THE COVARIANCE FUNCTION OF A REGENERATIVE PROCESS

by

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ABSTRACT

In this paper, we develop regenerative representations for the covariance function of both a non-delayed regenerative process and a stationary regenerative process. These results are used to obtain conditions under which the lag-t covariance vanishes as $t \to \infty$, together with associated rates of convergence. The spectral density of a stationary regenerative process is then calculated explicitly in terms of quantities expressed over regenerative cycles. The paper concludes with an application of the theory developed here to the steady-state simulation problem.

KEYWORDS: Regenerative process, renewal function, stationary process, steady-state, simulation.

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1. Introduction

Let $X = (X(t) : t \geq 0)$ be a real-valued regenerative stochastic process. Our goal in this paper is to study the covariance function $c(s,t)$ defined by

$$c(s,t) = cov(X(s), X(s+t))$$

for $s, t \geq 0$. As will be discussed in Section 6, our motivation for studying this problem arose from our interest in the steady-state simulation problem. The covariance function of the stationary version of the regenerative process X plays an important role in development of confidence interval algorithms for estimating the steady-state mean of X.

In Section 2, we review the basic theory of regenerative processes that will be used throughout the paper and develop an appropriate notation and framework for our study. Section 3 obtains a regenerative representation for the covariance function and considers limit theory for $c(s,t)$, both as $s \to \infty$ and as $t \to \infty$ (separately). In Section 4, analogous results are obtained for the covariance function of the stationary version of the regenerative process. In addition, we show that the covariance function typically goes to zero at infinity exponentially fast, whenever certain moment conditions are in force and τ_1 is suitably aperiodic.

Section 5 describes the spectral density of a stationary regenerative process; this representation forms a bridge between the regenerative process literature and that of stationary processes. As indicated earlier, Section 6 describes an application of our results to the steady-state simulation problem. Finally, Section 7 collects the proofs of the main results developed in this paper.

2. A Brief Review of Regenerative Process Theory

We start with a brief description of the probability space that we shall be studying throughout the remainder of this paper. Let $D[0,\infty)$ be the Skorohod space of right-continuous real-valued functions $x : [0,\infty) \to \mathbb{R}$, having left limits everywhere. Our basic sample space Ω will take the form $\Omega = D[0,\infty) \times \mathbb{R}_+^\infty$, where $\mathbb{R}_+ = [0,\infty)$. To equip Ω with a σ-field $\mathcal{F}$, we use the σ-field associated with the product topology on $D[0,\infty) \times \mathbb{R}_+^\infty$. We denote a generic element ω of Ω as $\omega = (x, s_0, s_1, \ldots)$, where $x \in D[0,\infty)$ and $s_i \in \mathbb{R}_+$. Given the sample space Ω, we may define the process $X = (X(t) : t \geq 0)$ via the co-ordinate projections $X(t,\omega) = x(t)$; the sequence of r.v.'s $\tau = (\tau_n : n \geq 0)$ is obtained by setting $\tau_n(\omega) = s_n$.

Set $T(-1) = 0$ and $T(n) = \tau_0 + \cdots + \tau_n$ for $n \geq 0$; the sequence $(T(n) : n \geq 0)$ will play the role of regeneration times for X. To define the cycles that are basic to the analysis of regenerative processes, we take a "cemetary point" $\Delta \notin \mathbb{R}_+$ and set

$$X_n(t) = \begin{cases} X(T(n-1)+t) & ; 0 \leq t < \tau_n \\ \Delta & ; t \geq \tau_n. \end{cases}$$

Note that $X_n \in D_\Delta[0,\infty)$, the space of right-continuous functions x_Δ with left limits, taking values for $\mathbb{R}_+ \cup \{\Delta\}$. Observe that τ_n can be represented as $\tau_n = h_1(X_n)$, where $h_1(x_\Delta) = \inf\{t \geq 0 : x_\Delta(t) = \Delta\}$.

Given a probability measure P on $(\Omega, \mathcal{F})$, we say that X is regenerative under P (with respect to the sequence $(T(n) : n \geq 0)$) if:

(2.1) i) $\{X_n : n \geq 0\}$ is a sequence of independent random elements.

ii) $\{X_n : n \geq 1\}$ is a sequence of identically distributed random elements with $E\tau_1 > 0$.

Conditions i) and ii) together imply that $T(n) \to \infty$ P a.s. The process X is said to be non-delayed under P if $T(0) = 0$ P a.s.; otherwise, X is said to be delayed under P. Finally, we say that X is positive recurrent under P if $E\tau_1 < \infty$ ($E(\cdot)$ corresponds to expectation under P); otherwise, we say that X is null recurrent under P.

Before continuing, we wish to point out that the above specification of a probability space can be made without any essential loss of generality. Suppose that on the probability triple $(\Omega', \mathcal{F}', P')$, we wish to study the real-valued process X' having $D[0,\infty)$ paths and associated regeneration times $(T'(n) : n \geq 0)$. Set $\tau'_n = T'(n) - T'(n-1)$ and note that $(X', (\tau'_n : n \geq 0))$ induces a probability measure P on $D[0,\infty) \times \mathbb{R}_+^\infty$. The analysis of the structure of X' may then be done on the probability space $(\Omega, \mathcal{F}, P)$, where Ω has the required form $\Omega = D[0,\infty) \times \mathbb{R}_+^\infty$. To obtain results for X', we use the fact that the co-ordinate process X on $(\Omega, \mathcal{F}, P)$ is distributionally equivalent to X'.

Let $N(t,\omega) = \max\{n \geq -1 : T(n,\omega) \leq t\}$. For $t \geq 0$, we can then define a shift on the sample space Ω via

$$\theta_t\omega = (x(t+\cdot), s_{N(t)+1} - t, s_{N(t)+2}, s_{N(t)+3}, \ldots).$$

Let W be a real-valued r.v. defined on Ω. For $n \geq 0$, put $\lambda = 1/E\tau_1$ and

$$Y_n(W) = \int_{T(n-1)}^{T(n)} W \circ \theta_t dt.$$

Note that $Y_n(W)$ can be represented in the form $Y_n(W) = h_2(X_n, X_{n+1}, \ldots)$ for some function $h_2 : D_\Delta[0,\infty)^\infty \to \mathbb{R}$. Thus, $(\tau_n, Y_n(W)) = (h_1(X_n), h_2(X_n, X_{n+1}, \ldots))$. Definition (2.1) then implies that the sequence $\{(\tau_n, Y_n(W)) : n \geq 1\}$ is a strictly stationary ergodic sequence of random vectors under P. A routine argument, based on applying Birkhoff's ergodic theorem separately to the τ_n's and $Y_n(W)$'s, yields the following law of large numbers.

