0\theta^k, t_0\theta^{k+1})} e^{-\delta t}\alpha^{-\beta}\left|\alpha\Lambda(t/\alpha)-\lambda t\right| > \delta_0\right) \\
\leq\ &\sum_{k=0}^{\infty} P\left(\sup_{t\in[t_0\theta^k, t_0\theta^{k+1})} \alpha^{-\beta}\frac{\left|\alpha\Lambda(t/\alpha)-\lambda t\right|}{\left(t_0\theta^{k+1}\right)^{\beta}} > \frac{t_0\exp\left(\delta\theta^k\right)\delta_0}{\left(t_0\theta^{k+1}\right)^{\beta}}\right) \\
\leq\ &\sum_{k=0}^{\infty} P\left(\sup_{t\in[0, t_0\theta^{k+1})} \alpha^{-\beta}\frac{\left|\alpha\Lambda(t/\alpha)-\lambda t\right|}{\left(t_0\theta^{k+1}\right)^{1-\beta}} > \frac{t_0^{\beta}\exp\left(\delta\theta^k\right)\delta_0}{\theta^{(k+1)\beta}}\right). \qquad (19)
\end{aligned}$$

Put $r_k = t_0\theta^{k+1}$ and note that making $ur_k = t$ and $\alpha/r_k = \widetilde{\alpha}$

$$\begin{aligned}
\sup_{t\in[0,r_k]} \alpha^{-\beta}\frac{\left|\alpha\Lambda(t/\alpha)-\lambda t\right|}{r_k^{1-\beta}} &= \sup_{u\in[0,1]} \frac{\left|\alpha\Lambda(ur_k/\alpha)-\lambda ur_k\right|}{\alpha^{\beta} r_k^{1-\beta}} \\
&= \sup_{u\in[0,1]} \frac{\left|\alpha\Lambda(ur_k/\alpha)-\lambda ur_k\right|}{(\alpha/r_k)^{\beta} r_k} \\
&= \sup_{u\in[0,1]} \frac{\left|\alpha\Lambda(ur_k/\alpha)-\lambda ur_k\right|}{\widetilde{\alpha}^{\beta}}.
\end{aligned}$$

Therefore, by the indicated property of $W(\cdot)$ we obtain from (19) that there exists a constant $b > 0$ such that for all α sufficiently small

$$P\left(\sup_{t\geq t_0} e^{-\delta t}\alpha^{-\beta}\left|\alpha\Lambda(t/\alpha)-\lambda t\right| > \delta_0\right)$$

$$\leq \sum_{k=0}^{\infty} P\left(\sup_{t\in[0,t_0\theta^{k+1})} \alpha^{-\beta}\frac{|\alpha\Lambda(t/\alpha)-\lambda t|}{\left(t_0\theta^{k+1}\right)^{1-\beta}} > \frac{t_0^{\beta}\exp\left(\delta\theta^k\right)\delta_0}{\theta^{(k+1)\beta}}\right)$$

$$\leq b\sum_{k=0}^{\infty}\frac{1}{\beta\log(t_0)+\delta\theta^k-(k+1)\beta\log(\theta)-\log\delta_0}.$$

Since $\theta > 1$ the previous quantity is finite and it goes to zero (because $\beta \in (0,1)$) as $t_0 \nearrow \infty$. The corresponding property for Γ follows completely analogous steps and therefore is ommitted. This concludes the proof of the lemma. ■

In most applications $\beta = 1/2$ and Z corresponds to a Gaussian process driven by Brownian motion, but in more general settings Z is driven by a stable process. More precisely, in many applications we can expect

$$\alpha^{1/2}\left(\begin{pmatrix}\Lambda_\alpha(\cdot)\\ \Gamma_\alpha(\cdot)\end{pmatrix}-\begin{pmatrix}\lambda\cdot/\alpha\\ \gamma\cdot/\alpha\end{pmatrix}\right)\Rightarrow G\begin{pmatrix}B_1(\cdot)\\ B_2(\cdot)\end{pmatrix},$$

where $B = (B_1, B_2)$ is a two dimensional Brownian motion and $GG^T = C$ is the corresponding covariance matrix. Typically, one would have

$$\begin{aligned}
C_{1,2} &= \lim_{t\to\infty} E\frac{(\Lambda(t)-\lambda t)(\Gamma(t)-\tau)}{t},\\
C_{1,1} &= \lim_{t\to\infty} E\frac{(\Lambda(t)-\lambda t)^2}{t},\\
C_{2,2} &= \lim_{t\to\infty} E\frac{(\Gamma(t)-\gamma t)^2}{t}.
\end{aligned}$$

Therefore, Theorem 3 guarantees that in great generality

$$\alpha^{-1/2}(\alpha D(\alpha)-\lambda/\gamma)\Rightarrow Z(\infty)\overset{D}{=}\sigma/\gamma^{1/2}N(0,1),$$

where

$$\sigma^2=\frac{1}{2}\left(C_{11}-2\frac{\lambda}{\gamma}C_{12}+\frac{\lambda^2}{\gamma^2}C_{22}\right).$$

The expression for σ^2 was obtained using integration by parts to represent

$$Z(\infty)=\int_0^\infty e^{-\gamma s}d(G_1.B(s))-\frac{\lambda}{\gamma}\int_0^\infty e^{-\gamma s}d(G_2.B(s)),$$

in combination with Ito's isometry. In practical situations (where $\alpha > 0$ does not appear naturally in the problem structure) Theorem 3 provides rigorous support for the formal approximation

$$D=\int_0^\infty \exp(-\Gamma(t-))\,d\Lambda(t)\overset{D}{\approx}\frac{\lambda}{\gamma}+\sigma/\gamma^{1/2}N(0,1), \tag{20}$$

the symbol " $\overset{D}{\approx}$ " means "approximately equal in distribution to" and the precise meaning of approximation (20) is given by Theorem 3.

4 Edgeworth Expansions

In this section, we provide refined versions for some of the approximations given in the previous sections. The refined approximation takes the form of an Edgeworth expansion for the distribution of D. We shall derive these approximations in the iid setting for the discrete time case and under Markovian assumptions for the continuous time case. More precisely, in the discrete time case, motivated by the applications to ARCH processes described in Section 2, we consider

$$D = \sum_{k=0}^{\infty} \exp\left(-\sum_{j=0}^{k-1} Z_j\right) X_k,$$

where $(X_k, Z_k)_{k\geq 1}$ is a sequence of iid random vectors satisfying certain assumptions to be described later (see assumptions ED1 to ED4 below); while in the continuous time context, we work with

$$D = \int_0^{\infty} \exp\left(-\int_0^t \gamma\left(Y\left(s\right)\right) ds\right) d\Lambda\left(t\right),$$

where $Y = (Y_s : s \geq 0)$ is a suitably defined homogeneous Markov process Λ is a stationary independent increment process, this setting is commonly used in the risk theory example discussed in Section 2 (see Ch. 7 of Asmussen (2001)).

4.1 The discrete time setting

In this section, we shall consider the following set of assumptions.

