

# Empirical Performance of Bias-Reducing Estimators for Regenerative Steady-State Simulations

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When simulating a stochastic system, simulationists often are interested in estimating various steady-state performance measures. The classical point estimator for such a measure involves simply taking the time average of an appropriate function of the process being simulated. Since the simulation can not be initiated with the (unknown) steady-state distribution, the classical point estimator is generally biased. In the context of regenerative steady-state simulation, a variety of other point estimators have been developed in an attempt to minimize the bias. In this paper, we provide an empirical comparison of these estimators in the context of four different continuous-time Markov chain models. The bias of the point estimators and the coverage probabilities of the associated confidence intervals are reported for the four models. Conclusions are drawn from this experimental work as to which methods are most effective in reducing bias.

Categories and Subject Descriptors: G.3 [**Probability and Statistics**]: *probabilistic algorithms*; I.6.1 [**Simulation and Modeling**]: Simulation Theory

General Terms: Algorithms, Performance, Theory

Additional Key Words and Phrases: Bias-reducing estimators, regeneration, simulation, steady-state estimation

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## 1. INTRODUCTION

Let  $Y = (Y(t) : t \geq 0)$  be a real-valued stochastic process representing the output of a simulation. Suppose that  $Y$  satisfies a law of large numbers (LLN)

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This research was supported by the Army Research Office under Contract no. DAAG55-97-1-0377 and the National Science Council, Taiwan, under Grant no. NSC 93-2213-E-004-011.

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of the form

$$\bar{Y}(t) \triangleq \frac{1}{t} \int_0^t Y(s) ds \Rightarrow \alpha \quad (1)$$

as  $t \rightarrow \infty$ , for some (deterministic) constant  $\alpha$ , where  $\Rightarrow$  denotes convergence in distribution. The steady-state simulation problem is concerned with the efficient estimation of the steady-state mean  $\alpha$ , and the construction of associated confidence intervals. Because steady-state performance measures are fundamental to the analysis of an enormous variety of stochastic systems arising in production, inventory, telecommunications, computing, and service industries, the steady-state simulation problem is of great practical importance.

One of the major difficulties associated with steady-state simulation is that the system is typically initialized in a state that is atypical of steady-state behavior. For example, a factory flow simulation is often initialized with no work-in-process on the factory floor. Such an initial condition gives rise to an “initial transient” during which the values of  $Y$  observed are not characteristic of steady-state. As a consequence, the obvious estimator of  $\alpha$  suggested by (1), namely the time-average  $\bar{Y}(t)$ , is biased as an estimator of  $\alpha$ . In other words,  $E\bar{Y}(t) \neq \alpha$ .

As might be expected, this bias problem is greatly magnified when parallel processors are used to accelerate the steady-state calculation. In particular, if independent replications of  $Y$  are simulated on each of the processors, the initial transient will be replicated on each processor, so that the presence of potentially massive parallelism then has no mitigating impact on the initial transient problem; see Glynn and Heidelberger [1991a, 1991b, 1992a] for details. In this article, we focus on the traditional computing environment, in which only a single processor is available.

In such a context, the bias in  $\bar{Y}(t)$  as an estimator of  $\alpha$  (also known as the “initial bias”) can be mitigated in two different ways. One approach is to delete that initial segment of the simulation that is “contaminated” by initial bias. Such an initial bias deletion approach has been studied by many authors; see, for example, Cash et al. [1992], Glynn [1995], Goldsman et al. [1994], Nelson [1992], Schruben [1982], Schruben et al. [1983], White [1997], and White et al. [2000]. An alternative is to consider an estimator, based on simulating  $Y$  over  $[0, t]$ , that attempts to compensate for the bias present in  $\bar{Y}(t)$ . We refer to such estimators as “bias reducing” estimators.

In this article, we provide an empirical investigation of six bias-reducing estimators that have been proposed in the context of processes  $Y$  that are regenerative. A great variety of stochastic systems exhibit regenerative structure, including discrete and continuous time Markov chains, as well as a broad class of discrete-event simulations; see Glynn [1989] for a discussion of the regenerative structure of discrete event simulations. Theoretically all “well-posed” steady-state simulation problem are regenerative, see Glynn [1994]. However, identifying the regeneration times can be difficult for a general discrete-event simulation, see Henderson and Glynn [2001]. Therefore, one should notice the limitation that the bias-reducing estimators we studied can only apply to simulations with identifiable regeneration times. This article may be viewed as

an update of an empirical study of Iglehart [1975] for regenerative processes, in which three of the six estimators considered here were investigated on a different time scale.

## 2. THEORETICAL BACKGROUND

Suppose that  $Y$  is the simulation output that is derived from the simulation of a stochastic system that can be modeled in terms of a Markov process  $X$ . In particular, suppose that  $Y(t) = f(X(t))$ , where  $X = (X(t) : t \geq 0)$  is a Markov process living on a state space  $S$ , and  $f : S \rightarrow \mathfrak{R}$  is a real-valued performance measure. Assuming that  $X$  exhibits positive recurrent behavior, it can be shown in substantial generality that

$$E\bar{Y}(t) = \alpha + \frac{b}{t} + o(\exp(-\beta t)) \quad (2)$$

as  $t \rightarrow \infty$ , for some constants  $b$  and  $\beta$  (where  $\beta > 0$ ), where  $o(a(t))$  denotes a function  $\psi(t)$  such that  $\psi(t)/a(t) \rightarrow 0$  as  $t \rightarrow \infty$ . See, for example, Glynn [1984], for such a result in the setting of finite-state continuous-time Markov chains (CTMC's).

Here, the time parameter  $t$  corresponds to simulated time. In other words,  $\bar{Y}(t)$  is obtained by averaging the values of  $Y(\cdot)$  associated with simulating  $X$  to the deterministic time horizon  $t$ . It is important to recognize that different bias expansions hold when the time horizon is specified differently. For example, when  $X$  is simulated to the completion of  $n$ th regenerative cycle at (simulated) time  $T(n)$ , one obtains a bias expansion of the form

$$E\bar{Y}(T(n)) = \alpha + \frac{\tilde{b}_1}{n} + \frac{\tilde{b}_2}{n^2} + o\left(\frac{1}{n^2}\right) \quad (3)$$

as  $n \rightarrow \infty$ , where  $\tilde{b}_2$  is generally nonzero. See Iglehart [1975] for a discussion of (3). Note that because  $\tilde{b}_2 \neq 0$ , the bias expansion (3) is qualitatively (and quantitatively) different from the expansion (2). The key point here is that bias expansions are heavily impacted by the time scale which one chooses to use for specifying the termination time for the simulation. In this article, we will focus exclusively on estimators obtained from a given simulated time horizon  $[0, t]$ , so that the relevant bias expansion is (2).

A corresponding expansion to (2), focused on the variance of  $\bar{Y}(t)$ , also exists in substantial generality:

$$\text{Var}\bar{Y}(t) = \frac{\sigma^2}{t} + \frac{c}{t^2} + o\left(\frac{1}{t^2}\right) \quad (4)$$

as  $t \rightarrow \infty$ ; see Glynn [1984] for details in the context of continuous-time Markov chains. It turns out that the time-average variance constant  $\sigma^2$  is unaffected by the choice of the initial condition for  $X$  (although  $c$  is impacted by the initialization). Thus, the mean square error (MSE) of  $\bar{Y}(t)$  can be easily computed via the formula

$$\text{MSE}(\bar{Y}(t)) = \text{Var}\bar{Y}(t) + (E\bar{Y}(t) - \alpha)^2 = \frac{\sigma^2}{t} + \frac{b^2 + c}{t^2} + o\left(\frac{1}{t^2}\right) \quad (5)$$

as  $t \rightarrow \infty$ . Note that the first-order term in the MSE expression, namely  $\sigma^2/t$ , is unaffected by the choice of the initial condition. Thus, the initial transient has only a second-order impact on the quality of the sample mean estimator  $\bar{Y}(t)$ . This suggests that use of a bias-reducing estimator, in preference to  $\bar{Y}(t)$ , will offer only second-order improvements to the quality of the simulation output procedure.

This brings us to the question of how to measure the quality of a particular steady-state estimator. Given an estimator  $\hat{\alpha}(t)$  for the steady-state mean  $\alpha$ , this paper will consider the two following measures:

- (1) mean absolute percentage error,

$$100 \cdot |E\hat{\alpha}(t) - \alpha|/|\alpha|;$$

- (2) median absolute percentage error,

$$100 \cdot |(\text{median of the distribution of } \hat{\alpha}(t)) - \alpha|/|\alpha|;$$

We will evaluate both of these quality measures via simulation. In particular, given a simulated time horizon  $t$  and an estimator  $\hat{\alpha}(t)$ , we perform  $m$  independent identically distributed (i.i.d.) replications of the random variable (r.v.)  $\hat{\alpha}(t)$ , thereby producing  $\hat{\alpha}_1(t), \dots, \hat{\alpha}_m(t)$ , and estimate (1) via

$$100 \cdot \frac{|\frac{1}{m} \sum_{i=1}^m \hat{\alpha}_i(t) - \alpha|}{|\alpha|}$$

and estimate (2) via

$$100 \cdot |(\text{sample median of } \{\hat{\alpha}_1(t), \dots, \hat{\alpha}_m(t)\}) - \alpha|/|\alpha|$$

In most carefully designed simulations, the simulator will wish to produce a confidence interval for  $\alpha$ , in addition to an estimator for  $\alpha$ . Approximately valid confidence intervals can easily be derived from the central limit theorem (CLT) that holds for all the bias reducing estimators  $\hat{\alpha}(t)$  considered in this paper, namely

$$\sqrt{t}(\hat{\alpha}(t) - \alpha) \Rightarrow \sigma N(0, 1) \quad (6)$$

as  $t \rightarrow \infty$ . The constant  $\sigma$  appearing here is precisely the time-average variance constant of expressions (4) and (5). As noted earlier,  $\sigma^2$  does not depend on the choice of the initial condition. Consequently, the quality of the confidence interval is affected by the initialization only in a second-order sense.