(2.2) Proposition. Suppose that X is regenerative under P. If $EY_1(|W|) < \infty$, then

$$\frac{1}{t}\int_0^t W \circ \theta_s ds \to \lambda EY_1(W) \; P \; a.s.$$

as $t \to \infty$.

Proposition 2.3 provides conditions under which expectations can be passed through the above law of large numbers.

(2.3) Proposition. Suppose that X is regenerative under P. If $EY_0(|W|) + Y_1(|W|) < \infty$ then

$$\frac{1}{t}\int_0^t EW \circ \theta_s ds \to \lambda EY_1(W)$$

as $t \to \infty$.

A proof of Proposition 2.3 can be found in Section 7.

Suppose that X is a positive recurrent regenerative process under P. Define the probability measure P^* on the measurable space $(\Omega, \mathcal{F})$ via the formula

$$P^*(A) = \lambda E\int_{T(0)}^{T(1)} I(\theta_s \in A)ds.$$

The probability P^* plays an important role in the analysis of the regenerative process X Observe that by specializing Proposition 2.3 to indicator r.v.'s, we find that

$$\frac{1}{t}\int_0^t P\{\theta_s \in \cdot\}ds \to P^*(\cdot)$$

as $t \to \infty$. Thus, the probability measure P^* characterizes the ergodic behavior of th process under P.

Under P^*, the process X is regenerative. However, X has the additional nice propert that X is a strictly stationary process under P^*; the next proposition summarizes th situation.

(2.4) Proposition. Let X be a positive recurrent regenerative process under P. Then,

i) $P^*\{X \circ \theta_t \in \cdot\} = P^*\{X \in \cdot\}$ for $t \geq 0$,

ii) X is a positive recurrent delayed regenerative process (with respect to $(T(n) : n \geq 0)$ under P^*,

iii) $P^*\{X_n \in \cdot\} = P\{X_n \in \cdot\}$ for $n \geq 1$.

For the proof, see Section 7.

We conclude this section with a brief description of the central limit theorem that goes with the law of large numbers described above. If $E^*|X(0)| < \infty$ ($E^*(\cdot)$ is the expectation operator corresponding to the probability P^*), we define the sequence of r.v.'s

$$Y_n = \int_{T(n-1)}^{T(n)} X(s)ds$$
$$Z_n = Y_n - E^*X(0) \cdot \tau_n.$$

(2.5) Proposition. Let X be a positive recurrent regenerative process under P, for which $E^*|X(0)| < \infty$. If $EZ_1^2 < \infty$, then

$$t^{1/2}\left(\frac{1}{t}\int_0^t X(s)dx - E^*X(0)\right) \Rightarrow \sigma N(0,1) \quad P - \text{weakly}$$
$$t^{1/2}\left(\frac{1}{t}\int_0^t X(s)ds - E^*X(0)\right) \Rightarrow \sigma N(0,1) \quad P^* - \text{weakly}$$

as $t \to \infty$, where $\sigma^2 = \lambda E Z_1^2 = \lambda E^* Z_1^2$.

The proof of this proposition follows in much the same way as does Theorem 1 of Section 16 of CHUNG (1967).

3. The Covariance Function of a Non-Delayed Regenerative Process

Our first objective is to determine conditions under which the covariance function is finite-valued. Let

$$V_n = \sup\{X^2(t) : T(n-1) \leq t < T(n)\}$$

for $n \geq 0$.

(3.1) Proposition. Let X be a non-delayed positive recurrent regenerative process under P.

i) If $E^*X(0)^2 < \infty$, then $E|X(s)X(t)| + E|X(s)| \cdot E|X(t)| < \infty$ for a.e. s,t, so that the covariance function $c(s,t)$ is finite-valued for a.e. $s,t \geq 0$.

ii) If $EV_1 < \infty$, then $E|X(s)X(t)| + E|X(s)| \cdot E|X(t)| < \infty$ for all $s,t \geq 0$, so that the covariance function $c(s,t)$ is finite-valued everywhere.

This proof, as well as those of all the other results of Sections 3 through 6, is deferred to Section 7.

Before proceeding, we wish to point out that the hypothesis of part i) of Proposition 3.1 implies that of part ii), when X is a discrete-time regenerative process. We say that X is a discrete-time regenerative process under P if:

(3.2) i) $X(t) = X(\lfloor t \rfloor)$ P a.s. for $t \geq 0$,

ii) $T(n) \in \mathbb{Z}^+$ P a.s. for $n \geq 0$.

Recall that $E^*X(0)^2 < \infty$ is equivalent to asserting that $E\int_{T(0)}^{T(1)} X^2(t)dt < \infty$. But, for a discrete-time process,

$$\begin{aligned} V_1 &= \max\{X^2(k) : T(0) \leq k < T(1)\} \\ &\leq \sum_{k=T(0)}^{T(1)-1} X^2(k) = \int_{T(0)}^{T(1)} X^2(t)dt, \end{aligned}$$

so that i) does indeed imply ii). Of course, for a general regenerative process, hypotheses i) and ii) are typically noncomparable.

This trick can also be used when X has piece-wise constant sample paths P a.s. Letting the sequence $(\kappa_n : n \geq 0)$ represent the transition epochs of the process X, note that

$$\begin{aligned} V_1 &= \max\{X^2(\kappa_n) : T(0) \leq \kappa_n < T(1)\} \\ &\leq \sum_{n:T(0)\leq\kappa_n<T(1)} X^2(\kappa_n), \end{aligned}$$

so that the hypothesis $EV_1 < \infty$ may be phrased in terms of an additive functional.