ED1 Assume that $Z_1 \geq 0$, $E\left(Z_1\right) = \gamma < \infty$, $E\left(Z_1^2\right) = \mu_Z^{(2)} < \infty$, and $E\left(|Z_1|^3\right) < \infty$. Let σ_Z^2 be the variance of Z_1 and $\kappa_Z^{(3)}$ its third order cumulant, which can be written as

$$\kappa_Z^{(3)} = \mu_Z^{(3)} - 3\mu_Z^{(2)}\gamma + 2\gamma^3.$$

ED2 Suppose that X_1 has non-lattice distribution with $E\left(X_1\right) = \lambda$, $Var\left(X_1^2\right) = \sigma_X^2$, and $E\left(|X_1|^3\right) < \infty$. Let $E\left(X_1^3\right) = \mu_3^X$ and write $\kappa_X^{(3)}$ to denote the third order cumulant of X_1. In addition, assume that the distribution of X_1 given Z_1 is non-lattice.

ED3 Suppose that $E\left(|X_1|^j |Z_1|^k\right) < \infty$ for $0 < j + k \leq 3$ and for $j, k \geq 1$ denote $\mu_{jk} = E\left(X_1^j Z_1^k\right)$. Moreover, let us define,

$$\delta\left(\theta, Z_1\right) = \left|E\left(e^{i\theta X_1}\middle| Z_1\right)\right|$$

and assume that

$$\overline{\lim_{h\to 0}} \sup_{\varepsilon\leq|\theta|\leq 1/\varepsilon} \frac{P\left(\delta\left(\theta, Z_1\right) > 1 - h\right)}{h} < \infty, \tag{21}$$

for $\varepsilon > 0$.

Condition (21) is technical, and may be seen as a form of strong non-latticity of X_1 given Z_1. Notice that, in the important special case in which the X_k's are independent of the Z_k's, assumption ED3 is an immediate consequence of ED2. Indeed, if X_1 is non-lattice, we have that $\delta(\theta, Z_1) = \delta(\theta) < 1$. Therefore, for all $h > 0$ sufficiently small, $\delta(\theta) < 1 - h$. This implies that the limit in (21) is zero.

As a remark, we also note that, alternatively, the non-negativity of Z_1 required in assumption ED1 can be replaced by the existence of exponential moments, we record this observation as our alternative assumption ED1'.

ED1' Assume that $E\exp(\rho Z_1) < \infty$ for ρ in a vicinity of the origin.

Under these assumptions, we improve the approximation (20) by providing an Edgeworth expansion for the distribution of D when $(X_k, Z_k)_{k\geq 1}$ is a sequence of *i.i.d.* random vectors and the discount rate γ is small. In particular, by defining

$$\sigma^2 = \frac{1}{2}\left(\sigma_X^2 - 2\frac{\lambda}{\gamma}\sigma_{XZ} + \frac{\lambda^2}{\gamma^2}\sigma_Z^2\right),$$

we can write the approximation proposed as

$$\begin{aligned} P(D \leq y) \quad \approx \quad & P\left(N\left(\lambda/\gamma, \sigma^2/\gamma\right) \leq y\right) - \sqrt{\gamma}\beta_1 \eta\left((y - \lambda/\gamma)\frac{\sqrt{\gamma}}{\sigma}\right) \\ & -\frac{\sqrt{\gamma}}{18}\beta_2 H\left((y - \lambda/\gamma)\frac{\sqrt{\gamma}}{\sigma}\right). \end{aligned} \tag{22}$$

The constants β_1 and β_2 satisfy

$$\begin{aligned} \beta_1 &= \frac{\mu_Z^{(2)}\lambda}{2\gamma^2\sigma}, \\ \sigma^3\beta_2 &= \kappa_X^{(3)} - 2\kappa_{21}\frac{\lambda}{\gamma} + 3\kappa_{12}\frac{\lambda^2}{\gamma^2} - 3\frac{\kappa_{11}}{\gamma}\left(\sigma_X^2 - 2\frac{\lambda}{\gamma}\sigma_{XZ} + \frac{\lambda^2}{\gamma^2}\sigma_Z^2\right) \\ & \quad +3\sigma_Z^2\frac{\lambda}{\gamma^2}\left(\sigma_X^2 - 2\frac{\lambda}{\gamma}\sigma_{XZ} + \frac{\lambda^2}{\gamma^2}\sigma_Z^2\right) - \frac{\kappa_Z^{(3)}\lambda^3}{\gamma^3}, \end{aligned}$$

with

$$\begin{aligned} \kappa_{12} &= \mu_{12} + \mu_{11} - \mu_Z^{(2)} - 3\gamma\mu_{11} + 2\gamma^2\lambda, \\ \kappa_{21} &= \mu_{21} + \mu_{11} - \mu_X^{(2)} - 3\lambda\mu_{11} + 2\lambda^2\gamma, \\ \kappa_{11} &= \mu_{11} - \lambda\gamma = \sigma_{XZ} \triangleq cov(X, Z); \end{aligned}$$

and

$$\begin{aligned} \eta(y) &= \frac{1}{\sqrt{2\pi}}\exp\left(-y^2/2\right) \\ H(y) &= \left(y^2 - 1\right)\eta(y). \end{aligned}$$

The application of the approximation (22), requires estimation of the joint moments μ_{ij}, which can be easily done (even non-parametrically) using standard methods. Also, observe that in the case in which the sequences $(X_k)_{k>0}$ and $(Z_k)_{k>0}$ are independent, the constants σ^2, β_1 and β_2 take the simplified form

$$\sigma^2 = \frac{1}{2}\left(\sigma_X^2 + \frac{\lambda^2}{\gamma^2}\sigma_Z^2\right)$$

and

$$\beta_1 = \frac{\mu_Z^{(2)}\lambda}{2\gamma^2\sigma}, \quad \beta_2 = \frac{1}{\sigma^3}\left(\kappa_X^{(3)} + 3\frac{\sigma_Z^2\lambda\sigma_X^2}{\gamma^2} - \frac{\kappa_Z^{(3)}\lambda^3}{\gamma^3} + 3\frac{\sigma_Z^4\lambda^3}{\gamma^4}\right).$$

In order to understand the nature of approximation (22), we introduce a small scaling parameter $\alpha > 0$ and define

$$D(\alpha) = \sum_{k=0}^{\infty} \exp\left(-\alpha \sum_{j=0}^{k-1} Z_j\right) X_k.$$

approximation (22) becomes (since the quantities σ, β_1 and β_2 are not affected by the scaling)

$$\begin{aligned} P(D(\alpha) \leq y) &\approx P\left(N\left(\lambda/\alpha\gamma, \sigma^2/\alpha\gamma\right) \leq y\right) - \sqrt{\gamma\alpha}\beta_1\eta\left((y - \lambda/\gamma\alpha)\frac{\sqrt{\gamma\alpha}}{\sigma}\right) \quad (23) \\ &\quad - \frac{\sqrt{\gamma\alpha}}{18}\beta_2 H\left((y - \lambda/\gamma\alpha)\frac{\sqrt{\gamma\alpha}}{\sigma}\right). \end{aligned}$$