To gain an appreciation for the magnitude of this second-order effect, it is appropriate to consider the coverage error of the confidence interval. Specifically, an approximate  $100(1 - \delta)\%$  confidence interval for  $\alpha$  is characterized via a choice of a point estimator  $\hat{\alpha}(t)$  for  $\alpha$  and a time-average variance constant estimator  $\hat{v}(t)$  for  $\sigma^2$ . Given  $\hat{\alpha}(t)$  satisfying (6) and  $\hat{v}(t)$  satisfying  $\hat{v}(t) \Rightarrow \sigma^2$  as  $t \rightarrow \infty$ , a corresponding  $100(1 - \delta)\%$  confidence interval for  $\alpha$  is given by

$$\left[ \hat{\alpha}(t) - z \cdot \sqrt{\frac{\hat{v}(t)}{t}}, \hat{\alpha}(t) + z \cdot \sqrt{\frac{\hat{v}(t)}{t}} \right],$$

where  $z$  is chosen so that  $P(-z \leq N(0, 1) \leq z) = 1 - \delta$ . Denote the above random interval by  $[L(t), R(t)]$ , so that  $L(t)$  and  $R(t)$  are, respectively, the left and right end points of the confidence interval. The coverage error associated with the

interval for simulated time horizon  $t$  is then given by

$$|P(\alpha \in [L(t), R(t)]) - (1 - \delta)|.$$

The one-side coverage error (to the left) is  $|P(\alpha < L(t)) - \delta/2|$ , whereas the one-sided error (to the right) is  $|P(\alpha > R(t)) - \delta/2|$ .

In this article, we will numerically investigate the coverage associated with a particular choice of  $\hat{\alpha}(t)$  and  $\hat{\nu}(t)$ , in addition to studying the quality of the point estimator  $\hat{\alpha}(t)$  itself.

### 3. DESCRIPTION OF THE NUMERICAL EXPERIMENT

In this section, we will provide the details of our numerical experiment. We start by describing three continuous-time Markov chain (CTMC) queueing models and one CTMC inventory model that were used as test vehicles for our simulations. We choose these four models both because they are somewhat representative of real applications and because they are simple enough to permit us to compute exactly the theoretical values of certain parameters associated with the simulation.

*Model 1.* The first model considered is a single-server queueing model with a Poisson arrival process having an arrival rate of 0.95 customers per unit time, and exponential processing times with a mean of 1. Any customer that arrives to find at least 199 customers in the system balks and is permanently lost to the system. Thus, the number-in-system process  $X = (X(t) : t \geq 0)$  is a birth–death process on  $S = \{0, 1, \dots, 199\}$  (often denoted as M/M/1/199). In this model (as in all our others),  $X(t) \Rightarrow X(\infty)$  as  $t \rightarrow \infty$ . Our goal here is to compute  $\alpha = EX(\infty) = 18.993$ .

*Model 2.* The second model is a non-Markovian server analog of our first model. Here the arrival process is Poisson with arrival rate 0.95, and processing times are gamma distributed with parameters (2, 2). Any customer that arrives to find at least 199 customers in the system balks and is permanently lost to the system. The number-in-system process  $X = (X(t) : t \geq 0)$  is certainly not a CTMC. However, a gamma(2,2) random variable is equivalent to the sum of two independent exponential random variables with mean 0.5. If we use  $Z = (Z(t) : t \geq 0)$  to keep track of the current phase of the service time in process, i.e.  $Z(t) = 1$  if the server is consuming the first exponential random variable at time  $t$  and  $Z(t) = 2$  if the server is consuming the second one at time  $t$ . Let  $\vec{X}(t) = (X(t), Z(t))$ . Then,  $\vec{X} = (\vec{X}(t) : t \geq 0)$  is a CTMC. Our goal here is to compute  $\alpha = Ef(\vec{X}(\infty)) = 14.487$ , where  $f(\vec{X}(t)) = X(t)$ .

*Model 3.* This model is the queue-length process for two M/M/1/19 queues in tandem. Here the arrival process is Poisson with arrival rate 1. The arrived job is served at server 1 first. The service discipline is FCFS at all servers and all servers generate independent identically distributed (i.i.d.) exponentially distributed service times with mean  $1/\mu_i$ . When a job completes service at server 1, it goes to server 2. If it finds at least 19 customers in the second queue, it is permanently lost to the system. When a job completes service at server 2, it goes to server 1 with probability  $p$  and leaves the system with probability

$1 - p$ . The parameter values we have selected are  $\mu_1 = 2.3$ ,  $\mu_2 = 2.15$ , and  $p = 0.5$ . Let  $X_i(t)$ ,  $i = 1, 2$ , denote the number of jobs in server  $i$  at time  $t$ , and  $X(t) = (X_1(t), X_2(t))$ . Our performance measure is  $\alpha = Ef(X(\infty)) = 11.523$ , where  $f(X(t)) = X_1(t) + X_2(t)$ .

*Model 4.* This model is the number-in-stock process of a single item inventory management system. The system is operated under a stationary  $(s, S)$  policy. When the number-in-stock decreases to a fixed number  $s$ , an order amount of  $S - s$  is placed. The delivery times are i.i.d. exponentially distributed with a mean of  $4/3$ . The demands for the item occur according to a Poisson process with arrival rate 1. Demands that cannot be satisfied immediately are lost. We select  $S = 15$  and  $s = 2$ . Let  $X(t)$  denote the number-in-stock at time  $t$ . Our performance measure is the steady-state operating cost  $\alpha = Ef(X(\infty)) = 401.696$ , where

$$f(X(t)) = \begin{cases} 50X(t), & \text{if } X(t) > 0; \\ 300, & \text{if } X(t) = 0. \end{cases}$$

We turn next to a discussion of the specific estimators for  $\alpha$  that we shall consider in this article. The generic steady-state simulation problem that we are considering in this article involves the computation of the steady-state mean  $\alpha$  of a real-valued process  $Y = (Y(t) : t \geq 0)$ . Given  $Y$  and a simulated time horizon  $t$ , our first steady-state estimator is the “classical” (time-average) sample mean, namely

$$\alpha_1(t) = \frac{1}{t} \int_0^t Y(s) ds.$$

As mentioned in the Introduction, all our bias reducing estimators take advantage of the presence of regenerative structure. In particular, the times at which the underlying continuous-time Markov chain returns to any fixed state  $z \in \mathcal{S}$  is a sequence of regeneration times for  $Y$ . Let  $(T(n) : n \geq 0)$  be the sequence of times at which  $Y$  regenerates, and set

$$\begin{aligned} \Gamma_i &\triangleq \int_{T(i-1)}^{T(i)} Y(s) ds, \\ \tau_i &= T(i) - T(i-1), \\ N(t) &= \max\{n \geq 0 : T(n) \leq t\}, \end{aligned}$$

for  $i \geq 1$  and  $t \geq 0$ . A fundamental theoretical assumption for the validity of the bias reducing estimators that we shall study is that the regenerative process  $Y$  be non-delayed, so that a regeneration occurs at  $t = 0$  (and so that  $T(0) = 0$ ). Thus, once the initial state for the simulation is chosen, this necessarily determines the regenerative return state to be used.

Our first bias-reducing estimator is due to Meketon and Heidelberger [1982]. It is defined as

$$\alpha_2(t) = \frac{\sum_{i=1}^{N(t)+1} \Gamma_i}{\sum_{i=1}^{N(t)+1} \tau_i}.$$

They established theoretically that  $\alpha_2(t)$  does indeed reduce bias, as evidenced by the bias expansion

$$E\alpha_2(t) = \alpha + o(1/t) \quad (7)$$

as  $t \rightarrow \infty$ . (Compare with (2).) This bias expansion can be derived by using Taylor series argument, see Glynn and Heidelberger [1990] for such an argument. All the bias-reducing estimators that we shall discuss here have the same theoretical characteristic, namely, that their bias is of order  $o(1/t)$  as  $t \rightarrow \infty$ .

The second bias-reducing estimator is obtained by using Taylor expansions for the bias of a general function of sample means based on  $N(t)$  observations. It was introduced by Glynn and Heidelberger [1990] and is given by

$$\alpha_3(t) = \begin{cases} \alpha_1(t), & N(t) = 0; \\ \alpha_1(t) + \frac{1}{t^2} \left[ \sum_{i=1}^{N(t)} \Gamma_i \tau_i - \alpha_1(t) \sum_{i=1}^{N(t)} \tau_i^2 \right], & N(t) \geq 1. \end{cases}$$

See Glynn and Heidelberger [1990] for a discussion of its bias-reducing properties.