We turn now to obtaining an expression for $EX(s)X(s+t)$ in terms of the regenerative cycle structure of the process. Let

$$\begin{aligned} F(t) &= P\{\tau_1 \leq t\} \\ U(t) &= \sum_{n=0}^{\infty} F^{(n)}(t) \\ b(t) &= E\{X(t); \tau_1 > t\} \\ b_t(s) &= E\{X(s)X(s+t); \tau_1 > s+t\} \\ H_s(u) &= \begin{cases} 0, & u \leq s \\ E\{X(s); s < \tau_1 \leq u\}, & u > s. \end{cases} \end{aligned}$$

(3.3) Theorem. Let X be a non-delayed regenerative process under P. If $E|X(s)X(s+t)| < \infty$ $(s, t \geq 0)$, then

$$EX(s)X(s+t) = \int_{[0,s]} (H_{s-u} * U * b)(s+t-u)U(du) + (U * b_t)(s).$$

Each of the two terms appearing in the right-hand side of the above expression for $EX(s)X(s+t)$ has a simple probabilistic interpretation. The first term is the contribution to $EX(s)X(s+t)$ from outcomes in which there is a regeneration in the interval $(s, s+t]$;

in this case, the regeneration "splits" the history, so that $X(s+t)$ is conditionally independent of $X(s)$ (conditional on τ_1). The second term reflects those outcomes in which no regeneration occurs in the interval $(s, s+t]$, so that no "splitting" occurs. The next proposition makes rigorous the above discussion.

(3.4) Proposition. Let X be a non-delayed regenerative process under P. If $E|X(s)X(s+t)| < \infty$ $(s, t \geq 0)$, then

$$EX(s)X(s+t) = \int_{(s,s+t]} EX(s+t-u)E\{X(s); T(N(s)+1) \in du\}$$
$$+ E\{X(s)X(s+t); T(N(s)+1) > s+t\}.$$

By using the fact that $EX(t) = (U * b)(t)$ whenever $E|X(t)| < \infty$, it follows that Theorem 3.3 permits the covariance function $c(s,t)$ to be expressed in terms of regenerative-type quantities.

$$c(s,t) = \int_{[0,s]} (H_{s-u} * U * b)(s+t-u)U(du)$$
$$+ (U * b_t)(s) - (U * b)(s) \cdot (U * b)(s+t).$$

Note that $EV_1^{1/2} < \infty$ is a sufficient condition for $E|X(u)| < \infty$ for all $u \geq 0$. (The proof of Proposition 3.1 ii) can be easily modified to fit the present circumstances.)

We turn now to the "mixing" structure of the regenerative process, as represented by the covariance function. The next theorem presents conditions under which $c(s,t) \to 0$ as $t \to \infty$.

(3.5) Theorem. Let X be a non-delayed positive recurrent regenerative process under P, for which $E(V_1 + V_1^{1/2}\tau_1) < \infty$. Then, $c(s,t) \to 0$ as $t \to \infty$ if either of the two following additional conditions holds:

i) F is spread-out,

ii) X is a discrete-time regenerative process under P and F has unit span.

One ingredient in the proof of Theorem 3.5 is that the conditions stated there imply that $EX(u) \to E^*X(0)$ as $u \to \infty$. Letting $X_c(u) = X(u) - E^*X(0)$, we see that this, in turn, implies that $c(s,t) - EX_c(s)X_c(s+t) \to 0$ as $s \to \infty$. Setting $W = X_c(0)X_c(t)$ and applying Proposition 2.3, we find that if $E^*(|X(0)| + |X(0)| \cdot |X(t)|) < \infty$,

$$\frac{1}{T}\int_0^T EX_c(s)X_c(s+t)ds \to c^*(t) \tag{3.6}$$

as $T \to \infty$, where $c^*(t) = E^*X(0)X(t) - E^*X(0)E^*X(t) = cov^*(X(0), X(t))$ ($cov^*(\cdot)$ is the covariance operator corresponding to $E^*(\cdot)$). Our next theorem presents an unaveraged version of (3.6); it shows that $c(s,t)$ can often be approximated by $c^*(t)$ when s is large.

(3.7) Theorem. Let X be a non-delayed positive recurrent regenerative process under P for which $E(V_1+V_1\tau_1+V_1^{1/2}\tau_1)<\infty$. Then, $c(s,t)\to c^*(t)$ as $s\to\infty$ if either of the two following additional conditions holds:

i) F is spread-out,

ii) X is a discrete-time regenerative process under P and F has unit span.

In the next section, we continue the study of the covariance function $c^*(t)$.

4. The Covariance Function of a Stationary Regenerative Process

Recall that under P^*, X is a strictly stationary process. In this section, we study the covariance function $c^*(\cdot)$ of this stationary process. We start by establishing conditions under which $c^*(\cdot)$ is finite-valued.

(4.1) Proposition. Let X be a positive recurrent regenerative process under P. Then, the following three statements are equivalent.

i) $c^*(t)$ is finite-valued for $t\geq 0$,

ii) $E^*X(0)^2<\infty$,

iii) $EY_1(X(0)^2)<\infty$.

The next result expresses the covariance function $c^*(t)$ in terms of probabilistic quantities expressed over regenerative cycles.

(4.2) Theorem. Let X be a positive recurrent regenerative process under P. If $E^*X(0)^2<\infty$, then

$$c^*(t)=E^*\{X_c(0)X_c(t);\tau_0>t\}+\int_{[0,t]}E^*X_c(T(0)+t-s)E^*\{X_c(0);\tau_0\in ds\}. \tag{4.3}$$

Furthermore, $E^*X_c(T(0)+t-s)=EX_c(T(0)+t-s)$,

$$E^*\{X_c(0)X_c(t);\tau_0>t\}=\lambda E\left\{\int_{T(0)}^{(T(1)-t)\vee T(0)}X(s)X(s+t)ds\right\}, \tag{4.4}$$

and the finite signed measure $E^*\{X_c(0);\tau_0\in ds\}$ can be expressed as

$$E^*\{X_c(0);\tau_0\in ds\}=\lambda E\{X_c(T(1)-s);\tau_1>s\}ds\cdot I(s\geq 0). \tag{4.5}$$

Observe that (4.3) breaks up the covariance into two pieces. The first is a contribution from those paths in which there is no regeneration in $[0,t]$. The second term involves a factorization, which reflects the independence over regenerative cycles.