Or, in other words,

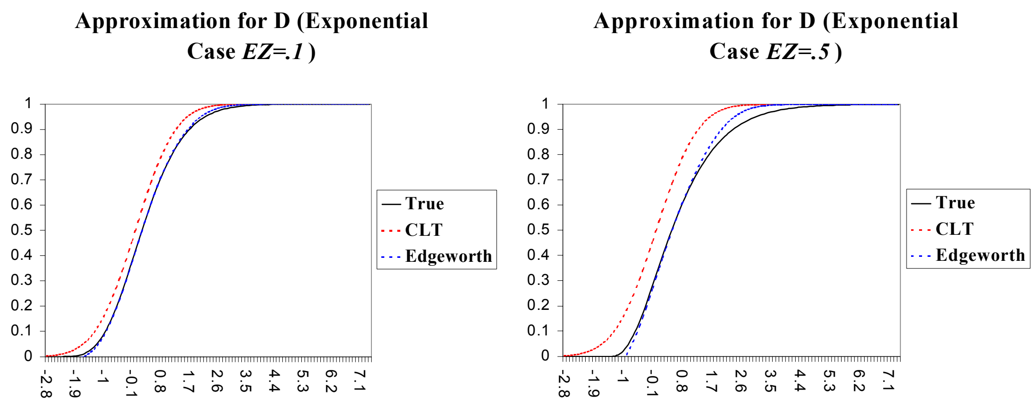
$$\begin{aligned} P\left(\sqrt{\alpha}\left(D(\alpha) - \lambda/\alpha\gamma\right) \leq y\right) &\approx P\left(N\left(0, \sigma^2/\gamma\right) \leq y\right) - \sqrt{\gamma\alpha}\beta_1\eta\left(\frac{\sqrt{\gamma}}{\sigma}y\right) \\ &\quad - \frac{\sqrt{\gamma\alpha}}{18}\beta_2 H\left(\frac{\sqrt{\gamma}}{\sigma}y\right) \end{aligned}$$

with an error of order $o(\sqrt{\alpha})$ (uniformly on y). The precise mathematical statement concerning the previous approximations is the content of Theorem 6 below, which provides the first order correction in the Edgeworth expansion for $D(\alpha)$. However, before moving on to Theorem 6, we present a simple example to illustrate the accuracy of the approximations proposed.

Example 5 *Suppose that $X_1 \sim \lambda \exp(1)$ and $Z_1 \sim \gamma \exp(1)$. Under these assumptions it follows (see Gjessing and Paulsen (1997)) that*

$$D = \sum_{k=0}^{\infty} \exp\left(-\sum_{j=0}^{k-1} Z_j\right) X_k \sim \lambda\Gamma(1/\gamma + 1, 1),$$

where $\Gamma(1/\gamma+1,1)$ *represents a random variable with distribution gamma with the parameters given. In order to illustrate the numerical fit of the approximation provided we consider the case in which* $\lambda = 1$ *and* $\gamma = .1$ *and* $\gamma = .5$ *respectively. The following graphs compare the CLT and Edgeworth approximations developed against the true distribution of* D



CLT and Edgeworth Based Approximations

We now provide the rigorous statement supporting approximation (22).

Theorem 6 *If the set of assumptions* $ED1$ *(or* $ED1'$*) to* $ED4$ *are in force, then*

$$P\left(\sqrt{\alpha}\left(D(\alpha)-\frac{\lambda}{\gamma\alpha}\right)\leq y\right) = P\left(N\left(0,\frac{\sigma^2}{\gamma}\right)\leq y\right)-\sqrt{\alpha}\beta_1 n(y) - \frac{\sqrt{\alpha}}{18}\frac{\beta_2}{\gamma}H(y)+G_\alpha(y); \quad (24)$$

where G_α *represents a signed measure with* $G_\alpha^+(R)+G_\alpha^-(R)\triangleq\|G_\alpha(dy)\|=o(\sqrt{\alpha})$.

In order to prove this theorem, we need some preliminary results. As it is standard in obtaining Edgeworth expansions via Fourier analytic methods (see Feller (1968) p. 512), one first proceeds to obtain an asymptotic expansion for the cumulant moment generating function of interest. Hence, our first result provides an asymptotic expansion for $\psi_\alpha(\theta)\triangleq\log E\exp\left(i\theta\alpha^{-1/2}(\alpha D(\alpha)-\lambda/\gamma)\right)$ in powers of $\sqrt{\alpha}$.

Lemma 7 *Assume* $ED1$ *(or* $ED1'$*) to* $ED3$*. Then, there exists* $\delta>0$ *for which we have that*

$$\begin{aligned}\psi_\alpha(\theta) &= \left(\frac{\mu_Z^{(2)}\lambda}{2\gamma^2}+O(\alpha)\right)i\theta\alpha^{1/2}\\ &+\left(\frac{1}{2\gamma\alpha}\left(\sigma_X^2-2\frac{\lambda}{\gamma}\sigma_{XZ}+\frac{\lambda^2}{\gamma^2}\sigma_Z^2\right)+O(1)\right)\frac{(i\theta)^2}{2}\alpha\\ &+\left(\frac{C_3}{\alpha}+O(1)\right)\frac{(i\theta)^3}{6}\alpha^{3/2}+o\left(\alpha^{1/2}\right),\end{aligned}$$

(uniformly in $\theta \in (-\delta, \delta)$, $\delta > 0$) where

$$\begin{aligned} 3\gamma C_3 &= \kappa_X^{(3)} - 2\kappa_{21}\frac{\lambda}{\gamma} + 3\kappa_{12}\frac{\lambda^2}{\gamma^2} \\ &\quad +3\left(\sigma_Z^2\frac{\lambda}{\gamma^2} - \frac{\sigma_{XZ}}{\gamma}\right)\left(\sigma_X^2 - 2\frac{\lambda}{\gamma}\sigma_{XZ} + \frac{\lambda^2}{\gamma^2}\sigma_Z^2\right) - \frac{\kappa_Z^{(3)}\lambda^3}{\gamma^3}. \end{aligned}$$

Proof. The idea is to write

$$\phi_\alpha(\theta) = \exp\left(i\theta\lambda/\gamma\sqrt{\alpha}\right)\phi\left(\theta\sqrt{\alpha}, \alpha\right),$$

where $\phi_\alpha(\theta) \triangleq \exp(\psi_\alpha(\theta))$ and $\phi(\theta, \alpha) \triangleq E\exp(i\theta D(\alpha))$. Notice that $\phi(\theta, \alpha)$ satisfies

$$\phi(\theta, \alpha) = E\left(\exp\left(i\theta\left(X_1 + \exp(-\alpha Z_1) D_1(\alpha)\right)\right)\right),$$

with $D_1(\alpha)$ independent of (X_1, Z_1). Thus, we have,

$$\begin{aligned} \phi(\theta, \alpha) &= E\left(\exp\left(i\theta\left(X_1 + \exp(-\alpha Z_1) D_1(\alpha)\right)\right)\right) \\ &= E\left(E\left(\exp\left(i\theta\left(X_1 + \exp(-\alpha Z_1) D_1(\alpha)\right)\right)\middle| X_1, Z_1\right)\right) \\ &= E\left(E\left(\exp(i\theta X_1)\phi\left(\theta\exp(-\alpha Z_1), \alpha\right)\middle| X_1, Z_1\right)\right) \\ &= E\left(\exp(i\theta X_1)\phi\left(\theta\exp(-\alpha Z_1), \alpha\right)\right). \end{aligned}$$

Using the Taylor development for characteristic functions (see Feller (1968) App. Sec. XV.5 and Breiman (1992) Prop. 8.44) applied to $\phi(\theta, \alpha)$ and $\phi_\alpha(\theta)$, together with the moment conditions implied by assumptions $ED1$ (or $ED1'$) to $ED3$, we arrive at the expression stated for $\psi_\alpha(\theta)$. ■

Lemma 8 *Under assumptions $ED1$ (or $ED1'$) to $ED4$, $\phi(\theta, \alpha) \triangleq E\exp(i\theta D(\alpha))$ satisfies*

$$|\phi(\theta, \alpha)| = o\left(\alpha^{1/2}\right)$$

as $\alpha \to 0$ uniformly in θ over compact sets not containing the origin.

