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Independent sampling of a stochastic process

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Abstract

We investigate the question of when sampling a stochastic process $X=\{X(t):\ t\geqslant 0\}$ at the times of an independent point process ψ leads to the same empirical distribution as the time-average limiting distribution of X. Two main cases are considered. The first is when X is asymptotically stationary and ergodic, and ψ satisfies a mixing condition. In this case, the path-wise limiting distributions in function space are shown to be the same. The second main case is when X is only assumed to have a constant finite time average and ψ is assumed a positive recurrent renewal processes with a spread-out cycle length distribution. In this latter case, the averages are shown to be the same when some further conditions are placed on X and ψ.

Keywords: Time average; Event average; Independent sampling; Asymptotically stationary ergodic

1. Introduction

Quite a bit of literature has been devoted to establishing suitable conditions under which the time-average distribution of a stochastic process $X=\{X(t):\ t\geqslant 0\}$ is the same as that obtained when averaging over the sampling times of an underlying point process $\psi=\{t_n:\ n\geqslant 0\}$.

The main emphasis in the literature has been on the case when X and ψ are dependent; for example, when t_n is the arrival time of the nth customer to a queueing system (with which each arrival interacts), and $X(t)$ is the state of the system at time t. Formally, this amounts to showing that for all measurable sets A in the state space of X

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t I(X(s)\in A)\,\mathrm{d}s=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} I(X(t_n-)\in A),\tag{1.1}$$

where $I(X(s)\in A)$ denotes the indicator function for the event $\{X(s)\in A\}$, and $X(t_n-)$ is then interpreted as the state right before the nth arrival. The classic and fundamental

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result in this regard is that Poisson arrivals see time averages (PASTA) (Wolff, 1982) which states that under a so-called *lack of anticipation property*, sampling by a Poisson process yields Eq. (1.1). In such cases, path regularity assumptions (such as left or right continuity) are placed on X because $X(t_n-)$ need not be equal to $X(t_n+)$.

Many papers generalizing, extending and giving converses to PASTA have appeared in recent years, giving rise to the general notion of arrivals see time averages (ASTA) (Miyazawa and Wolff, 1990; Melamed and Whitt, 1990a; Melamed and Whitt, 1990b; Green and Melamed, 1990; Wolff, 1990; König and Schmidt, 1989; Brémaud et al., 1992). Nevertheless, it seems that perhaps the most fundamental case has not been seriously studied: the case when X and ψ are independent (but not necessarily stationary).

Results of the form (1.1) are, of course, relevant to statisticians who wish to estimate the time average of a process by sampling it at a sequence of random epochs (Masry and Lui, 1976; Masry, 1988; Lii and Masry, 1994). One can find applications of this kind in mathematical finance (see Duffie, 1996 for example). But such results are also pertinent to some of the recently developed theory of Meyn and Tweedie (1993). This theory takes advantage of the smoothing properties of a sampled Markov process to develop theoretical insights into their associated recurrence structure. The analysis of Markov processes using such techniques goes back at least as far as Azèma et al. (1967), and was exploited in a more recent paper of Sigman (1990).

To illustrate some of the subtleties that can arise with sampling in this independent setting, we present the following examples of processes for which the sample average does not converge to the time average.

1.1. Examples

1. *Sampling with a renewal process having a non-lattice cycle-length distribution, F*: Define a cyclic deterministic process $X(t)=t-n$; $n\leqslant t<n+1$, (the fractional part of t). Let A denote the set of irrational numbers in $(0,1)$ and take F to have mass only on the rationals (with positive mass on each rational). We take a non-delayed version of the renewal process so that each arrival time t_n is rational. Then the time average of $I\{X(s)\in A\}$ is 1 but the sampled average of $I\{X(t_n-)\in A\}$ is 0.
2. *Sampling with a stationary renewal process*: Taking a stationary version of a renewal process does not generally help the problem. Break up $\mathbb{R}_+$ into the odd half-intervals $[n,n+\frac{1}{2})$, and even half-intervals $[n+\frac{1}{2},n+1]$, $n\geqslant 0$, and define $X(t)$ to be 1 on the even ones and 0 on the odd ones so that the time average of $\{X(s)=1\}$ is 0.5 a.s. For sampling, take a deterministic renewal process with interevent times identically 1, but let t_1 have a Unif$(0,1)$ distribution (this results in making the renewal process time stationary). Then with probability 0.5, the event $\{t_1\leqslant 0.5\}$ will occur in which case $X(t_n)=0$ for all n, yielding the sampled average as 0. Even if X is replaced by a stationary version of itself, the same problem occurs, and in this case it is important to observe that whereas both X and ψ are each ergodic, they are not so jointly, and this is partly what causes the problem.