Our third bias-reducing estimator is based on jack-knifing and is given as

$$\alpha_4(t) = \begin{cases} \alpha_1(t), & N(t) \leq 1; \\ N(t) \left[ \frac{\sum_{i=1}^{N(t)} \Gamma_i}{\sum_{i=1}^{N(t)} \tau_i} \right] - \frac{N(t)-1}{N(t)} \left[ \sum_{i=1}^{N(t)} \frac{\sum_{j \neq i} \Gamma_j}{\sum_{j \neq i} \tau_j} \right], & N(t) \geq 2. \end{cases}$$

The bias-reducing theoretical behavior of  $\alpha_4(t)$  was established by Glynn and Heidelberger [1992b].

Our final bias-reducing estimator is based on a renewal-theoretical asymptotic expansion for the bias of the sample mean up to time  $t$  of a regenerative process. It was proposed in Glynn [1994] and is given by

$$\alpha_5(t) = \begin{cases} \alpha_1(t), & N(t) = 0; \\ \alpha_1(t) + \frac{1}{t^2} \sum_{i=1}^{N(t)} (A_i - \alpha_1(t) \tau_i^2 / 2), & N(t) \geq 1, \end{cases}$$

where

$$A_i = \int_0^{\tau_i} sY(T(i-1) + s) ds.$$

We also have an interest in studying the degree to which initial bias affects the quality of an estimator. To this end, consider the estimator

$$\alpha_6(t) = \alpha_1(t) + (\alpha - E\alpha_1(t)).$$

Note that  $\alpha_6(t)$  is unbiased as an estimator for  $\alpha$ . However,  $\alpha_6(t)$  can not be employed practically, because it requires the ability to pre-compute both the steady-state mean  $\alpha$  and the “transient” expectation  $E\alpha_1(t)$ . Of course, for the class of continuous-time Markov chain models we have selected, both  $\alpha$  and  $E\alpha_1(t)$  can be computed numerically. (The quantity  $\alpha$  can be found from the linear equations that characterize the steady-state of a continuous-time Markov chain, whereas  $E\alpha_1(t)$  can be computed, as we shall see next, by solving numerically a system of linear differential equations.) By empirically studying  $\alpha_6(t)$ , we can see the degree of improvement that is theoretically possible when the initialization bias is entirely eliminated.

Suppose that  $Y$  takes the form  $Y(t) = f(X(t))$ , where  $X = (X(t), t \geq 0)$  is a  $S$ -valued continuous-time Markov chain with finite state space and  $f : S \rightarrow \mathfrak{R}$  is a “performance measure” that we choose to encode as a column vector (i.e.,  $f = (f(x) : x \in S)$  is a column vector.) Let  $w(t, x) = tE_x\alpha_1(t) = E_x \int_0^t f(X(s)) ds$ , where  $E_x(\cdot)$  denotes the expectation operator under which  $X(0) = x$ . For  $h > 0$ , the Markov property implies that

$$\begin{aligned} w(t+h, x) &= E_x \int_0^{t+h} f(X(s)) ds + E_x w(t, X(h)) \\ &= f(x)h + o(h) + E_x w(t, X(h)). \end{aligned}$$

But  $E_x w(t, X(h)) = w(t, x) + (Aw(t))(x)h + o(h)$ , where  $A = (A(x, y) : x, y \in S)$  is the generator (or “rate matrix”) of  $X$  and  $w(t) = (w(t, x) : x \in S)$  is encoded as a column vector. Hence,

$$h^{-1}(w(t+h, x) - w(t, x)) = f(x) + (Aw(t))(x) + o(1) \quad (8)$$

as  $h \downarrow 0$ . So, letting  $h \downarrow 0$  in (8), we conclude that  $w'(t)$  exists and equals  $f + Aw(t)$ . So,  $E_x\alpha_1(t)$  can be computed by solving the initial value problem

$$\begin{aligned} w'(t) &= f + Aw(t) \\ \text{subject to } w(0) &= 0. \end{aligned}$$

This can be solved numerically by a finite difference method. An alternative is available when  $X$  is irreducible. Then,  $X$  possesses a unique stationary distribution  $\pi = (\pi(x) : x \in S)$ , which we encode as a row vector. If  $\Pi$  is the matrix in which each row equals  $\pi$ , it is known that  $(A - \Pi)^{-1}$  exists and

$$w(t) = (A - \Pi)^{-1}[(\exp(At) - I)(f - \Pi f)] + \Pi f t;$$

See Glynn [1994].

As indicated in Section 2, the quality of the above estimators needs to be assessed. Based on Section 2’s discussion, we will compute, for each combination of estimator and model, the mean absolute percentage error and the median absolute percentage error. This calculation will be based on 5000 independent replications of the model. In addition, our experiments are concerned with coverage error, and the degree to which coverage error is affected by initialization issues. Thus, we will also numerically investigate the coverage errors described in Section 2.

#### 4. NUMERICAL RESULTS

We have run simulations for a variety of cases by varying the time interval. Since we are interested in evaluating the performance of bias-reducing steady-state estimators, we selected, for each model, three combinations of initial state and simulated time that have non-trivial initial bias (1% to 25%), see Figures 1–4 for details. For each combination of simulated time and model, we run 5000 simulations. For each simulation run, we compute the estimators  $\alpha_i$ ,  $i = 1, \dots, 6$ , described in Section 3. In addition, we also run another set of simulations starting from steady-state. We set the resulting estimator  $\alpha_7(t)$ .



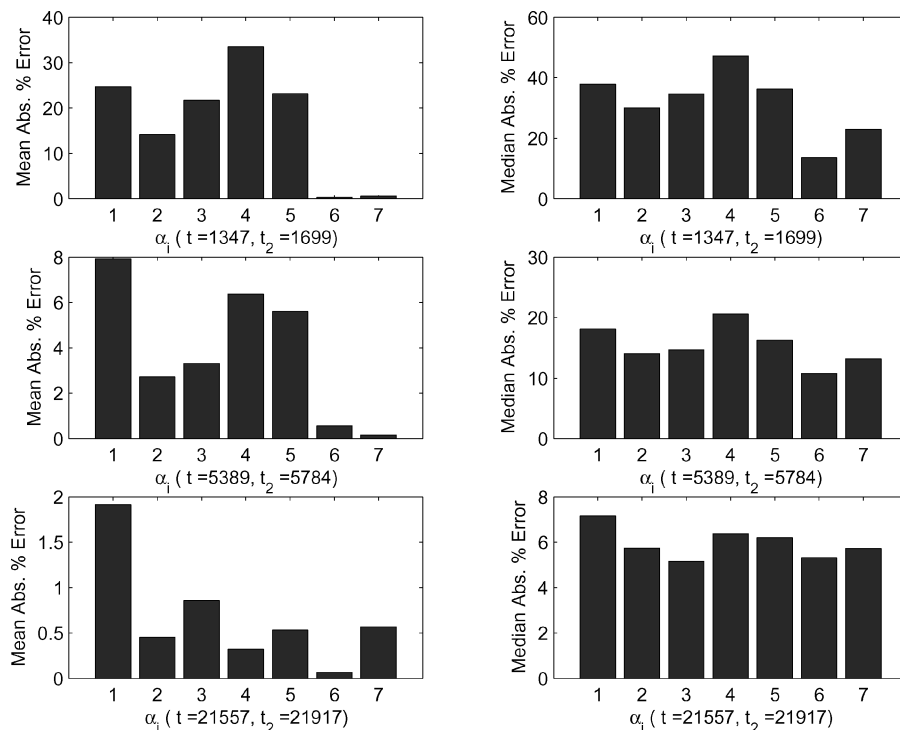


Fig. 1. Mean APE and median APE for various estimators in Model 1 with  $\lambda = .95$  and  $\mu = 1$  and initialized at  $X(0) = 0$ . Here,  $t$  is the simulated time horizon (chosen to be  $64E\tau_1$ ,  $256E\tau_1$ , and  $1024E\tau_1$ ) for estimators  $\alpha_1(t)$  and estimators  $\alpha_3(t)$  to  $\alpha_7(t)$ . For  $\alpha_2(t)$ , the simulated time  $t_2$  is the average (over 5000 replications) of the time required to complete the cycle in progress at  $t$ .

Recall that  $\alpha_1(t)$  is the classical point estimator which having “high-bias” (bias of order  $1/t$ , see Eq. (2));  $\alpha_2(t)$  to  $\alpha_5(t)$  are “low-bias” estimators (i.e., bias of order  $o(1/t)$ , see Eq. (7)); and  $\alpha_6(t)$  and  $\alpha_7(t)$  are unbiased estimators. The estimators  $\alpha_2(t)$  to  $\alpha_6(t)$  can be regarded as “bias-reducing” estimators and  $\alpha_6(t)$  is the “best” one. Initial transient deletion methods aim to collect sample paths that are approximately in steady-state. Viewed in this light,  $\alpha_7(t)$  and other initial transient deletion methods can be regarded as “approximated steady-state” estimators and  $\alpha_7(t)$  is the “best” one. Therefore, we may consider that we have done comparisons between bias-reducing estimators and a “perfect” initial transient deletion estimator (i.e.,  $\alpha_7(t)$ ).