Proof. Let $\phi_X(\theta, Z_1) = E\left(e^{i\theta X_1}\middle| Z_1\right)$, and let $T_\alpha = \inf\{k : S_k > 1/\alpha\}$. Then,

$$\begin{aligned} |\phi(\theta, \alpha)| &= \left|E\left(E\left(\exp\left(i\theta\sum_{k=1}^{\infty} X_k \exp(-\alpha S_{k-1})\right)\middle| Z\right)\right)\right| \\ &= \left|E\left(\Pi_{k=1}^{\infty}\phi_X\left(\theta e^{-\alpha S_{k-1}}, Z_k\right)\right)\right| \\ &\leq E\left(\Pi_{k=1}^{\infty}\left|\phi_X\left(\theta e^{-\alpha S_{k-1}}, Z_k\right)\right|\right) \\ &\leq E\left(\Pi_{k=1}^{T\alpha-1}\left|\phi_X\left(\theta e^{-\alpha S_{k-1}}, Z_k\right)\right|\right) \\ &\leq E\left(\Pi_{k=1}^{T\alpha-1}\left|\Delta(\theta, Z_k)\right|\right), \end{aligned}$$

where $\Delta(\theta, Z_1) = \sup\{|\phi_X(\theta^*, Z_1)| : |\theta^*| > |\theta e^{-1}|\}$. Since the distribution of X_1 given Z_1 is non-lattice, we must have that $0 < \Delta(\theta, Z_1) < 1$. So,

$$\begin{aligned} |\phi(\theta, \alpha)| &\leq E\left(\Pi_{k=1}^{T\alpha-1} |\Delta(\theta, Z_k)|\right) \\ &\leq P\left(\alpha \left|T_\alpha - \frac{1}{\alpha\gamma}\right| > \varepsilon\right) + E\left(\Pi_{k=1}^{T\alpha-1} |\Delta(\theta, Z_k)|; \alpha |T_\alpha - 1/\alpha\gamma| \leq \varepsilon\right) \\ &\leq P\left(\alpha \left|T_\alpha - \frac{1}{\alpha\gamma}\right| > \varepsilon\right) + E\left(|\Delta(\theta, Z_1)|^{1/\alpha(1/\gamma-\varepsilon)-1}\right). \end{aligned}$$

Since condition $AF1$ ($AF1'$) imply that $0 < EZ_1 < \infty$ and $Var(Z_1) < \infty$, we have that $\left(\alpha^{1/2}\left|T_\alpha - \frac{1}{\alpha\gamma}\right|\right)^2$ is uniformly integrable (see Gut (1988) p. 92.) In particular, this implies, using Chebyshev's inequality, that

$$P\left(\alpha \left|T_\alpha - \frac{1}{\alpha\gamma}\right| > \varepsilon\right) = O(\alpha).$$

Finally, if we choose $\varepsilon > 0$ small enough so that $c \triangleq 1/\gamma - \varepsilon > 0$, we must show (for θ not in a neighborhood of the origin) that

$$E\left(|\Delta(\theta, Z_1)|^{c/\alpha}\right) = o\left(\sqrt{\alpha}\right).$$

Let $W = -\log(|\Delta(\theta, Z_1)|)$ and $\beta = c/\alpha$. Then,

$$E\left(|\Delta(\theta, Z_1)|^\beta\right) = E(\exp(-\beta W)) = \int_0^\infty \exp(-u) P(u/\beta > W) \, du.$$

Thus,

$$\beta E\left(|\Delta(\theta, Z_1)|^\beta\right) = \int_0^\infty \exp(-u) \beta P(u/\beta > W) \, du.$$

Fix $\varepsilon > 0$ and write

$$\begin{aligned} \beta E\left(|\Delta(\theta, Z_1)|^\beta\right) &= \int_0^\varepsilon \exp(-u) \beta P(u/\beta > W) \, du \\ &\quad + \int_\varepsilon^\infty u \exp(-u) \beta/u P(u/\beta > W) \, du \qquad (25) \\ &\leq \beta P(\varepsilon/\beta > W) + \int_\varepsilon^\infty u \exp(-u) \beta/u P(u/\beta > W) \, du. \end{aligned}$$

We want to apply Fatou's Lemma in the form

$$\begin{aligned} &\overline{\lim}_{\beta \longrightarrow \infty} \int_\varepsilon^\infty u \exp(-u) \beta/u P(u/\beta > W) \, du \\ \leq &\int_\varepsilon^\infty \overline{\lim}_{\beta \longrightarrow \infty} u \exp(-u) \beta/u P(u/\beta > W) \, du. \end{aligned}$$

In order to do this, we must show that

$$0 \leq \beta/uP\left(u/\beta > W\right) \leq M$$

for some $M > 0$ for $u \in [\varepsilon, \infty]$, and β large. So, by right continuity and the existence of left limits, it suffices to show that

$$\overline{\lim}_{\beta \longrightarrow \infty} \frac{P\left(h > W\right)}{h} < \infty.$$

But

$$\begin{aligned}
\overline{\lim}_{h \longrightarrow 0} \frac{P\left(h > W\right)}{h} &= \overline{\lim}_{h \longrightarrow 0} \frac{P\left(h > -\log\left(\left|\Delta\left(\theta, Z_1\right)\right|\right)\right)}{h} \\
&= \overline{\lim}_{h \longrightarrow 0} \frac{P\left(\exp(-h) < \left|\Delta\left(\theta, Z_1\right)\right|\right)}{h} \\
&= \overline{\lim}_{h \longrightarrow 0} \frac{P\left(\left|\Delta\left(\theta, Z_1\right)\right| > 1 - h\right)}{h} < \infty,
\end{aligned}$$

by virtue of assumption $ED4$. This is what we require in order to apply Fatou's lemma. Consequently, we have

$$\overline{\lim}_{\beta \longrightarrow \infty} \beta E\left(\left|\Delta\left(\theta, Z_1\right)\right|^{\beta}\right) < \infty,$$

which implies

$$\overline{\lim}_{\beta \longrightarrow \infty} \sqrt{\beta} E\left(\left|\Delta\left(\theta, Z_1\right)\right|^{\beta}\right) = 0,$$

and this is what we needed to conclude the proof of the lemma. ■

We now are ready to proof Theorem 6.

Proof of Theorem 6. . The proof of this theorem follows closely the steps of Feller (1968) p.512. To simplify the notation, let us consider $E\left(X_1\right) = 0$ and $E\left(X_1^2\right) = 2\gamma$ and the X_k's independent of the Z_k's (as we shall see from the proof, these are just simplifying assumptions and the adaptation of the present proof is straightforward using the corresponding local expansion given in Lemma 7)). Let $\gamma\left(\theta\right) = \widehat{G}\left(\theta\right) = e^{-\theta^2/2}\left(1 + \frac{(i\theta)^3 \kappa_X^{(3)}}{18\gamma}\sqrt{\alpha}\right)$. Esséen's lemma applies here since

$$G\left(x\right) = \Phi\left(x\right) - \frac{\kappa_X^{(3)}}{18}\sqrt{a}\left(x^2 - 1\right)\eta\left(x\right)$$

is bounded by some constant C. Also $\gamma\left(0\right) = 1$ and $\gamma'\left(0\right) = 0$. Therefore,

$$\left|F_{\alpha}\left(x\right) - G\left(x\right)\right| \leq \frac{1}{\pi}\int_{-T}^{T} \frac{1}{|\theta|}\left|\phi\left(\sqrt{\alpha}\theta, \alpha\right) - \gamma\left(\theta\right)\right| d\theta + \frac{24C}{\pi T}.$$