In this paper, we study this sampling problem and (partially) fill in this theoretical gap. We assume very little in terms of path regularity for X nor do we necessarily assume that X or ψ are stationary processes. We consider two set-ups. In the first

(Section 2) X is assumed time asymptotically stationary ergodic (TAS-E) and ψ is assumed mixing. We show that the sampled process has the same limiting distribution (Theorem 2.1). The distributions we deal with are those in function space (2.1) and (2.6), not just the *marginal* distribution as in Eq. (1.1). (When starting from the sampled average, we present a similar result but require much weaker conditions (Theorem 3.1).) As a corollary (Corollary 2.1), it is seen that if X is TAS-E and ψ is a positive recurrent renewal process with a spread-out cycle length distribution then the result holds. In Theorem 2.2 we place the mixing condition on X instead of on ψ, and allow ψ to have a stationary sequence of interarrival times (this case allows for sampling by a null recurrent renewal process, for example).

In the second set-up (Section 4) we are no longer interested in equating *distributions* as in Section 2 but only the average of a real-valued process

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t X(s)\,\mathrm{d}s=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}X(t_n). \tag{1.2}$$

For example, X could be of the form $X(t)=I\{Y(t)\in A\}$ where Y has a general state space and A is a fixed set of states. Or X could be a deterministic real-valued function $X(t)=x(t)$. Therefore, in this case, no further assumptions are placed on X other than that it has a finite constant Césaro limit. We assume, though, that the point process is renewal with a spead-out cycle length distribution (Theorem 4.1).

Our main results begin with Section 2.2, with the earlier subsections dealing largely with preliminary material on asymptotic stationarity (the details of which can be found in the book (Sigman, 1995)).

2. The asymptotically ergodic case

2.1. Preliminaries

Let $X=\{X(t)\colon t\geqslant 0\}$ be a stochastic process on some underlying probability space $(\Omega,\mathscr{B},P)$, with $X(t)$ taking values in the *state space* $\boldsymbol{S}$ (a measurable space endowed with σ-field $\mathscr{F}$). We view X as a random element of a subspace $\boldsymbol{X}$ of the function space $\boldsymbol{S}^{\mathbb{R}_+}$. $\boldsymbol{X}$ is assumed closed under the shift operator: $\theta_s x=\{x(s+t)\colon t\geqslant 0\}\in\boldsymbol{X}$, $x\in\boldsymbol{X}$, $s\geqslant 0$. We also assume that $\boldsymbol{X}$ is *shift measurable*: the mapping $(t,x)\to\theta_t x$ from $\mathbb{R}_+\times\boldsymbol{X}$ to $\boldsymbol{X}$ is measurable (see Sverchkov, 1993; Thorisson, 1992 for details). (If, for example, $\boldsymbol{S}$ is a complete separable metric space and $\boldsymbol{X}$ is the space of functions that are right continuous, then this is satisfied.)

We wish to *sample* X at the times of an independent point process $\psi=\{t_n\colon n\geqslant 0\}$ on $\mathbb{R}_+=[0,\infty)$. To start with, we assume that X and ψ are on the same probability space, and that ψ is *simple*, that is, that the t_n are strictly increasing (to infinity) as $n\to\infty$.

We view ψ as a random point measure on $\mathbb{R}_+$, where for any Borel set $A\subset\mathbb{R}_+$,

$$\psi(A)=\sum_{n=0}^{\infty}I\{t_n\in A\}.$$

We let $\psi(t)\stackrel{\text{def}}{=}\psi((0,t])$ denote the associated counting process.

$\boldsymbol{M}$ denotes the space of all point measures, $\mu=\mu(\cdot)$, that are bounded on compact inervals, equipped with the $\hat{w}$ topology (and associated Borel sets) defined via: $\mu_k\to\mu$ as $n\to\infty$ iff for each bounded continuous function $f:\mathbb{R}_+\to\mathbb{R}$ with compact support, $\mu_k(f)\to\mu(f)$. This makes $\boldsymbol{M}$ into a complete separable metric space (consult, for example, p. 628 of Daley and Vere-Jones, 1988). ψ is assumed a random element of $\boldsymbol{M}$.

For $s\geqslant 0$, $\theta_s:\boldsymbol{X}\to\boldsymbol{X}$ denotes the *shift operator* $\theta_s x(t)=x(s+t)$, and it also will be used to denote the shift on $\boldsymbol{M}$; $\theta_s\psi(A)=\psi(s+A)$. For a pair, (x,ψ), we use notation $\theta_s(x,\psi)=(\theta_s x,\theta_s\psi)$.