The next result analyzes the "mixing" structure of a stationary regenerative process by identifying conditions under which $c^*(t) \to 0$ as $t \to \infty$.

(4.6) Theorem. Let X be a positive recurrent regenerative process under P, for which $E(Y_1(X(0)^2) + V_1^{1/2}\tau_1 + V_1^{1/2}) < \infty$. Then, $c^*(t) \to 0$ as $t \to \infty$ if either of the two following additional conditions holds:

i) F is spread-out,

ii) X is a discrete-time regenerative process under P and F has unit span.

In fact, under appropriate moment hypotheses, we can obtain rates of convergence for how fast $c^*(t)$ goes to zero.

(4.7) Theorem. Let X be a regenerative process under P for which there exists $\varepsilon, \alpha > 0$ such that $EV_1^{1+\varepsilon} < \infty$ and $E\exp(\alpha\tau_1) < \infty$. Then, there exists $\beta > 0$ such that $c^*(t) = O(e^{-\beta t})$ as $t \to \infty$, provided that either of the two following additional conditions holds:

i) F is spread-out,

ii) X is a discrete-time regenerative process under P and F has unit span.

An important ingredient of both Theorem 4.6 and Theorem 4.7 is the assumption of non-periodic behavior in the distribution of τ_1. For example, suppose that under P, X is a finite-state irreducible Markov chain having transition matrix $R = (R_{xy} : x, y \in E)$. Suppose that R has period $d > 1$ and let $E = \mathcal{P}(0) \cup \ldots \cup \mathcal{P}(d-1)$ be the cyclic decomposition of the state space. We also adopt the convention that $\mathcal{P}(i) = \mathcal{P}(i \bmod d)$ for $i \geq d$ and set $x_c = x - E^*X(0)$ for $x \in E$. Finally, let $\pi = (\pi_x : x \in E)$ be the (unique) stationary distribution of R. Then, it follows from elementary Markov chain theory that for $0 \leq r < d$,

$$
\begin{aligned}
c^*(nd + r) &= \sum_{x,y} \pi_x x_c R_{xy}^{nd+r} y_c \\
&\to \sum_{i=0}^{d-1} \sum_{x \in \mathcal{P}(i)} \pi_x x_c \sum_{y \in \mathcal{P}(i+r)} d\pi_y y_c
\end{aligned}
$$

as $n \to \infty$, where the limit is typically non-zero and depends on r. The fact that $c^*(n)$ does not vanish as $n \to \infty$ is an indication of the "non-mixing" behavior of X that is typical of periodic regenerative processes. The key characteristic of stationary periodic regenerative processes is that they never forget the "initial point" in the zero'th cycle, where the term "initial point" is to be interpreted as the value of τ_0.

It is worth noting that the covariance function does vanish asymptotically when the initial state is fixed. Specifically, if X is the chain described above, it is easy to verify that if $X(0) = x$ P a.s., then

$$c(m, m+n) = cov(X_c(m), X_c(m+n)) \rightarrow 0$$

as $n \rightarrow \infty$. Basically, we can expect the covariance to vanish asymptotically so long as the initial distribution of X is supported on only one of the $\mathcal{P}(i)$'s, since this guarantees that all trajectories see the same "initial point" (mod d) in the zero'th cycle.

Our next result is a different expression of the fact that $c^*(t)$ often vanishes at infinity. Specifically, conditions are obtained which guarantee that $c^*(\cdot)$ is an element of $L^1[0,\infty)$ ($L^1[0,\infty)$ is the family of Lebesgue integrable functions on $[0,\infty)$); these conditions also suffice to ensure that $EX_c(T(0)+\cdot) \in L^1[0,\infty)$.

(4.8) Theorem. Let X be a regenerative process under P, for which $E(\tau_1^2 + Y_1(|X(0)|)^2 + Y_1(X(0)^2)) < \infty$. Then, $\int_0^\infty |c^*(t)|dt < \infty$ and $\int_0^\infty |EX_c(T(0)+t)|dt < \infty$ if either of the two following additional conditions holds:

i) F is spread-out,

ii) X is a discrete-time regenerative process under P and F has unit span.

In the next section, we express the integral $\int_0^\infty c^*(t)dt$ in terms of regenerative quantities expressed over cycles.

5. The Spectral Density of a Stationary Regenerative Process

Our interest in this section is to obtain a regenerative representation for the spectral density of X under P^*. The spectral density $f : \mathbb{R} \rightarrow [0,\infty)$ is defined by

(5.1) $$f(\alpha) = \frac{1}{\pi}\int_0^\infty \cos(\alpha, t)c^*(t)dt;$$

note that under the hypotheses of Theorem 4.8, $c^*(\cdot) \in L^1[0,\infty)$, so that (5.1) makes sense. The spectral density plays a central role in the analysis of stationary processes (see ANDERSON (1971)). Thus, the results of this section form an important bridge between regenerative process research and the stationary process literature.

Recall that (4.7) expresses the covariance function as the sum of two quantities. The next proposition integrates $\cos(\alpha t)$ against the first term.

(5.2) Proposition. Let X be a positive recurrent regenerative process under P. If $EY_1(|X(0)|)^2 < \infty$, then

$$\int_0^\infty \cos(\alpha t)E^*\{X_c(0)X_c(t); \tau_0 > t\}dt = \frac{\lambda}{2}E\left\{\int_0^{\tau_1}\int_0^{\tau_1} \cos(\alpha(t-s))X_c(T(0)+s)X_c(T(0)+t)dsdt\right\}$$

Before analyzing the second term, we need the following result concerning the cosine transform of $EX_c(T(0)+\cdot)$. For $\alpha \in \mathbb{R}$, let

$$\begin{aligned} \chi(\alpha) &= E\exp(i\alpha\tau_1) \\ \Gamma(\alpha) &= E\int_0^{\tau_1} \exp(i\alpha s)X_c(T(0)+s)ds \\ \beta(\alpha) &= \lambda E\int_0^{\tau_1} \exp(i\alpha s)X_c(T(1)-s)ds. \end{aligned}$$

Also, if F is a spread-out distribution, let $\mathcal{D}(T)=\{0\}$; if F has unit span, set $\mathcal{D}(F)=\{k\pi : k \in \mathbb{Z}\}$.