Let $T = M/\sqrt{\alpha}$, for some $M > 0$ big. Then, for any $\delta > 0$ small, we have

$$|F_\alpha(x) - G(x)| \le I_1 + I_2 + I_3 + \sqrt{\alpha}\frac{24C}{\pi M},$$

where

$$\begin{aligned} I_1 &= \frac{1}{\pi}\int_{-\delta/\sqrt{\alpha}}^{\delta/\sqrt{\alpha}} \frac{1}{|\theta|}\left|\phi\left(\sqrt{\alpha}\theta, \alpha\right) - \gamma(\theta)\right| d\theta, \\ I_2 &= \frac{1}{\pi}\int_{\delta/\sqrt{\alpha}}^{M/\sqrt{\alpha}} \frac{1}{|\theta|}\left|\phi\left(\sqrt{\alpha}\theta, \alpha\right) - \gamma(\theta)\right| d\theta, \\ I_3 &= \frac{1}{\pi}\int_{-M/\sqrt{\alpha}}^{\delta/\sqrt{\alpha}} \frac{1}{|\theta|}\left|\phi\left(\sqrt{\alpha}\theta, \alpha\right) - \gamma(\theta)\right| d\theta. \end{aligned}$$

Observe that

$$\begin{aligned} I_2 &\le \frac{1}{\pi}\int_{\delta/\sqrt{\alpha}}^{M/\sqrt{\alpha}} \frac{1}{|\theta|}\left|\phi\left(\sqrt{\alpha}\theta, \alpha\right)\right| d\theta + \frac{1}{\pi}\int_{\delta/\sqrt{\alpha}}^{M/\sqrt{\alpha}} \frac{1}{|\theta|}|\gamma(\theta)|\, d\theta \\ &= \frac{1}{\pi}\int_{\delta}^{M} \frac{1}{|\theta|}|\phi(\theta, \alpha)|\, d\theta + \frac{1}{\pi}\int_{\delta/\sqrt{\alpha}}^{M/\sqrt{\alpha}} \frac{1}{|\theta|}|\gamma(\theta)|\, d\theta. \end{aligned}$$

By virtue of our previous lemma, it is clear that I_2 goes to zero faster than $\sqrt{\alpha}$, similarly for I_3. Thus, we just have to study I_1. Let

$$\begin{aligned} \zeta(\theta, \alpha) &\triangleq \log(\phi(\theta, \alpha)) + \frac{\theta^2 2\gamma}{2(1 - m(-2\alpha))} \\ &= \log(\phi(\theta, \alpha)) + \frac{\theta^2\gamma}{(1 - m(-2\alpha))} \end{aligned}$$

where $m(-\lambda) = E\left(e^{-\lambda Z_1}\right)$. Hence, we can write

$$\begin{aligned} I_1 &= \frac{1}{\pi}\int_{-\delta/\sqrt{\alpha}}^{\delta/\sqrt{\alpha}} \frac{1}{|\theta|}\left|\phi\left(\sqrt{\alpha}\theta, \alpha\right) - \gamma(\theta)\right| d\theta \\ &= \frac{1}{\pi}\int_{-\delta/\sqrt{\alpha}}^{\delta/\sqrt{\alpha}} \frac{1}{|\theta|}\left|\exp\left(\zeta\left(\sqrt{\alpha}\theta, \alpha\right) - \frac{\theta^2\gamma}{(1 - m(-2\alpha))}\right) - \gamma(\theta)\right| d\theta \\ &= \frac{1}{\pi}\int_{-\delta/\sqrt{\alpha}}^{\delta/\sqrt{\alpha}} \frac{1}{|\theta|}e^{-\theta^2/2}\left|e^{\zeta\left(\sqrt{\alpha}\theta,\alpha\right) - \frac{\theta^2}{2}\left(\frac{\alpha\gamma}{(1-m(-2\alpha))} - 1\right)} - 1 - \frac{(i\theta)^3\mu_3\sqrt{\alpha}}{18}\right| d\theta. \end{aligned}$$

Using Feller (1968), p. 507, we have that for any $\widetilde{\beta}_1$ and $\widetilde{\beta}_2$ complex numbers,

$$\left|e^{\widetilde{\beta}_1} - 1 - \widetilde{\beta}_2\right| \le \left(\left|\widetilde{\beta}_1 - \widetilde{\beta}_2\right| + \frac{1}{2}\widetilde{\beta}_2^2\right)\exp(\upsilon), \tag{26}$$

where $\upsilon \geq \max\left(\left|\widetilde{\beta}_1\right|, \left|\widetilde{\beta}_2\right|\right)$. Given $\varepsilon > 0$, we can choose $\delta > 0$ small enough so that $|\theta\sqrt{\alpha}| < \delta$ (as in Feller (1968), p. 507) and

$$\left|\zeta\left(\theta\sqrt{\alpha}, \alpha\right) - \frac{\alpha^{3/2}\left(i\theta\right)^3 \kappa_X^{(3)}}{3!\left(1 - m\left(-3\alpha\right)\right)}\right| \leq \varepsilon \frac{\theta^3 \alpha^{3/2}}{\left|\left(1 - m\left(-3\alpha\right)\right)\right|} \leq \varepsilon K \theta^3 \alpha^{1/2}$$

for α small enough and some constant K_1 independent of α (because $\frac{\alpha^{3/2}\kappa_X^{(3)}}{(1-m(-3\alpha))}$ is the cumulant of order 3 for the random variable $\sqrt{\alpha}D\left(\alpha\right)$). At the same time, δ can also be chosen satisfying

$$\left|\zeta\left(\theta\sqrt{\alpha}, \alpha\right)\right| < \frac{1}{2}\frac{\gamma\alpha\theta^2}{\left(1 - m\left(-2\alpha\right)\right)} \leq \frac{K_2}{3}\theta^2$$

for some $K_2 \leq 1$ for α small enough. Now, δ can be chosen also with the property that

$$\left|\frac{\alpha^{3/2}\left(i\theta\right)^3 \kappa_X^{(3)}}{3!\left(1 - m\left(-3\alpha\right)\right)}\right| < \frac{K_2}{3}\theta^2.$$

Notice that

$$\begin{aligned} & \left|e^{\zeta\left(\sqrt{\alpha}\theta,\alpha\right) - \frac{\theta^2}{2}\left(\frac{\alpha\gamma}{(1-m(-2\alpha))} - 1\right)} - 1 - \frac{\left(i\theta\right)^3 \kappa_X^{(3)}}{18}\right| \\ \leq & \left|e^{\zeta\left(\sqrt{\alpha}\theta,\alpha\right) - \frac{\theta^2}{2}\left(\frac{\alpha\gamma}{(1-m(-2\alpha))} - 1\right)} - 1 - \frac{\alpha^{3/2}\left(i\theta\right)^3 \kappa_X^{(3)}}{3!\left(1 - m\left(-3\alpha\right)\right)}\right| + \\ & \left|\frac{\alpha^{3/2}\left(i\theta\right)^3 \kappa_X^{(3)}}{3!\left(1 - m\left(-3\alpha\right)\right)} - \frac{\left(i\theta\right)^3 \kappa_X^{(3)}}{18}\sqrt{\alpha}\right|, \end{aligned}$$

and observe that

$$\left|\frac{\alpha^{3/2}\left(i\theta\right)^3 \kappa_X^{(3)}}{3!\left(1 - m\left(-3\alpha\right)\right)} - \frac{\left(i\theta\right)^3 \kappa_X^{(3)}}{18}\sqrt{\alpha}\right| \leq \sqrt{\alpha}\theta^3 o\left(1\right).$$