2.1.1. Time asymptotic stationarity (TAS)

We say that X is *time asymptotically stationary* (TAS) if there exists a *limiting probability distribution* Q on $\boldsymbol{X}$ in the Césaro sense: for all measurable sets $A\subset\boldsymbol{X}$

$$Q(A)=\lim_{t\to\infty}\frac{1}{t}\int_0^t P(\theta_s\circ X\in A)\,\mathrm{d}s. \tag{2.1}$$

Similarly, ψ is TAS if there exists a probability measure M on $\boldsymbol{M}$ such that for all measurable sets $B\subset\boldsymbol{M}$

$$M(B)=\lim_{t\to\infty}\frac{1}{t}\int_0^t P(\theta_s\circ\psi\in B)\,\mathrm{d}s. \tag{2.2}$$

X^* denotes a process with distribution Q, $Q(A)=P(X^*\in A)$, and X^* is a time stationary process, called a *time stationary version* of X: $\theta_s X^*$ has the same distribution as X^* for any $s\geqslant 0$. Similarly, ψ^* denotes a random point process with distribution M, and ψ^* is time stationary, called a *time stationary version* of ψ.

We shall always assume in this paper that $E_{\mathscr{I}}(\psi^*(1))<\infty$, a.s. (where $\mathscr{I}$ denotes the invariant σ-field *of $\boldsymbol{M}$* (see Sigman, 1995)). *A sufficient condition for this is that $E(\psi^*(1))<\infty$.*

TAS can be defined for (X,ψ) jointly as well: (X,ψ) is TAS if there exists a probability measure J on $\boldsymbol{X}\times\boldsymbol{M}$ such that for all measurable sets $C\subset\boldsymbol{X}\times\boldsymbol{M}$ (from the product σ-field)

$$J(C)=\lim_{t\to\infty}\frac{1}{t}\int_0^t P(\theta_s(X,\psi)\in C)\,\mathrm{d}s. \tag{2.3}$$

In this case (X^*,ψ^*) denotes a time stationary version with distribution J.

2.1.2. Event asymptotic stationarity (EAS)

Analogously to TAS, the point process ψ is called *event asymptotically stationary* (EAS) if there exists a probability measure M^0 such that

$$M^0(B)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}P(\theta_{t_k}\psi\in B). \tag{2.4}$$

ψ^0 denotes a point process (called an event stationary version of ψ) and it is *event stationary*: $\theta_{t_k}\psi^0$ has the same distribution as ψ^0 for all $k\geqslant 0$.

The EAS definition extends to a pair (X,ψ) as well, with event stationary distribution J^0 and event stationary version (X^0,ψ^0):

$$J^0(C) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P(\theta_{t_k}(X,\psi)\in C). \tag{2.5}$$

The marginal distributions of the pair are denoted by $Q^0(A)=P(X^0\in A)$, $M^0(B)=P(\psi^0\in B)$.

Finally, we say that ψ admits coupling to a stationary version ψ^* if there exists versions of ψ and ψ^* and a proper random time S (all on the same probability space) such that $\theta_S\psi=\theta_S\psi^*$ (a.s.). (This is equivalent to the distribution of $\theta_t\psi$ converging in total variation to the distribution of ψ^*, as $t\to\infty$.)

Lemma 2.1. *If X is TAS with stationary version X^*, and if ψ is independent of X and admits coupling to a stationary version ψ^*, then the pair (X,ψ) is jointly TAS with stationary version (X^*,ψ^*) in which ψ^* and X^* are independent.*

Proof. Consider the pair (X^*,ψ^*) in which ψ^* and X^* are independent and time stationary. From this independence, the pair is time stationary: $\theta_s(X^*,\psi^*)$ has the same distribution for all $s\geq 0$. From Theorem 2.1, p. 31 in Sigman (1995), it thus suffices to show that (X,ψ) *shift couples* to such a stationary version: there exists a probability space that supports versions of the two pairs, (X,ψ), (X^*,ψ^*) and proper random times T, T^* such that $\theta_T(X,\psi)=\theta_{T^*}(X^*,\psi^*)$.

We proceed in two steps: **1** We show that (X,ψ) shift couples to (X,ψ^*) and that **2** (X,ψ^*) shift couples to (X^*,ψ^*); for then the result follows by Corollary 2.5, p. 35 in Sigman (1995).