(5.3) Proposition. Let X be a regenerative process under P, for which $E(\tau_1^2 + Y_1(|X(0)|)^2 + Y_1(X(0)^2)) < \infty$. Then, $EX_c(T(0)+\cdot) \in L^1[0,\infty)$ and

$$\int_0^\infty \cos(\alpha t)EX_c(T(0)+t)dt = \begin{cases} \frac{1}{2}\left(\frac{\Gamma(\alpha)}{1-\chi(\alpha)} + \frac{\Gamma(-\alpha)}{1-\chi(-\alpha)}\right), & \alpha \notin \mathcal{D}(F) \\ -\lambda E\left\{\int_0^{\tau_1} sX_c(T(0)+s)ds\right\}, & \alpha \in \mathcal{D}(F) \end{cases} \tag{5.4}$$

for $\alpha \in \mathbb{R}$, if either of the two following additional conditions holds:

i) F is spread-out,

ii) X is a discrete-time regenerative process under P and F has unit span.

We can now state our spectral density representation theorem.

(5.5) Theorem. Let X be a regenerative process under P, for which $E(\tau_1^2 + Y_1(|X(0)|)^2 + Y_1(X(0)^2)) < \infty$. If either of the two following additional conditions holds:

i) F is spread-out,

ii) X is a discrete-time regenerative process under P and F has unit span.

then $c^*(\cdot) \in L^1[0,\infty)$ and

$$\begin{aligned} 2\pi f(\alpha) = \lambda E&\left\{\int_0^{\tau_1}\int_0^{\tau_1} \cos(\alpha(t-s))X_c(T(0)+s)X_c(T(0)+t)dsdt\right\} \\ &+ \frac{\Gamma(\alpha)\beta(\alpha)}{1-\chi(\alpha)} + \frac{\Gamma(-\alpha)\beta(-\alpha)}{1-\chi(-\alpha)} \end{aligned} \tag{5.6}$$

for $\alpha \notin \mathcal{D}(F)$, whereas for $\alpha \in \mathcal{D}(F)$, $2\pi f(\alpha) = \lambda EZ_1^2$.

Note that for $\alpha \in \mathcal{D}(F)$, $2\pi f(\alpha) = \lambda EZ_1^2$; this identical constant arose in Proposition 2.5. This identity will be discussed in more detail in the next section.

6. Applications to Steady-State Simulation

Our interest in obtaining the results of the previous sections arose from our involvement in the development of output analysis algorithms for steady-state simulations.

Given a real-valued regenerative stochastic process X, it is frequently of interest to calculate the steady-state parameter $E^*X(0)$. Simulation is often a powerful method for numerically estimating such parameters. The law of large numbers (2.2) asserts that if $EY_1(|X(0)|) < \infty$, then

$$\bar{X}(t) \equiv \frac{1}{t}\int_0^t X(s)ds \rightarrow E^*X(0) \ \ P \text{ a.s.}$$

as $t \rightarrow \infty$. Thus, if the process X is simulated over a long enough time interval $[0,t]$, the estimator $\bar{X}(t)$ formed from the resulting simulation output will be close to the desired quantity $E^*X(0)$.

For a numerical algorithm to be useful, it is important to have error bounds that indicate the amount of deviation of the estimator from the true solution $E^*X(0)$. For simulations, this error bound is typically captured by the notion of a confidence interval for $E^*X(0)$. A confidence interval for the parameter $E^*X(0)$, based on estimation via $\bar{X}(t)$, can be obtained from the central limit theorem given by Proposition 2.5:

$$t^{1/2}(\bar{X}(t) - EX^*(0)) \Rightarrow \sigma N(0,1) \tag{6.1}$$

P-weakly, as $t \rightarrow \infty$, where $\sigma^2 = \lambda E Z_1^2$. Specifically, select z so that $P\{N(0,1) \leq z\} = 1 - \delta/2$. If an estimator $\{s(t) : t \geq 0\}$ can be constructed so that $s(t) \Rightarrow |\sigma|$ P-weakly as $t \rightarrow \infty$, the limit theorem (6.1) asserts that

$$\left[\bar{X}(t) - \frac{s(t)z}{t^{1/2}}, \bar{X}(t) + \frac{s(t)z}{t^{1/2}}\right]$$

will be an asymptotic $100(1-\delta)\%$ confidence interval of $\sigma \neq 0$. Thus, the central issue in the output analysis of steady-state simulations is the construction of an estimator $s(t)$, based on the evolution of X up to time t, that converges to $|\sigma|$ P-weakly.

One approach to constructing such an estimator $s(t)$ explicitly uses the regenerative structure of X. Let

$$v(t) = \begin{cases} \frac{1}{t}\sum_{k=1}^{N(t)}(Y_k - \bar{X}(t)\tau_k)^2 & ; N(t) \geq 1 \\ 0 & ; N(t) \leq 0, \end{cases}$$

and set $s(t) = v(t)^{1/2}$. If $E(Y_1(|X(0)|)^2 + \tau_1^2) < \infty$, it is easily verified that

$$s(t) \rightarrow |\sigma| \qquad P \ a.s.$$

as $t \rightarrow \infty$. Let $A_i = Z_i^2 - \sigma^2\tau_i$ and $\varphi = 2EZ_1\tau_1/E\tau_1$. The following result, established in Glynn and Iglehart (1986), shows that the mean square error of $s(t)$ decreases at the canonical rate, namely $t^{-1/2}$ in the computational horizon t.