As mentioned in Section 2, we are interested in mean absolute percentage error (APE) for these estimators. In Figures 1–4 we displayed the simulation results for mean APE. The results are as expected: compared to the classical estimator  $\alpha_1(t)$ , the low-bias estimators generally reduced the bias. The only exception is  $\alpha_4(t)$  which has more bias than  $\alpha_1(t)$  for some cases in Model 1 to 3. This exception theoretically can relate to the fact that Eq. (7) (theoretical property of the low-bias estimators we considered) implies that the low-bias estimators must reduce bias when the chosen  $t$  is sufficiently large. Empirically, this exception can be explained as follows:  $\alpha_4(t)$  only uses simulated information

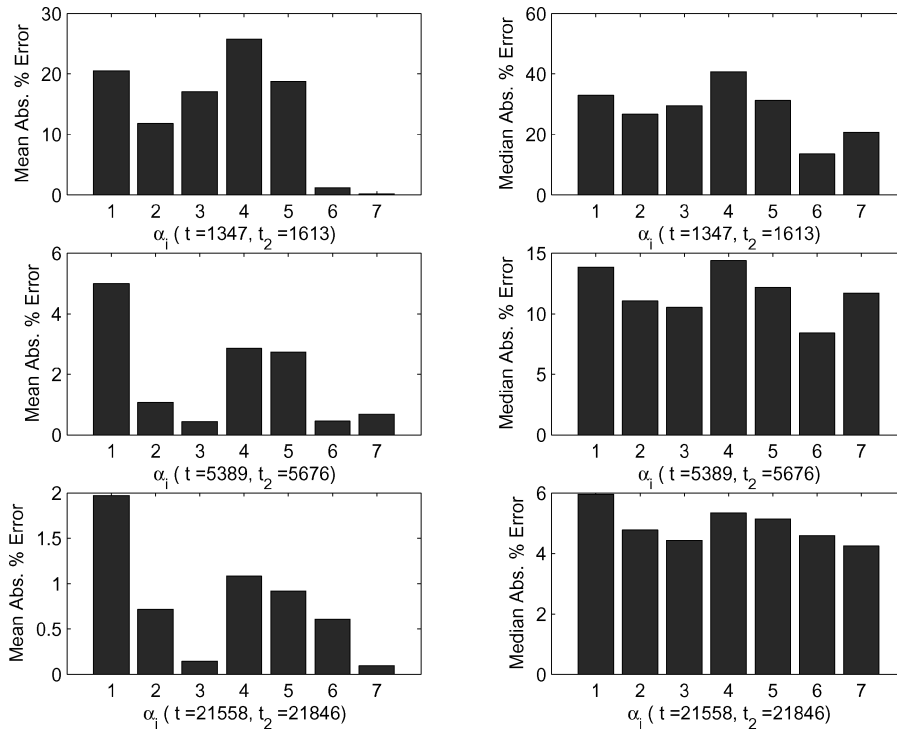


Fig. 2. Mean APE and median APE for various estimators in Model 2 with  $\lambda = .95$  and  $\text{gamma}(2,2)$  distributed processing times and initialized at  $X(0) = 0$ . Here,  $t$  is the simulated time horizon (chosen to be  $64E\tau_1$ ,  $256E\tau_1$ , and  $1024E\tau_1$  for estimators  $\alpha_1(t)$  and estimators  $\alpha_3(t)$  to  $\alpha_7(t)$ ). For  $\alpha_2(t)$ , the simulated time  $t_2$  is the average (over 5000 replications) of the time required to complete the cycle in progress at  $t$ .

up to  $T(N(t))$  which is significant shorter than  $t$  for the exceptional cases, for example,  $(ET(N(t)), t) \approx (990, 1347)$  for the exceptional case in Model 1 (see Figure 1). In short, the empirical results indicate that the low-bias estimators are generally effective in reducing mean APE.

We are also interested in the bias-reducing effect on median APE of these estimators. The experimental results are reported in Figures 1–4. From our empirical study, we found that the bias-reducing effect on median APE of these estimators is model dependent. For Models 1 and 2, the low-bias estimators have limited use in reducing median APE. This is because the distribution of  $\alpha_1(t)$  of these two models is highly skewed, and thus the magnitude of median APE of  $\alpha_1(t)$  is much larger than that of mean APE of  $\alpha_1(t)$ . Note that even the unbiased estimators have the same limitation for these two models. For Models 3 and 4, the low-bias estimators are effective in reducing median APE, since the distributions of  $\alpha_1(t)$  of these two models are more symmetric, and the magnitude of median APE of  $\alpha_1(t)$  and mean APE of  $\alpha_1(t)$  are about the same. In brief, the empirical results indicate that the low-bias estimators can effectively reduce median APE if the distribution of  $\alpha_1(t)$  is near symmetric and is less effective if the distribution of  $\alpha_1(t)$  is skewed.

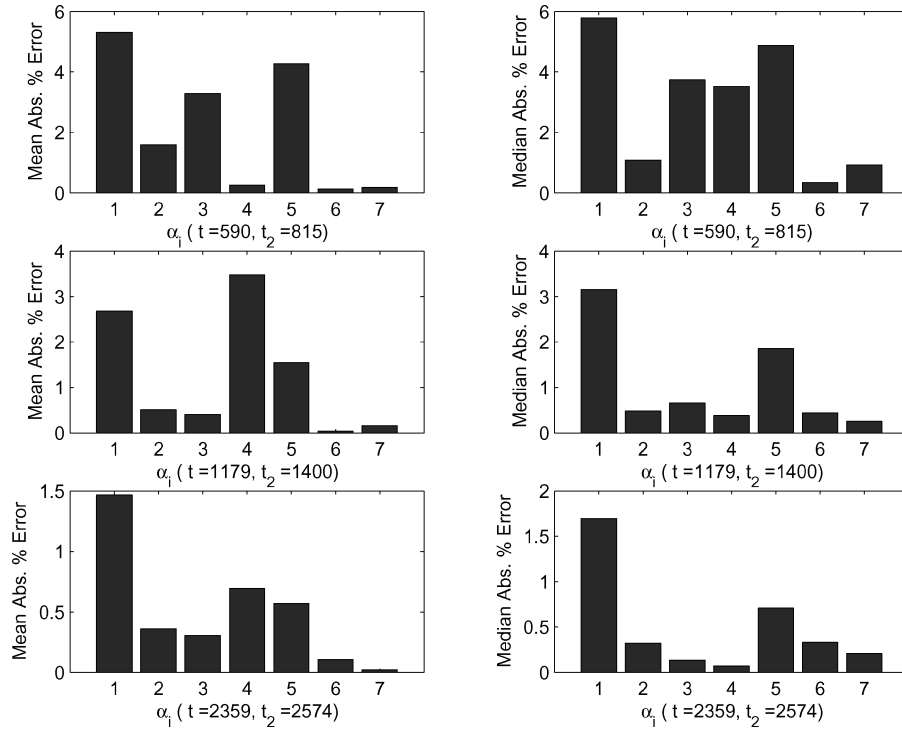


Fig. 3. Mean APE and Median APE for Various Estimators in Model 3 with  $\lambda = 1$ ,  $\mu_1 = 2.3$ ,  $\mu_2 = 2.15$ , and  $p = 0.5$  and initialized at  $X(0) = (0, 0)$ . Here,  $t$  is the simulated time horizon (chosen to be  $8E\tau_1$ ,  $16E\tau_1$ , and  $32E\tau_1$ ) for estimators  $\alpha_1(t)$  and estimators  $\alpha_3(t)$  to  $\alpha_7(t)$ . For  $\alpha_2(t)$ , the simulated time  $t_2$  is the average (over 5000 replications) of the time required to complete the cycle in progress at  $t$ .

We turn next to the coverage issues of the confidence interval for  $\alpha$ . Before proceeding to the experimental results, let us introduce the time-average variance constant (TAVC) estimators we used.

The first TAVC estimator  $\sigma^2$  is its theoretical value. We use it as a performance benchmark.

The second TAVC estimator  $t \cdot \widehat{\text{Var}}(\alpha_i(t))$  is based on Eq. (4). Here,  $\widehat{\text{Var}}(\alpha_i(t))$  is the sample variance of  $\alpha_i(t)$ . That is,

$$\widehat{\text{Var}}(\alpha_i(t)) = \frac{1}{4999} \sum_{k=1}^{5000} \alpha_i^{(k)}(t), \quad \text{for } i = 1, \dots, 7,$$

where  $\alpha_i^{(k)}(t)$  is the  $k$ th replication of  $\alpha_i(t)$ .

The third TAVC estimator  $v_i(t)$  is defined as

$$v_i(t) = \begin{cases} 0, & N(t) = 0, 1; \\ \frac{\sum_{j=1}^{N(t)} (\Gamma_j - \alpha_i(t)\tau_j)^2}{\sum_{j=1}^{N(t)} \tau_j}, & N(t) \geq 2. \end{cases}$$

for  $i = 1, \dots, 6$ . It is the classical TAVC estimator for the regenerative method.