1. Choosing versions of ψ,ψ^* and coupling time S, independent of X, so that $\theta_S\psi=\theta_S\psi^*$, we conclude that (X,ψ) couples, hence shift couples, to (X,ψ^*): $\theta_S(X,\psi)=\theta_S(X,\psi^*)$.
2. Since X is TAS, it shift couples to a stationary version X^* at proper times T_1,T_1^*: $\theta_{T_1}X=\theta_{T_1^*}X^*$. Now, take ψ^* as a two-sided (e.g. defined on all of the real line, positive and negative) stationary version that is independent of (X^*,X,T_1,T_1^*). Since T_1 and T_1^* are independent of ψ^* (which is stationary), we conclude that both $\theta_{-T_1}\psi^*$ and $\theta_{-T_1^*}\psi^*$ are versions of ψ^*. Moreover, it is immediate (by conditioning on $T_{-1}=t$ and $T_1^*=t$) that $\theta_{-T_1}\psi^*$ remains independent of X, and $\theta_{-T_1^*}\psi^*$ remains independent of X^*. Thus, our two versions $(X,\theta_{-T_1}\psi^*)$, $(X,\theta_{-T_1^*}\psi^*)$ shift couple at times T_1 and T_1^* and the proof is complete. □

As in our current paper, it is sometimes useful to consider X non-jointly, so we define X to be EAS *with respect to* ψ if there exists a probability measure Q^0 on $\mathbf{X}$ such that for all measurable A

$$Q^0(A) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P(\theta_{t_k}X\in A). \tag{2.6}$$

X^0 denotes a process with distribution Q^0 (called an event stationary version of X with respect to ψ) and it is *event stationary*: $\theta_{t_k}X^0$ has the same distribution as X^0 for all $k\geqslant 0$.

Note that if (X,ψ) is EAS then X is EAS with respect to ψ (with the same Q^0).

2.1.3. Ergodicity

If the convergence in Eqs. (2.1)–(2.3) is strengthened to a.s. convergence

$$Q(A)=\lim_{t\to\infty}\frac{1}{t}\int_0^t I\{\theta_s X\in A\}\,\mathrm{d}s \quad \text{a.s. } P, \tag{2.7}$$

$$M(B)=\lim_{t\to\infty}\frac{1}{t}\int_0^t I\{\theta_s\psi\in B\}\,\mathrm{d}s \quad \text{a.s. } P, \tag{2.8}$$

$$J(C)=\lim_{t\to\infty}\frac{1}{t}\int_0^t I\{\theta_s(X,\psi)\in C\}\,\mathrm{d}s \quad \text{a.s. } P, \tag{2.9}$$

then we call X (or ψ or (X,ψ)) *time asymptotically stationary ergodic* (TAS-E), in which case it holds that X^* (or ψ^* or (X^*,ψ^*)) is time stationary and ergodic. (TAS-E implies TAS from the bounded convergence theorem). ATS-E holds if (for example) X is *positive recurrent regenerative* or *stationary ergodic* or is a positive *Harris recurrent Markov process* (as defined in Azéma et al., 1967).

TAS-E for X (analogously defined for ψ or (X,ψ)) is equivalent to: for all measurable sets A_0, A

$$P(X\in A_0)Q(A)=\lim_{t\to\infty}\frac{1}{t}\int_0^t P(X\in A_0,\theta_s X\in A)\,\mathrm{d}s. \tag{2.10}$$

(See p. 101 in Loéve, 1978 or p. 38 in Sigman, 1995 for example.)

The same notion can be defined in the context of event stationarity. For example, (X,ψ) is *event asymptotically stationary ergodic* (EAS-E) if Eq. (2.5) is strengthened to

$$J^0(C)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} I\{\theta_{t_k}(X,\psi)\in C\} \quad \text{a.s. } P, \tag{2.11}$$

in which case,

$$Q^0(A)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} I(\theta_{t_k}\circ X\in A) \quad \text{a.s. } P \tag{2.12}$$

and we say that X is EAS-E w.r.t. ψ.

The following can be found as Theorem 2.9, p. 61 and Theorem 5.1, p. 107 in Sigman (1995).

Proposition 2.1. *ψ is TAS if and only if it is EAS. ψ is TAS-E if and only if it is EAS-E. The same holds for a pair (X,ψ).*

2.1.4. Mixing

We say that ψ is mixing if for all Borel sets B_0 and B,

$$P(\psi \in B_0, \theta_s \psi \in B) \to P(\psi \in B_0) M(B) \quad \text{as } s \to \infty, \tag{2.13}$$

where M denotes a probability measure on $\boldsymbol{M}$. Mixing implies TAS since the Césaro convergence in Eq. (2.1) is implied by Eq. (2.13). Mixing also implies ergodicity (via Eq. (2.10)); thus mixing implies TAS-E. $\lambda \stackrel{\text{def}}{=} E\psi^*(1)$ denotes the *intensity* of ψ which we assume is finite and non-zero.