(6.2) Theorem. Let X be a non-delayed regenerative process under P, for which $E(Y_1(|X(0)|)^8 + \tau_1^8) < \infty$. Then, if $\sigma \neq 0$,

$$tE(s(t) - \sigma)^2 \to (4\sigma^2)^{-1}(EA_1^2 - 2\varphi EA_1Z_1 + \varphi^2 EZ_1^2).$$

One difficulty with the regenerative method described above is that the identification of regeneration times can, in certain settings, be quite expensive computationally. Fortunately, a second class of algorithms for estimating σ exists. Note that Theorem 5.5 proves that under certain regularity hypotheses on the regenerative process, $2\pi f(0) = \sigma^2$. In other words, σ^2 is (up to a multiplicative constant) just the spectral density of the stationary version of the regenerative process X. Hence, Theorem 5.5 justifies the application of spectral density estimation methods to the estimation of σ^2 for regenerative processes. This approach enjoys considerable popularity in the simulation community and is described in more detail in Chapter 3 of Bratley, Fox, and Schrage (1987).

The discussion of Sections 4 and 5 of this paper shows that a critical ingredient in determining the existence of a spectral density is the lack of periodicity in the distribution of τ_1. If the distribution of τ_1 is periodic, the covariance function $c^*(t)$ typically does not vanish at infinity, and the function $c^*(\cdot)$ is not absolutely integrable; thus, the spectral density does not exist. An important conclusion of this paper is that standard spectral density estimation methods for calculating σ^2 may be inappropriate when the process is periodic.

7. Proofs of the Main Results

Proof of Proposition 2.3. Since

$$t^{-1}\left|\int_0^t W \circ \theta_s ds\right| \leq t^{-1}\int_0^{T(N(t)+1)} |W| \circ \theta_s ds = t^{-1}\sum_{k=0}^{N(t)+1} Y_k(|W|) \tag{7.1}$$

it suffices to show that the extreme right-hand side of (7.1) is uniformly integrable in t. The right-hand side is easily shown to converge P a.s. to $\lambda EY_1(|W|)$. So, we're done if we show that the right-hand side converges in expectation to $\lambda Y_1(|W|)$. Although we can not apply Wald's identity directly here (note that the $Y_k(|W|)$'s are dependent r.v.'s), a Wald-type argument does go through. Note that (2.1) implies that $\tau_0, \ldots, \tau_{k-1}$ are independent of $Y_K(|W|)$, so

$$E\sum_{k=0}^{N(t)+1} Y_k(|W|) = \sum_{k=0}^{\infty} EY_k(|W|)I(N(t) \geq k-1)$$
$$= \sum_{k=0}^{\infty} EY_k(|W|)P\{\tau_0 + \cdots + \tau_{k-1} \leq t\}$$
$$= EY_0(|W|) + EY_1(|W|)EN(t).$$

Since the τ_k's are i.i.d., $N(t)$ is a renewal process. The elementary renewal theorem then implies that $EN(t)/t \to \lambda$ as $t \to \infty$, completing the proof of the proposition. $\|$

Proof of Proposition 2.4. If W is a bounded r.v., so is $W \circ \theta_u$ and Proposition 2.3 then implies that

$$\frac{1}{t}\int_0^t EW \circ \theta_s ds \to E^* W,$$
(7.2)
$$\frac{1}{t}\int_0^t EW \circ \theta_u \circ \theta_s ds \to E^* W \circ \theta_u.$$

but, with a little thought, it is easy to see that $\theta_{s+u} = \theta_u \circ \theta_s$ for $s, u \geq 0$. Hence,

$$\frac{1}{t}\int_0^t EW \circ \theta_u \circ \theta_s ds = \frac{1}{t}\int_0^t EW \circ \theta_{s+u} ds$$
$$= \frac{1}{t}\int_u^{u+t} EW \circ \theta_s ds,$$

which converges to $E^* W \circ \theta_u = E^* W$ for $u \geq 0$, proving part i).

For $i = 0, 1, \ldots, n$, let $f_i : D_\Delta[0, \infty) \to I\!R$ be a family of bounded (measurable) functions. By definition of P^*, it is evident that

$$E^* \prod_{i=0}^{n} f_i(X_i) = \lambda E \int_{T(0)}^{T(1)} \prod_{i=0}^{n} f_i(X_i \circ \theta_s) ds.$$

Note that for $T(0) \leq s < T(1)$, $X_i \circ \theta_s = X_{i+1}$ for $i \geq 1$. Hence,

$$\lambda E \int_{T(0)}^{T(1)} \prod_{i=0}^{n} f_i(X \circ \theta_s) ds = \lambda E \int_{T(0)}^{T(1)} f_0(X_0 \circ \theta_s) ds \bullet \prod_{i=1}^{n} f_i(X_{i+1})$$
$$= \lambda E \int_{T(0)}^{T(1)} f_0(X_0 \circ \theta_s) ds \bullet \prod_{i=1}^{n} E f_i(X_{i+1})$$
$$= E^* f_0(X_0) \bullet \prod_{i=1}^{n} E f_i(X_i),$$

which proves ii) and iii). $\|$

Proof of Proposition 3.1. For part i), we apply Proposition 2.3 with $W = X(0)$. Noting that $Y_0(|W|) = 0$ and $EY_1(|W|) = \lambda^{-1} E^* X(0)^2$, we find that $\int_0^t EX^2(s) ds < \infty$ for $t \geq 0$.

Hence, $EX^2(s) < \infty$ for a.e. s. The result then follows by applying the Cauchy-Schwarz inequality at these (s,t) pairs for which $EX^2(s) + EX^2(t) < \infty$.