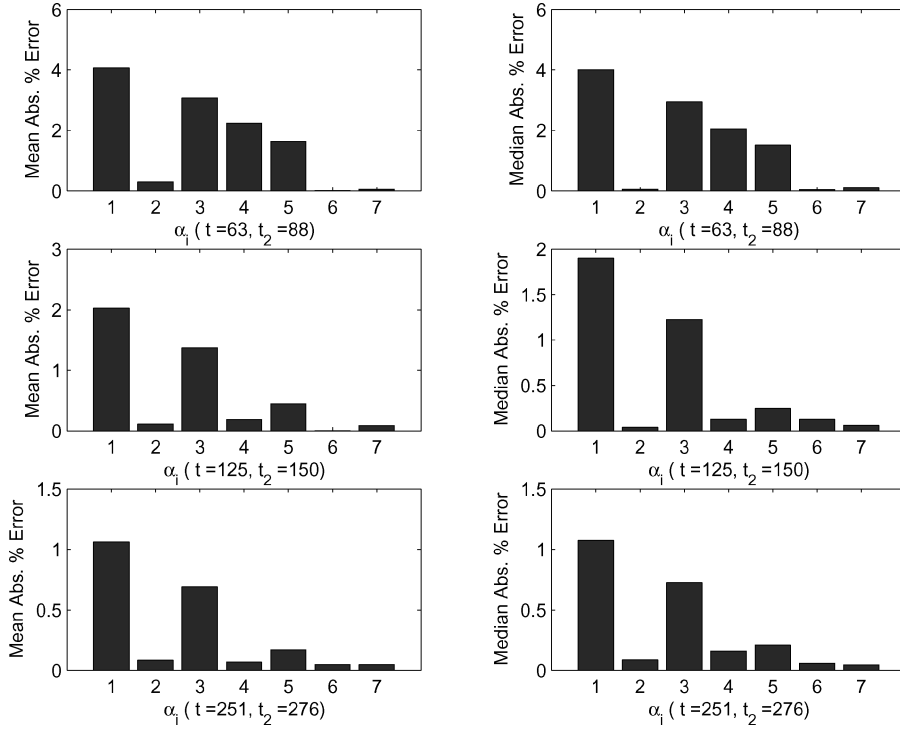


Fig. 4. Mean APE and median APE for various estimators in Model 4 with  $s = 2$  and  $S = 15$  and initialized in State  $S = 15$ . Here,  $t$  is the simulated time horizon (chosen to be  $2E\tau_1$ ,  $4E\tau_1$ , and  $8E\tau_1$ ) for estimators  $\alpha_1(t)$  and estimators  $\alpha_3(t)$  to  $\alpha_7(t)$ . For  $\alpha_2(t)$ , the simulated time  $t_2$  is the average (over 5000 replications) of the time required to complete the cycle in progress at  $t$ .

The last TAVC estimator  $t \cdot v_J(t)/N(t)$  is based on jackknife theory and  $v_J(t)$  is defined as

$$v_J(t) = \begin{cases} 0, & N(t) = 0, 1; \\ \frac{1}{N(t)-1} \sum_{j=1}^{N(t)} (\tilde{\alpha}_j(t) - \alpha_4(t))^2, & N(t) \geq 2, \end{cases}$$

where

$$\tilde{\alpha}_j(t) = N(t) \left[ \frac{\sum_{i=1}^{N(t)} \Gamma_i}{\sum_{i=1}^{N(t)} \tau_i} \right] - (N(t) - 1) \left[ \frac{\sum_{i \neq j} \Gamma_i}{\sum_{i \neq j} \tau_i} \right].$$

It is noteworthy that only the third and fourth TAVC estimators can be obtained from a single replication. Using these four TAVC estimators, we constructed four different  $100(1 - \delta)\%$  confidence intervals of  $\alpha$ . Namely,

$$\left[ \alpha_i(t) - z \frac{\sigma}{\sqrt{t}}, \alpha_i(t) + z \frac{\sigma}{\sqrt{t}} \right], \quad (9)$$

$$\left[ \alpha_i(t) - z \sqrt{\widehat{\text{Var}}(\alpha_i(t))}, \alpha_i(t) + z \sqrt{\widehat{\text{Var}}(\alpha_i(t))} \right], \quad (10)$$

$$\left[ \alpha_i(t) - z \sqrt{\frac{v_i(t)}{t}}, \alpha_i(t) + z \sqrt{\frac{v_i(t)}{t}} \right], \quad (11)$$

and

$$\left[ \alpha_i(t) - z \sqrt{\frac{v_J(t)}{N(t)}}, \alpha_i(t) + z \sqrt{\frac{v_J(t)}{N(t)}} \right], \quad (12)$$

where  $z$  is chosen so that  $P(-z \leq N(0, 1) \leq z) = 1 - \delta$ . Let  $[L(t), R(t)]$  denote any above random interval. To access the coverage quality of confidence intervals, we computed the following measures:

$$m = \hat{P}(\alpha \in [L(t), R(t)]) = \frac{1}{5000} \sum_{k=1}^{5000} I(\alpha \in [L^{(k)}(t), R^{(k)}(t)]),$$

$$l = \hat{P}(\alpha < L(t)) = \frac{1}{5000} \sum_{k=1}^{5000} I(\alpha < L^{(k)}(t)),$$

and

$$r = \hat{P}(\alpha > R(t)) = \frac{1}{5000} \sum_{k=1}^{5000} I(\alpha > R^{(k)}(t)),$$

where  $I(\cdot)$  is an indicator function and  $L^{(k)}(t)$  and  $R^{(k)}(t)$  are independent copies of  $L(t)$  and  $R(t)$ . We choose  $\delta = 0.1$ , thus the ideal value of  $(l, m, r)$  is  $(5\%, 90\%, 5\%)$ . Note that the coverage errors described in Section 2 are the absolute difference between  $(l, m, r)$  and  $(5\%, 90\%, 5\%)$ . The experimental results of these measures for various confidence intervals are reported in Tables I–IV. A few interesting things were observed as follows:

- (1) The coverage quality of confidence intervals is dominated by the quality of TAVC estimator. For example, in Model 1 and 2, confidence intervals formed by using  $\alpha_1(t)$  and  $v_i(t)$  have very bad coverage quality. Replacing  $\alpha_1(t)$  by unbiased estimators improves the cover quality only a bit. However, replacing  $v_i(t)$  by  $\sigma^2$  has dramatic improvement. This is as expected, since bias has only a second-order effect on the quality of confidence intervals, see Eq. (5).
- (2) The distribution of  $\alpha_1(t)$  plays an important role in coverage quality. This can be seen from the following observations. Confidence intervals constructed from unbiased estimators and  $\sigma^2$  are expected to have  $(l, m, r)$  close to  $(5\%, 90\%, 5\%)$ . This is observed in Models 3 and 4, but not for Models 1 and 2. The problem in Models 1 and 2 is the highly skewed distribution of  $\alpha_1(t)$ . The skewed distribution of  $\alpha_1(t)$  also leads to a high median APE.
- (3) Confidence intervals constructed from  $\widehat{Var}(\alpha_i)$  have comparable coverage quality of those formed by using  $\sigma^2$ .

Table I.  $l$ ,  $m$ , and  $r$  for Various Confidence Intervals in Model 1 (Displayed in Cells Below TAVC Estimator.)