Similarly, X is mixing if all measurable sets A_0 and A,

$$P(X \in A_0, \theta_s X \in A) \to P(X \in A_0) Q(B) \quad \text{as } s \to \infty, \tag{2.14}$$

where Q denotes a probability measure on $\boldsymbol{X}$.

Mixing can be defined analogously in discrete time for stochastic sequences.

2.2. Main results

Theorem 2.1. *Suppose X is TAS-E with limiting distribution Q (from* Eq. (2.7)). *If ψ is independent of X and is mixing, and admits coupling to a stationary version ψ^* then, sampling by ψ leads to the same limiting distribution: For each measurable $A \subset \boldsymbol{X}$*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^1 I\{\theta_s X \in A\}\, \mathrm{d}s = Q(A) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} I\{\theta_{t_k} \circ X \in A\} \quad \text{a.s. } P. \tag{2.15}$$

Proof. By Lemma 2.1, (X, ψ) is TAS with stationary version (X^*, ψ^*) in which X^* and ψ^* are independent. Moreover, since these versions shift couple, we conclude that Eq. (2.15) holds for (X, ψ) if and only if it holds for (X^*, ψ^*). Thus, without loss of generality, it suffices (by Proposition 2.1) to show that (X^*, ψ^*) is ergodic and that $Q = Q^0$ (recall Eq. (2.12)).

To prove that $J = Q \times M$ is ergodic, it suffices to show (see Lemma 10.3.II, p. 341 of Daley and Vere-Jones, 1988) that for all measurable *cylinder* sets (A_0, B_0), (A, B)

$$Q(A_0)Q(A)M(B_0)M(B) = \lim_{t\to\infty} \frac{1}{t} \int_0^t P(X^* \in A_0, \theta_s X^* \in A) P(\psi^* \in B_0, \theta_s \psi^* \in B)\, \mathrm{d}s.$$

This follows easily from assumed coupling of ψ to ψ^* and mixing (2.13), and the assumed ergodicity of Q (as given by Eq. (2.10)).

It now remains to show that $Q = Q^0$. From the Palm formula for stationary and ergodic processes (see Corollary 2.10, p. 63, and p. 98, Section 4.4 in Sigman, 1995, for example)

$$Q^0(A) = \lambda^{-1} E^* \sum_{n=0}^{\psi(1)} I\{\theta_{t_n} X \in A\},$$

where E^* denotes expectation under the time stationary ergodic measure $Q \times M$. But from the independence of X and ψ it follows (via conditioning first on ψ) that the right-hand side above is given by

$$\lambda^{-1}E(\psi^*(1))Q(A)=Q(A),$$

thus completing the proof. □

A distribution F on $\mathbb{R}_+$ is said to be *spread out* if for some integer $m\geqslant 1$, the convolution $F*\cdots*F$, m times, has an absolutely continuous component with respect to Lebesgue measure (see, for example, p. 140 in Asmussen, 1987).

Corollary 2.1. *Suppose ψ is a positive recurrent renewal process, independent of X, having a spread-out cycle length distribution. If X is TAS-E with limiting distribution Q, then Eq.* (2.15) *holds.*

Proof. It suffices to show that such as ψ is mixing. For each fixed t, let $\mathscr{F}_t$ denote the σ-field generated by the point measures during $[0,t]$: $\{\mu(s)\colon s\leqslant t\}$. The spread-out condition is both the sufficient and necessary one for renewal process ψ to admit coupling to a stationary version ψ^* and do so for any initial condition of the form $\psi\in B_0\in\mathscr{F}_t$ (see, for example, Theorem 2.3, p. 146 in Asmussen, 1987). This implies that $\theta_s\psi$ converges in total variation to limiting distribution M regardless of such initial conditions (see Corollary 1.5, p. 142 in Asmussen, 1987). In particular, for each fixed measurable B, mixing condition (2.13) holds for any $B_0\in\bigcup_{t\geqslant 0}\mathscr{F}_t$. We can assume, without loss of generality, that ψ is stationary (via coupling to ψ^*) and thus since $\bigcup_{t\geqslant 0}\mathscr{F}_t$ forms a semiring that generates the Borel sets of $\boldsymbol{M}$, the proof is complete by Lemma 10.3.II in Daley and Vere-Jones (1988). □

The following is motivated by the problem of sampling with a null recurrent renewal process. Here we place the mixing condition on X instead of on ψ.