Part ii) is immediate, provided that we can show that $k(t) = EX^2(t) < \infty$ for $t \geq 0$. A standard renewal argument shows that k satisfies the renewal equation $k = k_1 + F * k$, where $F(\cdot) = P\{\tau_1 \leq \cdot\}$ and $k_1(t) = E\{X^2(t); \tau_1 > t\}$. Note that $k_1(t) \leq E\{V_1; \tau_1 > t\}$ and that k_1 is bounded over finite intervals. Hence, $k = U * k_1$, where U is the renewal kernel $U = \sum_{n=0}^{\infty} F^{(n)}$ associated with F. It follows that $k(t) \leq EV_1 \cdot U(t)$, completing the proof. ||

Proof of Theorem 3.3. We assume initially that X is a bounded process P a.s. (i.e. there exists K such that $P\{|X(t)| \leq K\} = 1$ for all $t \geq 0$). Let $a(s,t) = EX(s)X(s+t)$. Conditioning on τ_1, we find that $a(\cdot,t)$ satisfies the renewal equation

$$a(s,t) = a_t(s) + \int_{[0,s]} a(s-u,t)F(du)$$

where $a_t(s) = E\{X(s)X(s+t); \tau_1 > s\}$. Since $a(\cdot,t)$ and $a_t(\cdot)$ are clearly bounded on finite intervals, we find that

$$a(s,t) = (U * a_t)(s) \tag{7.3}$$

(see KARLIN and TAYLOR (1975), p. 184). We now need to analyze $a_t(s)$. Note that

$$\begin{aligned}
a_t(s) &= E\{X(s)X(s+t); \tau_1 > s+t\} + \int_{(s,s+t]} E\{X(s)X(s+t); \tau_1 \in du\} \\
&= b_t(s) + \int_{(s,s+t]} EX(s+t-u)E\{X(s); \tau_1 \in du\} \\
&= b_t(s) + \int_{[0,s+t]} EX(s+t-u)H_s(du).
\end{aligned}$$

$EX(\cdot)$ can itself be analyzed by a renewal argument, yielding the expression $EX(t) = (U*b)(t)$. Hence,

$$a_t(s) = b_t(s) + (H_s * U * b)(s+t).$$

Substituting this expression into (7.3) proves the theorem when X is bounded P a.s. For general X, we approximate X by bounded processes, and take limits; the integrability condition justifies the limiting operation. ||

Proof of Proposition 3.4. As in the proof of Theorem 3.3, we first assume that X is bounded P a.s. We start by showing that $(U * b_t)(s) = E\{X(s)X(s+t); T(N(s)+1) > s+t\}$. Let $d(s) = E\{X(s)X(s+t); T(N(s)+1) > s+t\}$. By conditioning on τ_1, we find that d satisfies the renewal equation $d = b_t + F * d$. Using the boundedness of X, we may conclude that $d = U * b_t$.

For the first term appearing on the right-hand side of Theorem 3.3, note that it equals (recall, from the proof of Theorem 3.3, that $EX(t) = (U * b)(t)$)

$$\begin{aligned}
&\int_{[0,s]} U(du) \int_{[0,s+t-u]} H_{s-u}(dr) EX(s+t-u-r) \\
&= \int_{[0,s]} U(du) \int_{(s-u,s+t-u]} H_{s-u}(dr) EX(s+t-u-r) \\
&= \int_{[0,s]} U(du) \int_{(s,s+t]} H^u_{s-u}(dv) EX(s+t-v),
\end{aligned}$$

where $H^u_r(v) = H_r(v-u)$. Thus, we will obtain the required equality for the second term if we can show that

$$\int_{[0,s]} U(du) \int_{(s,s+t]} H^u_{s-u}(dv) = \begin{cases} E\{X(s); T(N(s)+1) \in dv\}, & v \leq s+t \\ 0, & v > s+t. \end{cases}$$

The two sides are clearly equal if $v \leq s$ or $v > s+t$. For $s < v \leq s+t$, it suffices to show that

$$\int_{[0,s]} U(du)(H^u_{s-u}(v) - H^u_{s-u}(s)) = E\{X(s); T(N(s)+1) \leq v\}. \tag{7.4}$$

Recall that $H^u_{s-u}(s) = H_{s-u}(s-u) = 0$. Also, if we set $e(s) = E\{X(s); T(N(s)+1) - s \leq r\}$, we find (by the renewal argument and boundedness of X) that $e(s) = \int_{[0,s]} U(du) H_{s-u}(s+r-u)$, proving (7.4) and the proposition for bounded processes. We complete the proof by removing the boundedness assumption as in the proof of Theorem 3.3. ||

Proof of Theorem 3.5. First, we observe that since $EV_1 < \infty$, Theorem 3.3 and Proposition 3.4 apply at every $s, t \geq 0$. For part i), we now use the fact that $|b(t)| \leq E\{|X(t)|; \tau_1 > t\} \leq E\{1 + V_1^{1/2}; \tau_1 > t\}$. Note that the bounding function is decreasing to zero and integrable, since

$$\begin{aligned}
\int_0^\infty E\{1 + V_1^{1/2}; \tau_1 > t\} dt &= E \int_0^\infty (1 + V_1^{1/2}) I(\tau_1 > t) dt \\
&= E\tau_1 (1 + V_1^{1/2}) < \infty.
\end{aligned}$$

Hence, by Theorem 1 of ARJAS, NUMMELIN, and TWEEDIE (1978), we may conclude that $(U * b)(t) \rightarrow E^* X(0)$ as $t \rightarrow \infty$. Let $X_c(t) = X(t) - E^* X(0)$. The result for part i) follows if we can show that $EX(s) X_c(s+t) \rightarrow 0$ as $t \rightarrow \infty$. From Proposition 3.4, we have that

$$\begin{aligned}
EX(s) X_c(s+t) = &\int_{(s,s+t]} EX_c(s+t-u) E\{X(s); T(N(s)+1) \in du\} \\
&+ E\{X(s) X_c(s+t; T(N(s)+1) > s+t\}.
\end{aligned} \tag{7.5}$$

For the first term, observe that $EX_c(s+t-u)$ is a bounded function which converges to zero as $t \to \infty$. On the other hand, the integrator is a finite signed measure, since

$$\int_{(s,\infty]} |E\{X(s); T(N(s)+1) \in du\}| \le E|X(s)| < \infty.$$

So, the bounded convergence theorem proves that the first term converges to zero. For the second term on the right-hand side of (7.5), we use Theorem 3.3 to represent it as

$$\int_{[0,s]} U(du) E\{X(s-u) X_c(s+t-u); \tau_1 > s+t-u\}. \tag{7.6}$$

Again, the measure $U(du)$ over $[0,s]$ has total mass $U(s) < \infty$ and the integrand is bounded and converges to zero as $t \to \infty$. Bounded convergence shows that (7.6) therefore converges to zero as $t \to \infty$, proving i).