Finally, we apply inequality (26) with $\widetilde{\beta}_1 = \zeta\left(\sqrt{\alpha}\theta, \alpha\right) - \frac{\theta^2}{2}\left(\frac{\alpha\gamma}{(1-m(-2\alpha))} - 1\right)$ and $\widetilde{\beta}_2 = \frac{\alpha^{3/2}(i\theta)^3\kappa_X^{(3)}}{3!(1-m(-3\alpha))}$ for $\delta > 0$ small enough so that

$$\begin{aligned} I_1 \leq & \frac{\varepsilon}{\pi}\kappa_1\sqrt{\alpha}\int_{-\infty}^{\infty}\theta^2 e^{-\theta^2/6}d\theta + \frac{\alpha}{\pi}K_1^2\int_{-\infty}^{\infty}e^{-\theta^2/6}\theta^6 d\theta + \\ & \frac{\sqrt{\alpha}}{\pi}o\left(1\right)\int_{-\infty}^{\infty}\left|\theta\right|^3 e^{-\theta^2/6}d\theta. \end{aligned}$$

Hence we conclude that

$$\limsup_{\alpha\to 0}\frac{1}{\sqrt{\alpha}}\sup_{x}\left|F_{\alpha}\left(x\right)-G\left(x\right)\right|\leq\varepsilon\kappa,$$

for some constant κ. Since ε was arbitrary, this concludes the proof of the theorem. ■

4.2 The continuous time setting

A popular model in the risk theory setting discussed in Section 2 consists of considering the processes Γ as Λ two independent Levy processes (i.e. two stationary independent increment processes, see Gjessing and Paulsen (1997)). The stationary independent increment assumption of the risk process Λ has been argued to hold by several authors in the risk theory community (this setting includes the so-called classical risk model, see Asmussen (2001) and Grandell (1991)). On the other hand, in finance, short rate processes usually are modelled as positive functions of a Markov process (typically with mean reverting characteristics). This motivates the following setting in which we develop the desired Edgeworth expansion.

Suppose that $\Lambda=(\Lambda\left(t\right):t\geq 0)$ is a Levy process. In addition, let $Y=(Y\left(s\right):s\geq 0)$ be a homogeneous Markov process taking values in a compact Polish space Ξ and let $\mathcal{B}\left(\Xi\right)$ be the Borel sigma-field in Ξ. Let $P\left(t,y,B\right)$ ($t\in\mathbb{R}_{+}$, $y\in\Xi$ and $B\in\mathcal{B}\left(\Xi\right)$) be the corresponding transition probability function. Assume that Y satisfies the Feller condition (i.e. $P\left(t,y,B_{\delta}\left(x\right)\right)\to 1$ as $t\searrow 0$, for all $\delta>0$) and that the mapping $y\to E_{y}f\left(Y\left(t\right)\right)$ is continuous for all $f\left(\cdot\right)\in C\left(\Xi\right)$ (the space of continuous function taking values on Ξ). Let A be the associated infinitesimal generator of the process Y, defined via the relation

$$Af\left(y\right)=\lim_{t\downarrow 0}\frac{E_{y}f\left(Y\left(t\right)\right)-f\left(y\right)}{t},$$

where $f\in C(\Xi)$. The domain $D\left(A\right)$ of A is composed by those functions $f\in C\left(\Xi\right)$ for which the previous limit exists (uniformly, for all $y\in\Xi$) (See Skorohod, Hoppensteadt and Salehi (2002)). In addition, suppose that $Y\left(\cdot\right)$ has right continuous with left limits sample paths and that it is geometrically ergodic (see Kontoyiannis and Meyn (2003), p. 9).

The following set of assumptions are in force throughout this section.

EC1 Λ and Γ are independent and the distribution of $\Lambda\left(1\right)$ is non-lattice with moments of order greater or equal than 3.

EC2 Suppose Y is geometrically ergodic (see Kontoyiannis and Meyn (2003) p. 9).

Suppose that $\widetilde{\gamma}\left(\cdot\right):\Xi\to\mathbb{R}$ is a continuous mapping such that $\widetilde{\gamma}\left(x\right)>0$ for all $x\in\Xi$ and define Γ as

$$\Gamma\left(t\right)=\int_{0}^{t}\widetilde{\gamma}\left(Y\left(s\right)\right)ds.$$

Under EC1 and EC2, we shall provide rigorous support for the approximation

$$\begin{aligned} P_{y_0}(D \le y) \;\approx\;& P\left(N\left(\lambda/\gamma, \chi^{(2)}(0)/2\right) \le y\right) - \sqrt{\gamma}\frac{\lambda}{\gamma}F(y_0)\,\eta\left((y-\lambda/\gamma)\sqrt{2/\chi^{(2)}(0)}\right) \\ & -\frac{\sqrt{\gamma}}{18}\chi^{(3)}(0)\,H\left((y-\lambda/\gamma)\sqrt{2/\chi^{(2)}(0)}\right), \end{aligned} \tag{27}$$

where (if $\pi(dy)$ denotes the stationary distribution of Y), F can be characterized as the solution of the Poisson equation

$$AF = E_\pi\gamma(Y(1)) - \gamma(y),$$

and $\chi(\cdot)$ depends on the log-moment generating function of Λ and the Perron-Frobenius eigenvalue associated with cumulative Markov reward Γ. More precisely, for every $\theta \in \mathbb{R}$ consider the (unique) solution pair $(u(y,\theta), \psi_\Gamma(\theta))$ (such that $u(y,0)=1$) satisfying

$$(Au)(y,\theta) = (\psi_\Gamma(\theta) - \theta\widetilde{\gamma}(y))\,u(y,\theta). \tag{28}$$

Note that the geometric ergodicity guarantees existence and uniqueness of the solution pair (u, ψ_Γ), see Kontoyiannis and Meyn (2003)). Let $\psi_\Lambda(i\theta) = \log(E\exp(i\theta\Lambda(1)))$ (we work with the branch $\{arg(z) \in [0,2\pi)\}$ when operating with complex logarithms) then $\chi(i\theta) = -\psi_\Gamma^{-1}(-\psi_\Lambda(i\theta))$ (note that $\chi'(0) = \lambda/\gamma$). Just as in the discrete time case, the approximation (27) will be supported in the context of small interest rates for a suitably parameterized family of discounted rewards. In particular, we shall prove that the approximation

$$\begin{aligned} & P\left(\sqrt{\alpha}\left(D(\alpha) - \lambda/(\gamma\alpha)\right) \le y\right) \\ \approx\;& P\left(N\left(0, \chi^{(2)}(0)/2\right) \le y\right) - \sqrt{\gamma\alpha}\frac{\lambda}{\gamma}F(y_0)\,\eta\left(y\sqrt{2/\chi^{(2)}(0)}\right) \\ & -\frac{\sqrt{\gamma\alpha}}{18}\chi^{(3)}(0)\,H\left(y\sqrt{2/\chi^{(2)}(0)}\right) \end{aligned}$$

holds with an error of order $o(\sqrt{\alpha})$ (uniformly on y), where

$$D(\alpha) = \int_0^\infty \exp(-\alpha\Gamma(t))\,d\Lambda(t).$$

(Note that the previous integral can be interpreted, via integration by parts, path by path as a Lebesgue-Stieltjes integral.)