Theorem 2.2. *Suppose X is stationary and mixing with distribution Q (from Eq.* (2.7)*). If ψ is independent of X and is event stationary, then sampling by ψ leads to the same limiting distribution.*

Proof. It suffices to show that $\{\theta_{t_n}X\colon n\geqslant 0\}$ is stationary and ergodic so that the sampled distribution Q^0 (as defined by Eq. (2.12)) exists, and is given by the distribution of $\theta_{t_0}X=\theta_0X=X$; hence $Q^0=Q$. Clearly, $\{\theta_{t_n}X\}$ is stationary because X is time stationary and ψ is independent of it. It thus suffices to prove mixing (which implies ergodicity). To this end, let $X_k\stackrel{\text{def}}{=}\theta_{t_k}X$, and for $\varepsilon>0$ choose $b>0$ such that

$$|P(X\in A_0,X_s\in A)-P(X\in A_0)P(X\in A)|\leqslant\varepsilon,\quad s\geqslant b.$$

Then conditioning on t_k yields

$$\begin{aligned}&|P(X\in A_0,X_k\in A)-P(X\in A_0)P(X\in A)|\\&\quad\leqslant\int_0^b|P(X\in A_0,\theta_sX\in A)-P(X\in A_0)P(X\in A)|t_k(\mathrm{d}s)\end{aligned}$$

$$+\int_b^{\infty} |P(X\in A_0, \theta_s X \in A) - P(X\in A_0)P(X\in A)| t_k(\mathrm{d}s)$$
$$\leqslant bP(t_k \leqslant b) + \varepsilon.$$

Since $P(t_k \leqslant b) \to 0$ as $k \to \infty$ (because $t_k \to \infty$ w.p.1.) and ε is arbitrary, the proof is complete. □

2.2.1. Comment

1. Theorem 2.1 can be extended to cover the case when X and ψ are TAS but non-ergodic, as long as for each ergodic component of (X^*, ψ^*), the ψ^* component is mixing. For example, suppose that X^* is ergodic, and ψ^* is a mixture of $\psi_1{}^*$ and ψ_2^*, both mixing. Then the result holds.

3. Starting with the sampled average

In Theorem 2.1, we assumed in advance that the time-average distribution Q for X existed, and then showed that the sampled distribution exists and is identical to Q. We next study the reverse direction and find that we require much weaker conditions; neither mixing of ψ nor asymptotic stationarity of X are required. We only require that ψ be TAS.

Theorem 3.1. *Suppose X is a stochastic process and ψ is TAS and independent of X. If for some fixed measurable set $A \subset X$,*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} I(\theta_{t_k} \circ X \in A) = \alpha \quad a.s.\ P \tag{3.1}$$

with α a constant, then

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t I(\theta_s \circ X \in A)\,\mathrm{d}s = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} I(\theta_{t_k} \circ X \in A) \quad a.s.\ P. \tag{3.2}$$

Corollary 3.1. *Suppose ψ is EAS and independent of X. If X is EAS-E (with respect to ψ), with event stationary Q^0 then, X is TAS-E with time stationary distribution Q and $Q = Q^0$, that is, Eq. (3.17) holds for every measurable set A.*

Proof. The existence of $\alpha = Q^0(A)$ is ensured for all A by the EAS-E assumption, and so the result follows from Theorem 3.1. □

Proof of Theorem 3.1. If Eq. (3.1) holds for (ψ, X), then it will also hold for (ψ^*, X) by shift coupling ψ to ψ^*; $\psi_T = \psi^*_{T^*}$ (see Theorem 2.1, p. 31 in Sigman, 1995). Thus, it suffices to prove Eq. (3.2) in the case when $\psi = \psi^*$ is stationary. From Eq. (3.1) it follows that as $t \to \infty$

$$\frac{1}{t}\sum_{k=1}^{\psi(t)} I(\theta_{t_k} \circ X \in A) \to \lambda\alpha \quad \text{a.s. } P.$$

Since the summation is bounded above by $\psi(t)/t$, which is uniformly integrable (UI) (due to stationarity of ψ),

$$\frac{1}{t}\int_0^t I(\theta_s \circ X \in A)\lambda\,\mathrm{d}s = E\left\{\frac{1}{t}\sum_{k=1}^{\psi(t)} I(\theta_{t_k} \circ X \in A)\,|\,X\right\} \to \lambda\alpha \quad \text{a.s. } P.$$

Dividing by λ gives Eq. (3.2). □

3.1. Comments

1. Even PASTA does not assert that $Q = Q^0$, it asserts that the *marginal* $X^0(0-)$ and $X^*(0)$ have the same distribution.