Estimator	$\bar{t}$	$\sigma^2$			$t \cdot \widehat{\text{Var}}(\alpha_i)$		
$\alpha_1(t)$	1347.3	0.00%	99.50% $\pm$ 0.16%	0.50%	1.44%	94.62% $\pm$ 0.52%	3.94%
$\alpha_2(t)$	1699.4	0.00%	98.98% $\pm$ 0.23%	1.02%	0.00%	94.28% $\pm$ 0.54%	5.72%
$\alpha_3(t)$	1347.3	0.00%	99.52% $\pm$ 0.16%	0.48%	0.98%	95.08% $\pm$ 0.50%	3.94%
$\alpha_4(t)$	1347.3	0.00%	99.54% $\pm$ 0.16%	0.46%	4.72%	91.28% $\pm$ 0.66%	4.00%
$\alpha_5(t)$	1347.3	0.00%	99.52% $\pm$ 0.16%	0.48%	1.26%	94.86% $\pm$ 0.51%	3.88%
$\alpha_6(t)$	1347.3	0.00%	99.12% $\pm$ 0.22%	0.88%	0.00%	93.38% $\pm$ 0.58%	6.62%
$\alpha_7(t)$	1347.3	0.00%	96.34% $\pm$ 0.44%	3.66%	0.00%	93.76% $\pm$ 0.56%	6.24%
$\alpha_1(t)$	5389.3	0.00%	96.94% $\pm$ 0.40%	3.06%	0.02%	94.98% $\pm$ 0.51%	5.00%
$\alpha_2(t)$	5784.1	0.00%	95.64% $\pm$ 0.48%	4.36%	0.00%	94.16% $\pm$ 0.55%	5.84%
$\alpha_3(t)$	5389.3	0.00%	96.00% $\pm$ 0.46%	4.00%	0.00%	94.16% $\pm$ 0.55%	5.84%
$\alpha_4(t)$	5389.3	0.00%	95.72% $\pm$ 0.47%	4.28%	0.02%	95.44% $\pm$ 0.49%	4.54%
$\alpha_5(t)$	5389.3	0.00%	96.48% $\pm$ 0.43%	3.52%	0.00%	94.60% $\pm$ 0.53%	5.40%
$\alpha_6(t)$	5389.3	0.00%	96.50% $\pm$ 0.43%	3.50%	0.00%	93.90% $\pm$ 0.56%	6.10%
$\alpha_7(t)$	5389.3	0.00%	94.88% $\pm$ 0.51%	5.12%	0.00%	93.84% $\pm$ 0.56%	6.16%
$\alpha_1(t)$	21557.1	0.16%	94.52% $\pm$ 0.53%	5.32%	0.36%	93.76% $\pm$ 0.56%	5.88%
$\alpha_2(t)$	21917.3	0.12%	93.92% $\pm$ 0.56%	5.96%	0.20%	93.56% $\pm$ 0.57%	6.24%
$\alpha_3(t)$	21557.1	0.10%	93.00% $\pm$ 0.59%	6.90%	0.02%	93.68% $\pm$ 0.57%	6.30%
$\alpha_4(t)$	21557.1	0.16%	93.22% $\pm$ 0.58%	6.62%	0.02%	94.04% $\pm$ 0.55%	5.94%
$\alpha_5(t)$	21557.1	0.10%	93.96% $\pm$ 0.55%	5.94%	0.16%	93.80% $\pm$ 0.56%	6.04%
$\alpha_6(t)$	21557.1	0.02%	94.20% $\pm$ 0.54%	5.78%	0.24%	93.50% $\pm$ 0.57%	6.26%
$\alpha_7(t)$	21557.1	0.10%	94.20% $\pm$ 0.54%	5.70%	0.22%	93.68% $\pm$ 0.57%	6.10%
Estimator	$\bar{t}$	$v_i(t)$			$t \cdot v_j(t)/N(t)$		
$\alpha_1(t)$	1347.3	65.16%	25.46% $\pm$ 1.01%	9.38%	54.34%	36.88% $\pm$ 1.12%	8.78%
$\alpha_2(t)$	1699.4	57.80%	27.84% $\pm$ 1.04%	14.36%	47.40%	38.38% $\pm$ 1.13%	14.22%
$\alpha_3(t)$	1347.3	64.44%	24.14% $\pm$ 1.00%	11.42%	52.86%	38.60% $\pm$ 1.13%	8.54%
$\alpha_4(t)$	1347.3	76.68%	14.46% $\pm$ 0.82%	8.86%	65.38%	34.04% $\pm$ 1.10%	0.58%
$\alpha_5(t)$	1347.3	64.86%	24.98% $\pm$ 1.01%	10.16%	53.64%	37.70% $\pm$ 1.13%	8.66%
$\alpha_6(t)$	1347.3	47.98%	33.68% $\pm$ 1.10%	18.34%	35.60%	51.94% $\pm$ 1.16%	12.46%
$\alpha_1(t)$	5389.3	43.48%	47.54% $\pm$ 1.16%	8.98%	37.46%	58.54% $\pm$ 1.15%	4.00%
$\alpha_2(t)$	5784.1	39.76%	47.06% $\pm$ 1.16%	13.18%	33.88%	58.88% $\pm$ 1.14%	7.24%
$\alpha_3(t)$	5389.3	41.54%	45.50% $\pm$ 1.16%	12.96%	35.82%	60.16% $\pm$ 1.14%	4.02%
$\alpha_4(t)$	5389.3	47.40%	41.46% $\pm$ 1.15%	11.14%	41.40%	57.70% $\pm$ 1.15%	0.90%
$\alpha_5(t)$	5389.3	42.54%	46.62% $\pm$ 1.16%	10.84%	36.82%	59.18% $\pm$ 1.14%	4.00%
$\alpha_6(t)$	5389.3	37.92%	49.82% $\pm$ 1.16%	12.26%	31.00%	64.22% $\pm$ 1.12%	4.78%
$\alpha_1(t)$	21557.1	26.66%	70.22% $\pm$ 1.06%	3.12%	24.50%	73.84% $\pm$ 1.02%	1.66%
$\alpha_2(t)$	21917.3	25.28%	70.00% $\pm$ 1.07%	4.72%	22.96%	74.52% $\pm$ 1.01%	2.52%
$\alpha_3(t)$	21557.1	25.26%	69.26% $\pm$ 1.07%	5.48%	23.08%	74.74% $\pm$ 1.01%	2.18%
$\alpha_4(t)$	21557.1	27.04%	68.40% $\pm$ 1.08%	4.56%	24.80%	73.90% $\pm$ 1.02%	1.30%
$\alpha_5(t)$	21557.1	26.00%	69.98% $\pm$ 1.07%	4.02%	23.76%	74.28% $\pm$ 1.02%	1.96%
$\alpha_6(t)$	21557.1	24.38%	71.82% $\pm$ 1.05%	3.80%	22.34%	75.74% $\pm$ 1.00%	1.92%

Note that  $m$  is reported in terms of a 95% confidence interval and  $\bar{t}$  is the average simulated time for each steady-state mean estimator.

- (4) Confidence intervals constructed from  $v_j(t)$  have consistently better coverage quality than those constructed from  $v_i(t)$ .
- (5) Confidence intervals constructed from low-bias estimators have similar coverage quality to those constructed from unbiased estimators, and have better coverage quality to those constructed from  $\alpha_1(t)$ .

Table II.  $l$ ,  $m$ , and  $r$  for Various Confidence Intervals in Model 2 (Displayed in Cells Below TAVC Estimator.)

Estimator	$\bar{t}$	$\sigma^2$			$t \cdot \widehat{\text{Var}}(\alpha_i)$		
$\alpha_1(t)$	1347.4	0.00%	99.34% ± 0.19%	0.66%	0.52%	94.70% ± 0.52%	4.78%
$\alpha_2(t)$	1613.4	0.00%	98.78% ± 0.26%	1.22%	0.00%	94.08% ± 0.55%	5.92%
$\alpha_3(t)$	1347.4	0.00%	99.30% ± 0.19%	0.70%	0.18%	94.70% ± 0.52%	5.12%
$\alpha_4(t)$	1347.4	0.00%	98.28% ± 0.30%	1.72%	0.86%	94.80% ± 0.52%	4.34%
$\alpha_5(t)$	1347.4	0.00%	99.30% ± 0.19%	0.70%	0.28%	94.70% ± 0.52%	5.02%
$\alpha_6(t)$	1347.4	0.00%	98.96% ± 0.24%	1.04%	0.00%	93.06% ± 0.59%	6.94%
$\alpha_7(t)$	1347.4	0.00%	95.96% ± 0.46%	4.04%	0.00%	93.86% ± 0.56%	6.14%
$\alpha_1(t)$	5389.5	0.00%	95.94% ± 0.46%	4.06%	0.00%	94.24% ± 0.54%	5.76%
$\alpha_2(t)$	5676.1	0.00%	94.88% ± 0.51%	5.12%	0.00%	93.72% ± 0.56%	6.28%
$\alpha_3(t)$	5389.5	0.00%	94.32% ± 0.54%	5.68%	0.00%	93.30% ± 0.58%	6.70%
$\alpha_4(t)$	5389.5	0.02%	94.10% ± 0.55%	5.88%	0.02%	94.24% ± 0.54%	5.74%
$\alpha_5(t)$	5389.5	0.00%	95.12% ± 0.50%	4.88%	0.00%	93.72% ± 0.56%	6.28%
$\alpha_6(t)$	5389.5	0.00%	95.30% ± 0.49%	4.70%	0.00%	93.32% ± 0.58%	6.68%
$\alpha_7(t)$	5389.5	0.00%	94.98% ± 0.51%	5.02%	0.00%	94.74% ± 0.52%	5.26%
$\alpha_1(t)$	21557.9	0.38%	94.86% ± 0.51%	4.76%	0.98%	93.56% ± 0.57%	5.46%
$\alpha_2(t)$	21845.6	0.34%	94.20% ± 0.54%	5.46%	0.48%	93.70% ± 0.57%	5.82%
$\alpha_3(t)$	21557.9	0.32%	93.50% ± 0.57%	6.18%	0.26%	93.70% ± 0.57%	6.04%
$\alpha_4(t)$	21557.9	0.36%	94.00% ± 0.55%	5.64%	0.26%	94.34% ± 0.54%	5.40%
$\alpha_5(t)$	21557.9	0.36%	94.22% ± 0.54%	5.42%	0.40%	93.90% ± 0.56%	5.70%
$\alpha_6(t)$	21557.9	0.20%	94.62% ± 0.52%	5.18%	0.50%	93.60% ± 0.57%	5.90%
$\alpha_7(t)$	21557.9	0.38%	93.44% ± 0.58%	6.18%	0.50%	93.08% ± 0.59%	6.42%
Estimator	$\bar{t}$	$v_i(t)$			$t \cdot v_J(t)/N(t)$		
$\alpha_1(t)$	1347.4	60.70%	29.28% ± 1.06%	10.02%	51.38%	41.12% ± 1.14%	7.50%
$\alpha_2(t)$	1613.4	53.86%	30.80% ± 1.07%	15.34%	45.12%	42.52% ± 1.15%	12.36%
$\alpha_3(t)$	1347.4	59.50%	27.58% ± 1.04%	12.92%	49.54%	43.18% ± 1.15%	7.28%
$\alpha_4(t)$	1347.4	70.36%	19.28% ± 0.92%	10.36%	59.86%	39.82% ± 1.14%	0.32%
$\alpha_5(t)$	1347.4	60.24%	28.40% ± 1.05%	11.36%	50.48%	42.16% ± 1.15%	7.36%
$\alpha_6(t)$	1347.4	47.38%	35.06% ± 1.11%	17.56%	34.94%	54.62% ± 1.16%	10.44%
$\alpha_1(t)$	5389.5	37.24%	54.92% ± 1.16%	7.84%	33.22%	63.64% ± 1.12%	3.14%
$\alpha_2(t)$	5676.1	34.36%	54.20% ± 1.16%	11.44%	30.50%	63.92% ± 1.12%	5.58%
$\alpha_3(t)$	5389.5	36.04%	51.80% ± 1.16%	12.16%	31.28%	65.16% ± 1.11%	3.56%
$\alpha_4(t)$	5389.5	40.50%	49.16% ± 1.16%	10.34%	35.56%	63.22% ± 1.12%	1.22%
$\alpha_5(t)$	5389.5	36.64%	53.62% ± 1.16%	9.74%	32.34%	64.36% ± 1.11%	3.30%
$\alpha_6(t)$	5389.5	33.62%	56.40% ± 1.15%	9.98%	28.24%	68.06% ± 1.08%	3.70%
$\alpha_1(t)$	21557.9	24.90%	72.18% ± 1.04%	2.92%	23.06%	75.32% ± 1.00%	1.62%
$\alpha_2(t)$	21845.6	23.68%	71.92% ± 1.05%	4.40%	21.68%	75.82% ± 1.00%	2.50%
$\alpha_3(t)$	21557.9	23.72%	71.80% ± 1.05%	4.48%	21.64%	76.20% ± 0.99%	2.16%
$\alpha_4(t)$	21557.9	25.30%	71.58% ± 1.05%	3.12%	23.28%	75.74% ± 1.00%	0.98%
$\alpha_5(t)$	21557.9	24.26%	72.10% ± 1.04%	3.64%	22.38%	75.70% ± 1.00%	1.92%
$\alpha_6(t)$	21557.9	23.30%	73.32% ± 1.03%	3.38%	20.96%	77.12% ± 0.98%	1.92%