Theorem 9 *Suppose that EC1 and EC2 hold. Then,*

$$\begin{aligned} & P\left(\sqrt{\alpha}\left(D(\alpha) - \chi'(0)/\alpha\right) \le y\right) \\ =\;& P\left(N\left(0, \chi^{(2)}(0)/2\right) \le y\right) - \sqrt{\gamma\alpha}F(y_0)\,\eta\left(y\sqrt{2/\chi^{(2)}(0)}\right) \\ & -\frac{\sqrt{\gamma\alpha}}{18}\chi^{(3)}(0)\,H\left(y\sqrt{2/\chi^{(2)}(0)}\right) + G_\alpha([-\infty, y)); \end{aligned}$$

where G_α represents a signed measure with $G_\alpha^+(\mathbb{R}) + G_\alpha^-(\mathbb{R}) \triangleq \|G_\alpha\| = o(\sqrt{\alpha})$.

The proof of the previous theorem parallels its corresponding continuous time analogue described in the previous section. We first obtain a local description of $\psi_\alpha(\theta) = \log E \exp(i\theta\sqrt{\alpha}(D(\alpha) - \lambda/(\gamma\alpha)))$.

Lemma 10 *Under assumptions EC1 and EC2 we have that*

$$\psi_\alpha(\theta) = -\frac{\chi^{(2)}(0)}{2}\theta^2 + \sqrt{\alpha}\left(\frac{\chi^{(3)}(0)}{18}(i\theta)^3 - \frac{\lambda}{\gamma}F(y_0)\, i\theta\right) + o\left(\sqrt{\alpha}\right)$$

(uniformly in $\theta \in (-\delta, \delta)$, $\delta > 0$).

Proof. It is known that for every $u \in D(A)$ such that $\inf_{x\in\Xi} |u(x)| > 0$ we have that

$$M_t(z) = \frac{u(Y(t), \theta)}{u(Y_0, \theta)} \exp\left(-\int_0^t \left(\frac{Au}{u}\right)(Y(s), \theta)\, ds\right) \quad (29)$$

is a Martingale with respect to the filtration generated by Y (see Lemma 2, p. 82 of Skorohod, Hoppensteadt and Salehi (2001)). Since Y is geometrically ergodic it follows that the generalized eigenvalue problem

$$(Au)(y, \theta) = (\psi_\Gamma(\theta) - \theta\widetilde{\gamma}(y))\, u(y, \theta), \quad u(y, 0) = 1 \quad (30)$$

has a unique solution pair $(u(y, \theta), \psi_\Gamma(\theta))$ for every $\theta \in \mathbb{R}$. In addition, $\inf_{\theta\in\Xi} u(y, \theta) > 0$ for all $\theta \in \mathbb{R}$ and $\psi_\Gamma(\cdot)$ is a strictly increasing function (since

$$\psi_\Gamma(\theta) = \lim_{t\to\infty} \frac{1}{t} \log E \exp(\theta\Gamma(t))).$$

Observe that the solution to (30) automatically provides the solution to the problem

$$\begin{aligned}
\frac{1}{\widetilde{\gamma}(y)}(Au)(y, \theta) &= \left(\frac{\psi_\Gamma(\theta)}{\widetilde{\gamma}(y)} - \theta\right) u(y, \theta) \\
&= \left(-\psi_\Gamma^{-1}(-\nu) - \frac{\nu}{\widetilde{\gamma}(y)}\right) u\left(y, \psi_\Gamma^{-1}(-\nu)\right),
\end{aligned}$$

(where $\nu = -\psi_\Gamma(\theta)$). In addition, Proposition 4.8 of Kontoyiannis and Meyn (2003) states that for each $\theta \in \Xi$, both $u(y, \cdot)$ and $\psi_\Gamma(\cdot)$ are analytic in $\mathcal{N} = \{z \in \mathbb{C} : |z| \le \delta\}$ for some $\delta > 0$ (which immediately implies the analyticity of $\zeta(\cdot) = -\psi_\Gamma^{-1}(-\cdot)$) and $\inf_{x\in\Xi, z\in\mathcal{N}} |u(x, z)| > 0$. Note that the Markov process $\widetilde{Y} = \left(\widetilde{Y}(t) : t \ge 0\right)$ defined as $\widetilde{Y}(t) = Y(\Gamma^{-1}(t))$ is also a geometrically ergodic Markov process with generator $\widetilde{A} = \frac{1}{\widetilde{\gamma}}A$ (the reason is that $\widetilde{\gamma}$ being continuous and positive implies $\inf_{x\in\Xi} \widetilde{\gamma}(x) > 0$, which yields that the Lyaponuv bound needed in the definition of geometric ergodicity is immediately satisfied after scaling factors (see Kontoyiannis and Meyn (2003)

p. 9). Therefore, by considering the Markov generator $\partial_t + \widetilde{A}$ and the function $u(y, \psi_\Lambda(i\theta e^{-\alpha t}))$, (for $\theta \in \mathbb{R}$ with $|\theta| < \delta$) in the relation (29) we can build the Martingales

$$\begin{aligned} M_t(i\theta) &= \frac{u\left(\widetilde{Y}(t), -\chi(i\theta e^{-\alpha t})\right)}{u(Y_0, -\chi(i\theta e^{-\alpha t}))} \exp\left(\int_0^t \frac{\psi_\Lambda(i\theta e^{-\alpha t})}{\widetilde{\gamma}\left(\widetilde{Y}(t)\right)} dt - \int_0^t \chi\left(i\theta e^{-\alpha t}\right) dt\right) \\ &\quad \exp\left(-\alpha \int_0^t i\theta e^{-\alpha t} \frac{u_\theta\left(\widetilde{Y}(t), -\chi(i\theta e^{-\alpha t})\right)}{u\left(\widetilde{Y}(t), -\chi(i\theta e^{-\alpha t})\right)} \dot{\chi}\left(i\theta e^{-\alpha t}\right) dt\right). \end{aligned}$$

Note that $M_t(i\theta)$ is a bounded martingale (in particular, uniformly integrable). Thus it possesses a last element $M_\infty(i\theta)$, which implies that

$$\exp\left(\int_0^\infty \chi\left(i\theta e^{-\alpha t}\right) dt\right) u(Y_0, i\theta) = E \exp\left(\int_0^\infty \frac{\psi_\Lambda(i\theta e^{-\alpha t})}{\widetilde{\gamma}\left(\widetilde{Y}(t)\right)} dt - \xi(\alpha, i\theta)\right),$$

where

$$\xi(\alpha, i\theta) = \alpha \int_0^t i\theta e^{-\alpha t} \frac{u_\theta\left(\widetilde{Y}(t), -\chi(i\theta e^{-\alpha t})\right)}{u\left(\widetilde{Y}(t), -\chi(i\theta e^{-\alpha t})\right)} \dot{\chi}\left(i\theta e^{-\alpha t}\right) dt.$$

Therefore, we conclude that

$$\begin{aligned} &\exp\left(\int_0^\infty \left(\chi\left(\sqrt{\alpha} i\theta e^{-\alpha t}\right) - \sqrt{\alpha} i\theta e^{-\alpha t} \lambda/\gamma\right) dt\right) u\left(Y_0, -\chi\left(\sqrt{\alpha} i\theta\right)\right) \\ &= E \exp\left(\int_0^\infty \frac{\psi_\Lambda(\sqrt{\alpha} i\theta e^{-\alpha t})}{\widetilde{\gamma}\left(\widetilde{Y}(t)\right)} dt - i\theta \frac{\lambda}{\gamma\sqrt{\alpha}} - \xi\left(\alpha, \sqrt{\alpha} i\theta\right)\right) \\ &= E \exp\left(\int_0^\infty \frac{\psi_\Lambda(\sqrt{\alpha} i\theta e^{-\alpha t})}{\widetilde{\gamma}\left(\widetilde{Y}(t)\right)} dt - i\theta \frac{\lambda}{\gamma\sqrt{\alpha}}\right) + o\left(\sqrt{\alpha}\right) \end{aligned} \tag{31}$$