4. The average value of $X(t)$ with renewal sampling

In this section, we are interested in equating time and sampled averages for a real-valued process X as mentioned in Eq. (1.2). To be precise

$$\alpha = \lim_{t\to\infty}\frac{1}{t}\int_0^t X(s)\,\mathrm{d}s \quad \text{a.s. } P \tag{4.1}$$

is assumed to exist with α a constant and we wish to give sufficient conditions ensuring that

$$\alpha = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} X(t_k) \quad \text{a.s. } P. \tag{4.2}$$

Whereas one would expect Eq. (4.2) to hold under fairly general conditions, we no longer have the power of asymptotic stationarity, ergodic theory and coupling at our disposal for use on X, and hence must resort to a different approach. For us this requires from the start assuming that ψ is renewal with a spread-out cycle length distribution. Let L denote a generic cycle length.

Theorem 4.1. *Suppose $X = \{X(t)\}$ is bounded and Eq. (4.1) holds for a finite constant α. If ψ is a renewal process independent of X, with a spread-out cycle length distribution, satisfying $EL^{1+\varepsilon} < \infty$, for some $\varepsilon > 0$, then Eq. (4.2) holds.*

Proof. Assume $\sup_t X(t) \leqslant M < \infty$, and let $\psi^* = \{t_n^*\}$ denote a time stationary version of ψ (to which, due to the spread-out assumption, we can assume that ψ admits coupling (Theorem 2.3, p. 146 in Asmussen, 1987): $\theta_T\psi = \theta_T\psi^*$). It suffices to show (via coupling to ψ^*) that

$$\lambda\alpha = \lim_{t\to\infty}\frac{1}{t}\sum_{k=1}^{\psi^*(t)} X(t_k^*) \quad \text{a.s. } P. \tag{4.3}$$

Let $U(t) = E\psi(t)$ denote the renewal measure for ψ. First, observe that

$$E\sum_{k=1}^{\psi^*(t)} X(t_k^*) = \int_0^t EX(s)E\psi^*(\mathrm{d}s) = \lambda\int_0^t EX(s)\,\mathrm{d}s$$

and that similarly (see p. 136 of Daley and Vere-Jones, 1988).

$$E\left\{\sum_{k=1}^{\psi^*(t)} X(t_k^*)\right\}^2 = \lambda\int_0^t EX^2(s)\,\mathrm{d}s + 2\lambda E\int_0^t\int_0^{t-s} X(s+u)U(\mathrm{d}u)X(s)\,\mathrm{d}s.$$

Consequently,

$$\begin{aligned}\mathrm{Var}\left\{\sum_{k=1}^{\psi^*(t)} X(t_k^*)/t\right\} &= \frac{\lambda}{t^2}\int_0^t EX^2(s)\,\mathrm{d}s \\ &\quad + 2\frac{\lambda}{t^2}E\int_0^t\int_0^{t-s} X(s+u)(U(\mathrm{d}u) - \lambda\,\mathrm{d}u)X(s)\,\mathrm{d}s \\ &\leqslant \lambda\frac{M^2}{t} + 2\frac{\lambda^2M^2}{t^2}\int_0^t\sum_{k=0}^{\lfloor t-s\rfloor}(|U(\mathrm{d}u) - \lambda\,\mathrm{d}u|(k+[0,1])).\end{aligned} \tag{4.4}$$

Using the hypothesis that $EL^{1+\varepsilon} < \infty$, we apply Theorem 1 of Stone and Wainger (1967) to deduce that the last integral in Eq. (4.5) reduces to

$$\int_0^t\sum_{k=0}^{\lfloor t-s\rfloor} \mathrm{o}(k^{-\varepsilon})\,\mathrm{d}s,$$

yielding (after integration)

$$\mathrm{Var}\left\{\sum_{k=1}^{\psi^*(t)} X(t_k^*)/t\right\} \leqslant \lambda\frac{M^2}{t} + 2\frac{\lambda^2M^2}{t^2}\mathrm{O}(t^{2-\varepsilon})$$

and finally

$$\mathrm{Var}\left\{\sum_{k=1}^{\psi^*(t)} X(t_k^*)/t\right\} = \mathrm{O}(t^{-\varepsilon}).$$