Note that  $m$  is reported in terms of a 95% confidence interval and  $\bar{t}$  is the average simulated time for each steady-state mean estimator.

### 5. CONCLUSIONS

We have studied seven estimation methods for four stochastic models. For the four low-bias estimators we have studied, we found

- (1) In terms of mean APE:  $\alpha_2(t)$  and  $\alpha_3(t)$  have superior performance to others. The performance of  $\alpha_5(t)$  is generally better than that of  $\alpha_1(t)$ , but on average

Table III.  $l$ ,  $m$ , and  $r$  for Various Confidence Intervals in Model 3 (Displayed in Cells Below TAVC Estimator.)

Estimator	$\bar{t}$	$\sigma^2$			$t \cdot \widehat{\text{Var}}(\alpha_i)$		
$\alpha_1(t)$	589.7	6.46%	90.80% ± 0.67%	2.74%	8.22%	88.30% ± 0.75%	3.48%
$\alpha_2(t)$	814.7	3.16%	94.84% ± 0.51%	2.00%	6.56%	89.60% ± 0.71%	3.84%
$\alpha_3(t)$	589.7	5.16%	91.58% ± 0.65%	3.26%	6.50%	89.44% ± 0.72%	4.06%
$\alpha_4(t)$	589.7	19.90%	60.78% ± 1.14%	19.32%	4.16%	88.66% ± 0.74%	7.18%
$\alpha_5(t)$	589.7	5.66%	91.38% ± 0.65%	2.96%	7.40%	88.76% ± 0.73%	3.84%
$\alpha_6(t)$	589.7	3.06%	92.02% ± 0.63%	4.92%	4.12%	89.94% ± 0.70%	5.94%
$\alpha_7(t)$	589.7	3.46%	91.42% ± 0.65%	5.12%	4.22%	90.08% ± 0.70%	5.70%
$\alpha_1(t)$	1179.3	5.50%	91.62% ± 0.64%	2.88%	7.02%	89.26% ± 0.72%	3.72%
$\alpha_2(t)$	1399.6	3.54%	93.56% ± 0.57%	2.90%	5.42%	90.08% ± 0.70%	4.50%
$\alpha_3(t)$	1179.3	4.26%	91.34% ± 0.65%	4.40%	4.94%	89.96% ± 0.70%	5.10%
$\alpha_4(t)$	1179.3	9.42%	75.88% ± 1.00%	14.70%	1.40%	92.24% ± 0.62%	6.36%
$\alpha_5(t)$	1179.3	4.90%	91.54% ± 0.65%	3.56%	5.86%	89.90% ± 0.70%	4.24%
$\alpha_6(t)$	1179.3	3.56%	92.04% ± 0.63%	4.40%	4.58%	89.98% ± 0.70%	5.44%
$\alpha_7(t)$	1179.3	4.22%	90.44% ± 0.68%	5.34%	4.38%	90.08% ± 0.70%	5.54%
$\alpha_1(t)$	2358.7	5.62%	90.92% ± 0.67%	3.46%	6.14%	90.18% ± 0.69%	3.68%
$\alpha_2(t)$	2573.9	4.10%	92.30% ± 0.62%	3.60%	5.36%	90.40% ± 0.69%	4.24%
$\alpha_3(t)$	2358.7	4.24%	90.52% ± 0.68%	5.24%	4.22%	90.62% ± 0.68%	5.16%
$\alpha_4(t)$	2358.7	6.00%	85.74% ± 0.81%	8.26%	2.92%	91.70% ± 0.64%	5.38%
$\alpha_5(t)$	2358.7	4.96%	90.80% ± 0.67%	4.24%	5.10%	90.52% ± 0.68%	4.38%
$\alpha_6(t)$	2358.7	4.30%	91.10% ± 0.66%	4.60%	4.58%	90.46% ± 0.68%	4.96%
$\alpha_7(t)$	2358.7	4.08%	90.86% ± 0.67%	5.06%	4.64%	89.86% ± 0.70%	5.50%
Estimator	$\bar{t}$	$v_i(t)$			$t \cdot v_j(t)/N(t)$		
$\alpha_1(t)$	589.7	29.86%	55.82% ± 1.16%	14.32%	16.08%	74.58% ± 1.01%	9.34%
$\alpha_2(t)$	814.7	21.12%	64.14% ± 1.12%	14.74%	11.18%	78.46% ± 0.96%	10.36%
$\alpha_3(t)$	589.7	29.72%	52.14% ± 1.16%	18.14%	15.70%	74.96% ± 1.01%	9.34%
$\alpha_4(t)$	589.7	39.30%	38.62% ± 1.13%	22.08%	26.84%	67.02% ± 1.09%	6.14%
$\alpha_5(t)$	589.7	29.70%	54.26% ± 1.16%	16.04%	15.86%	74.82% ± 1.01%	9.32%
$\alpha_6(t)$	589.7	23.36%	57.00% ± 1.15%	19.64%	11.80%	77.08% ± 0.98%	11.12%
$\alpha_1(t)$	1179.3	18.62%	70.72% ± 1.06%	10.66%	9.98%	87.06% ± 0.78%	2.96%
$\alpha_2(t)$	1399.6	14.24%	73.70% ± 1.02%	12.06%	7.22%	89.60% ± 0.71%	3.18%
$\alpha_3(t)$	1179.3	17.22%	67.90% ± 1.09%	14.88%	9.00%	87.76% ± 0.76%	3.24%
$\alpha_4(t)$	1179.3	22.82%	53.78% ± 1.16%	23.40%	12.90%	83.42% ± 0.87%	3.68%
$\alpha_5(t)$	1179.3	18.04%	69.92% ± 1.07%	12.04%	9.48%	87.44% ± 0.77%	3.08%
$\alpha_6(t)$	1179.3	15.24%	70.20% ± 1.06%	14.56%	7.64%	88.48% ± 0.74%	3.88%
$\alpha_1(t)$	2358.7	13.58%	78.40% ± 0.96%	8.02%	8.44%	88.62% ± 0.74%	2.94%
$\alpha_2(t)$	2573.9	11.18%	79.56% ± 0.94%	9.26%	6.94%	89.80% ± 0.70%	3.26%
$\alpha_3(t)$	2358.7	11.80%	76.18% ± 0.99%	12.02%	7.08%	89.36% ± 0.72%	3.56%
$\alpha_4(t)$	2358.7	14.50%	69.90% ± 1.07%	15.60%	8.50%	87.22% ± 0.78%	4.28%
$\alpha_5(t)$	2358.7	12.66%	77.76% ± 0.97%	9.58%	7.64%	89.08% ± 0.73%	3.28%
$\alpha_6(t)$	2358.7	11.48%	77.84% ± 0.97%	10.68%	6.80%	89.56% ± 0.71%	3.64%

Note that  $m$  is reported in terms of a 95% confidence interval and  $\bar{t}$  is the average simulated time for each steady-state mean estimator.

is not as good as that of  $\alpha_2(t)$  and  $\alpha_3(t)$ . The estimator  $\alpha_4(t)$  is unstable for a few simulation runs because it only uses sample information up to time  $T(N(t))$  which can be significant shorter than  $t$ . To sum up,  $\alpha_2(t)$  and  $\alpha_3(t)$  are the best choices.