(uniformly in $\theta \in (-\delta, \delta)$). The previous equality follows because

$$\begin{aligned} E\xi\left(\alpha, \sqrt{\alpha} i\theta\right) &= \sqrt{\alpha} i\theta\alpha E \int_0^\infty e^{-\alpha t} \frac{u_\theta\left(\widetilde{Y}(t), -\chi\left(\sqrt{\alpha} i\theta\right)\right)}{u\left(\widetilde{Y}(t), -\chi\left(\sqrt{\alpha} i\theta\right)\right)} \dot{\chi}\left(\sqrt{\alpha} i\theta e^{-\alpha t}\right) dt \\ &= \sqrt{\alpha} i\theta \frac{\lambda}{\gamma} E\alpha \int_0^\infty e^{-\alpha t} u_\theta\left(\widetilde{Y}(t), 0\right) dt + O(\alpha). \end{aligned}$$

Hence, using the bounded convergence theorem, we obtain

$$\begin{aligned} \alpha E \int_0^\infty e^{-\alpha t} u_\theta\left(\widetilde{Y}(t), 0\right) dt &= \int_0^\infty e^{-u} \alpha E \int_0^{u/\alpha} u_\theta\left(\widetilde{Y}(t), 0\right) dt du \\ &\rightharpoonup E u_\theta(Y(\infty), 0) = E_\pi F(Y(1)) = 0 \end{aligned}$$

(since $u_\theta(y,0) = F(y)$). The previous estimate combined with the asymptotic independence of $\xi(\alpha, \sqrt{\alpha} i\theta)$ and On the other hand, notice that

$$
\begin{aligned}
E\exp(i\theta D(\alpha)) &= E\left(E\left(\exp\left(i\theta\int_0^\infty \exp(-\alpha\Gamma(t))\,d\Lambda(t)\right)\middle|\,\Gamma\right)\right) \\
&= E\exp\left(\int_0^\infty \psi_\Lambda(i\theta\exp(-\alpha\Gamma(t)))\,dt\right) \\
&= E\exp\left(\int_0^\infty \frac{\psi_\Lambda(i\theta e^{-\alpha u})}{\widetilde{\gamma}\left(\widetilde{Y}(t)\right)}du\right). \qquad (32)
\end{aligned}
$$

Combining expressions (25) and (32) with a Taylor expansion of $\chi(\cdot)$ and $u(Y_0,\cdot)$ yields the conclusion of the Theorem. ■

The proof of Theorem 9 can be completed along the same lines as in the discrete time case after showing that $\phi(\theta,\alpha) \triangleq E\exp(i\theta D(\alpha))$ goes to zero fast enough for $|\theta| \in (w_0, w_1)$ for any $0 < w_0 < w_1 < \infty$.

Lemma 11 *Suppose that EC1 and EC2 are in force, then* $\phi(\theta,\alpha) \triangleq E\exp(i\theta D(\alpha))$ *satisfies*

$$
\sup_{|\theta|\in(\theta_0,\theta_1)} |\phi(\theta,\alpha)| = o\left(\sqrt{\alpha}\right),
$$

for all $0 < \theta_0 < \theta_1 < \infty$.

Proof. Note that

$$
\begin{aligned}
|\phi(\theta,\alpha)| &= \left|E\exp\left(\int_0^\infty \psi_\Lambda(i\theta\exp(-\alpha\Gamma(t)))\,dt\right)\right| \\
&\leq E\left|\exp\left(\int_0^{\Gamma^{-1}(1/\alpha)} \psi_\Lambda(i\theta\exp(-\alpha\Gamma(t)))\,dt\right)\right|.
\end{aligned}
$$

Define $\Delta(\theta) = \sup\{|\exp(\psi_\Lambda(\theta^*))| : |\theta^*| > |\theta e^{-1}|\}$. Since $\Lambda(1)$ is non-lattice, we have that $\Delta(\theta) \in (0,1)$. On the other hand, if $0 < b = \sup_{x\in\Xi}\widetilde{\gamma}(x) < \infty$, then $1/(\alpha b) \leq \Gamma^{-1}(1/\alpha)$. Hence, $|\phi(\theta,\alpha)| \leq \Delta(\theta)^{b/\alpha}$ and we actually obtain an exponential rate of convergence instead of the rate $o\left(\alpha^{1/2}\right)$ which is more than we need. ■

Remarks

a) The assumption that Ξ is compact does not really play an essential role. It was only used to ensure that the martingale property of $M_t(i\theta)$ in the proof of Lemma 10. A local description for $\psi_\alpha(i\theta)$ could also have been obtained by computing the moments of $D(\alpha)$, which is relatively easy in the present setting.

b) The independence between Γ and Λ can definitely be relaxed. For example, one could have assumed that both processes are conditionally independent given another

Markov process, say Z, provided that Λ remains a possibly non-time homogeneous Levy process with a non-lattice conditional distribution type assumption analogous to condition ED3 in the previous subsection.

c) Following the same ideas as in Lemma 10, a local expansion for $\psi_{\alpha}(\theta)$ can be obtained for the case in which

$$D(a) = \int_0^\infty \exp\left(-\alpha \int_0^t \widetilde{\gamma}(Y(s))\, ds\right) \widetilde{\lambda}(Y(s)).$$

(where $\widetilde{\lambda}$ is, say, continuous on the compact Polish space Ξ). In this case, the corresponding generalized eigenvalue problem takes the form

$$\frac{1}{\widetilde{\gamma}}(Au)(y,\theta) = \left(\chi(\theta) - \frac{\widetilde{\lambda}(y)}{\widetilde{\gamma}(y)}\right) u(y,\theta),\ u(y,0) = 1, \tag{33}$$

and a formal corrected approximation can be written as

$$\begin{aligned} P(D \le y) \;\approx\; & P\left(N\left(\lambda/\gamma, \chi^{(2)}(0)/2\right) \le y\right) - \sqrt{\gamma} u_\theta(y_0, 0)\, \eta\left((y - \lambda/\gamma)\sqrt{2/\chi^{(2)}(0)}\right) \\ & - \frac{\sqrt{\gamma}}{18}\chi^{(3)}(0)\, H\left((y - \lambda/\gamma)\sqrt{2/\chi^{(2)}(0)}\right). \end{aligned}$$

The only step (in addition to the existence of a solution to (33)) required to make the previous approximation rigorous is to show that for all $0 < \theta_0 < \theta_1 < \infty$, $\sup_{|\theta| \in (\theta_0,\theta_1)} |\phi(\theta,\alpha)| = o(\sqrt{\alpha})$ as in Lemma 11. This essentially involves assuming enough structure to ensure strongly non-lattice properties of D. We have chosen Levy processes in our exposition because they provide a convenient framework to easily verify, from the model primitives, the non-lattice conditions that yield the described Edgeworth expansions.

NOW THAT WE DIVIDED THE PAPER IN TWO PARTS I HAVE TO REMOVE SOME OF THE REFERENCES BELOW (I'LL TAKE CARE OF THAT).

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