Letting $t = s_n \stackrel{\mathrm{def}}{=} n^{2/\varepsilon}$ in the above equation yields

$$\mathrm{Var}\left\{\sum_{k=1}^{\psi^*(s_n)} X(t_k^*)/s_n\right\} = \mathrm{O}\left(\frac{1}{n^2}\right).$$

Applying Chebychev's inequality while using Borel–Cantelli, yields

$$\frac{1}{s_n}\sum_{k=1}^{\psi^*(s_n)} X(t_k^*) - \lambda\frac{1}{s_n}\int_0^{s_n} EX(s)\,\mathrm{d}s \to 0 \quad \text{a.s.} \tag{4.5}$$

as $n\to\infty$. By bounded convergence, $(1/s_n)\int_0^{s_n} EX(s)\,ds\to\alpha$, hence by Eq. (4.5)

$$\frac{1}{s_n}\sum_{k=1}^{\psi^*(s_n)} X(t_k^*)\to\alpha \quad \text{a.s.}$$

Let $n(t)$ denote the (deterministic) counting process for $\{s_n\}$. To prove Eq. (4.3) from Eq. (4.5), observe that for any t, there exists an $n(t)$ such that

$$n(t)^{2/\varepsilon}\leqslant t\leqslant (n(t)+1)^{2/\varepsilon},$$

so

$$\frac{1}{t}\sum_{k=1}^{\psi^*(t)} X(t_k^*)-\frac{1}{t_n(t)}\sum_{k=1}^{\psi^*(t_{n(t)})} X(t_k^*)\leqslant \frac{1}{t}-\frac{1}{t_{n(t)}}\sum_{k=1}^{\psi^*(t)}|X(t_k^*)|+\frac{1}{t_{n(t)}}\sum_{k=\psi^*(t_{n(t)}+1)}^{\psi^*(t)}|X(t_k^*)|$$

$$\leqslant M\frac{\psi^*(t)}{t}\frac{[(n(t)+1)^{2/\varepsilon}-n(t)^{2/\varepsilon}]}{n(t)^{2/\varepsilon}}$$

$$+\frac{1}{n(t)}\sum_{\psi^*(t_{n(t)+1})}^{\psi^*(t)}|X(t_k^*)|,$$

the second to last piece of which tends to zero. But the same argument as used above can be used on $|X(t)|$, implying that the last piece also tends to zero thus proving Eq. (4.3). □

4.1. Final comments

1. If X is not bounded, then Theorem 4.1 may fail even for Poisson arrivals as the following counter example shows: Define $X(t)=n^2$ for $t\in[n,n+1/n^2)$, $X(t)=0$ otherwise. Then the limit in Eq. (4.1) is 1 whereas the limit in Eq. (4.2) is 0. It is likely that this bounded condition can be relaxed; we expect the result to hold if $\{X(t)\colon t\geqslant 0\}$ is *stochastically bounded*: For all $0<\varepsilon<1$ there exists a closed interval $[a,b]$ (depending on ε) such that for all $t\geqslant 0$, $P(X(t)\in[a,b])\geqslant 1-\varepsilon$.
2. We expect that the conclusion of Theorem 4.1 remains valid for $E(T)<\infty$; but we have not been able to prove it.
3. A discrete-time process sampled at a discrete-time point process is easier to deal with than our continuous-time framework. In the renewal case, conditions like "spread-out cycle length" are replaced by "aperiodic cycle length".
4. We end here with an interesting (discrete-time) example (involving a null recurrent process sampled at the times of a null recurrent renewal process) that shows that one must be careful, in general, when sampling in the context of null recurrence. Consider three independent symmetric simple random walks on the integers; $X=\{X_n\colon n\geqslant 0\}$, $Y=\{Y_n\colon n\geqslant 0\}$, $Z=\{Z_n\colon n\geqslant 0\}$. Assume that $X_0=Y_0=Z_0=0$. It is well known that the origin is null recurrent for both the one-dimensional X and

the two-dimensional process (X,Y), but that it is transient for the three-dimensional process (X,Y,Z). Let the process to be sampled be (X,Y) and consider the fact that

$$\sum_{n=0}^{\infty} I\{X_n=0, Y_n=0\}=\infty \quad \text{a.s.}$$

Let the sampling times $\psi=\{\tau_k: k\geqslant 0\}$ be the consecutive times at which Z hits the origin; this forms a null recurrent renewal process that is independent of (X,Y). Then

$$\sum_{k=0}^{\infty} I\{X_{\tau_k}=0, Y_{\tau_k}=0\}<\infty \quad \text{a.s.}$$

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