- (2) In terms of median APE: the performance ranking of low-bias estimators is about the same of that in terms of mean APE. That is,  $\alpha_2(t)$  and  $\alpha_3(t)$  are



Table IV.  $l$ ,  $m$ , and  $r$  for Various Confidence Intervals in Model 4 (Displayed in Cells Below TAVC Estimator.)

Estimator	$\bar{t}$	$\sigma^2$			$t \cdot \widehat{\text{Var}}(\alpha_i)$		
$\alpha_1(t)$	62.7	1.08%	83.96% ± 0.85%	14.96%	1.06%	84.12% ± 0.85%	14.82%
$\alpha_2(t)$	87.9	1.84%	94.44% ± 0.53%	3.72%	3.60%	90.10% ± 0.69%	6.30%
$\alpha_3(t)$	62.7	1.68%	86.00% ± 0.81%	12.32%	1.56%	86.66% ± 0.79%	11.78%
$\alpha_4(t)$	62.7	3.02%	85.70% ± 0.81%	11.28%	2.06%	89.32% ± 0.72%	8.62%
$\alpha_5(t)$	62.7	2.36%	90.18% ± 0.69%	7.46%	2.48%	89.54% ± 0.71%	7.98%
$\alpha_6(t)$	62.7	4.88%	89.86% ± 0.70%	5.26%	4.76%	90.08% ± 0.70%	5.16%
$\alpha_7(t)$	62.7	5.22%	88.94% ± 0.73%	5.84%	4.56%	90.16% ± 0.69%	5.28%
$\alpha_1(t)$	125.4	1.70%	86.66% ± 0.79%	11.64%	1.70%	86.66% ± 0.79%	11.64%
$\alpha_2(t)$	150.1	2.88%	92.62% ± 0.61%	4.50%	4.50%	89.38% ± 0.72%	6.12%
$\alpha_3(t)$	125.4	2.62%	87.50% ± 0.77%	9.88%	2.50%	88.12% ± 0.75%	9.38%
$\alpha_4(t)$	125.4	7.62%	83.60% ± 0.86%	8.78%	4.16%	90.94% ± 0.67%	4.90%
$\alpha_5(t)$	125.4	3.74%	89.94% ± 0.70%	6.32%	3.86%	89.44% ± 0.72%	6.70%
$\alpha_6(t)$	125.4	4.74%	89.78% ± 0.70%	5.48%	4.74%	89.80% ± 0.70%	5.46%
$\alpha_7(t)$	125.4	4.86%	89.38% ± 0.72%	5.76%	4.40%	90.24% ± 0.69%	5.36%
$\alpha_1(t)$	250.8	2.64%	88.60% ± 0.74%	8.76%	2.64%	88.54% ± 0.74%	8.82%
$\alpha_2(t)$	276.1	4.00%	91.46% ± 0.65%	4.54%	4.78%	89.80% ± 0.70%	5.42%
$\alpha_3(t)$	250.8	3.68%	89.00% ± 0.73%	7.32%	3.38%	89.60% ± 0.71%	7.02%
$\alpha_4(t)$	250.8	6.74%	86.34% ± 0.80%	6.92%	4.88%	90.14% ± 0.69%	4.98%
$\alpha_5(t)$	250.8	4.70%	89.78% ± 0.70%	5.52%	4.70%	89.78% ± 0.70%	5.52%
$\alpha_6(t)$	250.8	4.98%	89.86% ± 0.70%	5.16%	5.00%	89.82% ± 0.70%	5.18%
$\alpha_7(t)$	250.8	5.02%	89.76% ± 0.71%	5.22%	4.88%	90.08% ± 0.70%	5.04%
Estimator	$\bar{t}$	$v_i(t)$			$t \cdot v_j(t)/N(t)$		
$\alpha_1(t)$	62.7	19.02%	34.40% ± 1.11%	46.58%	19.18%	37.00% ± 1.12%	43.82%
$\alpha_2(t)$	87.9	30.52%	44.18% ± 1.16%	25.30%	30.36%	45.60% ± 1.16%	24.04%
$\alpha_3(t)$	62.7	22.80%	33.26% ± 1.10%	43.94%	22.00%	38.86% ± 1.13%	39.14%
$\alpha_4(t)$	62.7	26.70%	27.92% ± 1.04%	45.38%	22.18%	39.24% ± 1.14%	38.58%
$\alpha_5(t)$	62.7	26.78%	37.04% ± 1.12%	36.18%	25.40%	42.30% ± 1.15%	32.30%
$\alpha_6(t)$	62.7	33.24%	40.02% ± 1.14%	26.74%	31.98%	43.54% ± 1.15%	24.48%
$\alpha_1(t)$	125.4	9.74%	68.38% ± 1.08%	21.88%	8.10%	73.68% ± 1.02%	18.22%
$\alpha_2(t)$	150.1	14.72%	73.54% ± 1.03%	11.74%	11.74%	78.94% ± 0.95%	9.32%
$\alpha_3(t)$	125.4	12.82%	66.26% ± 1.10%	20.92%	9.36%	74.48% ± 1.01%	16.16%
$\alpha_4(t)$	125.4	21.74%	57.22% ± 1.15%	21.04%	12.26%	73.50% ± 1.03%	14.24%
$\alpha_5(t)$	125.4	16.00%	67.96% ± 1.09%	16.04%	11.46%	76.62% ± 0.98%	11.92%
$\alpha_6(t)$	125.4	17.50%	68.52% ± 1.08%	13.98%	12.76%	77.36% ± 0.97%	9.88%
$\alpha_1(t)$	250.8	6.86%	78.70% ± 0.95%	14.44%	4.86%	83.56% ± 0.86%	11.58%
$\alpha_2(t)$	276.1	10.08%	81.32% ± 0.91%	8.60%	6.80%	86.78% ± 0.79%	6.42%
$\alpha_3(t)$	250.8	8.82%	77.64% ± 0.97%	13.54%	5.34%	84.34% ± 0.85%	10.32%
$\alpha_4(t)$	250.8	13.58%	74.04% ± 1.02%	12.38%	8.16%	83.46% ± 0.86%	8.38%
$\alpha_5(t)$	250.8	11.12%	78.56% ± 0.95%	10.32%	7.28%	85.38% ± 0.82%	7.34%
$\alpha_6(t)$	250.8	11.62%	78.84% ± 0.95%	9.54%	7.72%	85.62% ± 0.82%	6.66%

Note that  $m$  is reported in terms of a 95% confidence interval and  $\bar{t}$  is the average simulated time for each steady-state mean estimator.

also the best choices. However, for some models, the improvement can be quite limited.

- (3) In term of computational effort:  $\alpha_2(t)$  has the drawback of using longer simulated time, since it require  $(N(t) + 1)$ th regeneration cycle to complete. One might think the extra simulated time is negligible, but it is not true for some situations. For example, the extra simulated time  $T(N(t) + 1) - t \approx 15E\tau_1$

in Model 1. The estimator  $\alpha_5(t)$  has the drawback of needing extra computer memory for storing more detailed sample path information for computing  $A_i$ . The work for computing  $\alpha_3(t)$  and  $\alpha_4(t)$  is negligible, since the needed path information is also needed by TAVC estimators and thus the extra work is fairly minimal.

For the TAVC estimators, the jackknife variance estimator  $v_J(t)$  generally gives more accurate coverage than the classical variance estimator  $v_i(t)$ ; and both estimators use about the same amount of computer time (which is negligible). However, it must be noted that  $v_J(t)$  and  $v_i(t)$  both underestimated the TAVC and thus produced confidence intervals that under covered for the short simulation runs (unlike the second TAVC estimator which does quite well but requires much more computation due to replications).

From what has been said above, we may conclude that the confidence interval constructed from  $\alpha_3(t)$  and  $v_J(t)$  is our recommendation for estimating the steady-state mean  $\alpha$  in a regenerative steady-state simulation if the simulated time is sufficiently long. In addition, simulationists may wish to use  $\alpha_3(t)$  in place of  $\alpha_1(t)$ , since from the experimental results,  $\alpha_3(t)$  is consistently better than  $\alpha_1(t)$  in terms of mean APE and median APE, and the extra computational work in computing  $\alpha_3(t)$  is negligible. Although it may be risky to extrapolate the recommendations from experimental results of these four models to general discrete-event simulations, we feel these experimental experiences will be useful to other simulationists.

#### ACKNOWLEDGMENTS

We wish to thank the editors and the referees for suggestions that improved the exposition.

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Received January 2003; accepted January 2004