For part ii), the proof follows the same line, except that we apply the discrete renewal theorem (see FELLER (1970), p. 330) rather than the continuous-time version of ARJAS, NUMMELIN, and TWEEDIE (1978). ||

Proof of Theorem 3.7. The proof of Theorem 3.5 shows that $EX(t) \to E^*X(0)$ as $t \to \infty$ is guaranteed by i) and ii), as well as the moment hypothesis $EV_1^{1/2}\tau_1 < \infty$. The proof is therefore complete if we show that $EX(s)X(s+t) \to E^*X(0)X(t)$ as $s \to \infty$. Let $a(s,t) = EX(s)X(s+t)$. Since $EV_1 < \infty$, Proposition 3.1 implies that $E|X(s)X(s+t)| < \infty$ for $s,t \ge 0$. The proof of Theorem 3.3 shows that $a(\cdot,t)$ satisfies $a(s,t) = (U * a_t)(s)$. We can then apply Theorem 1 of ARJAS, NUMMELIN, and TWEEDIE (1978) to conclude (for part i)) that $EX(s)X(s+t) \to E^*X(0)X(t)$ as $s \to \infty$, provided that $|a_t(\cdot)|$ is bounded above by an integrable function which tends to zero at infinity. Now,

$$\begin{aligned} |a_t(s)| &\le E\{|X(s)X(s+t)|; \tau_1 > s+t\} \\ &\quad + \int_{(s,s+t]} EX(s+t-u) E\{X(s); \tau_1 \in du\} \\ &\le E\{V_1; \tau_1 > s\} + KE\{|X(s)|; \tau_1 > s\} \\ &\le E\{V_1 + KV_1^{1/2}; \tau_1 > s\} \end{aligned}$$

where $K = \sup\{|EX(t)| : t \ge 0\} < \infty$. (Recall that $|EX(t)|$ is bounded on finite intervals (by $EV_1^{1/2} \cdot U(t)$), and converges to $|E^*X(0)|$.) But

$$\int_0^\infty E\{V_1 + KV_1^{1/2}; \tau_1 > s\} ds = E(V_1\tau_1 + V_1^{1/2}\tau_1) < \infty;$$

This concludes the proof of i). For ii), the proof is identical except that the discrete renewal theorem is used instead. ||

Proof of Proposition 4.1. Recall that $c^*(0) = \text{var}\, X^*(0)$, so i) implies ii). On the other hand, the Cauchy-Schwarz inequality shows that ii) implies i). The equivalence of ii) and iii) follows by definition of P^*. ||

Proof of Theorem 4.1. The proof of (4.3) follows immediately from Proposition 2.4 (just condition on $\tau_0 = T(0)$); furthermore, since $P^*\{X_n \in \cdot\} = P\{X_n \in \cdot\}$ for $n \geq 1$, it is evident that $E^* X_c(T(0) + u) = EX_c(T(0) + u)$ for $u \geq 0$.

Using the definition of P^*, we have that

$$E^*\{X_c(0)X_c(t); \tau_0 > t\} = \lambda E \int_{T(0)}^{T(1)} (X_c(0)X_c(t)I(\tau_0 > t)) \circ \theta_s ds. \tag{7.7}$$

But $(X_c(0)X_c(t)I(\tau_0 > t)) \circ \theta_s = X_c(s)X_c(s+t)I(s < T(1) - t)$ for $T(0) < s \leq T(1)$. Substituting this relationship into the right-hand side of (7.7) yields (4.4). To prove (4.5), note that $\tau_0 \circ \theta_s = T(1) - s$ for $T(0) < s \leq T(1)$ so that for $u \geq 0$,

$$\begin{aligned} E^*\{X_c(0); \tau_0 \leq u\} &= \lambda E \int_{T(0)}^{T(1)} (X_c(0)I(\tau_0 \leq u)) \circ \theta_s ds \\ &= \lambda E \int_{T(0)}^{T(1)} X_c(s)I(T(1) - s \leq u) ds \\ &= \lambda E \int_{T(0)\vee(T(1)-u)}^{T(1)} X_c(r) dr. \end{aligned}$$

But the substitution $v = T(1) - r$ shows that

$$\begin{aligned} E \int_{T(0)\vee(T(1)-u)}^{T(1)} X_c(r) dr &= E \int_0^{\tau_1 \wedge u} X_c(T(1) - v) dv \\ &= E \int_0^u X_c(T(1) - v) I(\tau_1 > v) dv \\ &= \int_0^u E\{X_c(T(1) - v) I(\tau_1 > v)\} dv, \end{aligned}$$

proving the theorem. ||

Proof of Theorem 4.6. Recall that $EY_1(X(0)^2) = \lambda^{-1} E^* X(0)^2$ so that Theorem 4.2 may be applied. Note that the first term on the right-hand side of (4.3) can be bounded via the Cauchy-Schwarz inequality:

$$E^*\{X_c(0)X_c(t); \tau_0 > t\} \leq (E^*\{X_c(0)^2; \tau_0 > t\} \cdot E^* X(t)^2)^{1/2}.$$

But $E^*\{X(0)^2; \tau_0 > t\} \to 0$ as $t \to \infty$ by the dominated convergence theorem. For the second term, observe that $E^*\{X_c(0); \tau_0 \in ds\}$ is a finite signed measure. The proof will therefore be complete is we can prove that $EX_c(T(0)+u)$ is a bounded function which converges to zero as $u \to \infty$ (use the bounded convergence theorem). To accomplish this, we apply the renewal theorem, noting that $X_c(T(0)+\cdot)$ is a non-delayed regenerative process under P. Recall that $EX_c(T(0)+t) \le E^*|X_c(0)|+U(t)\cdot EV_1^{1/2}$ and $EX_c(T(0)+t) = (U*b_c)(t)$, where $b_c(t) = E\{X_c(t);\tau_1 > t\}$. Since $|b_c(t)| \le E\{V_1^{1/2} + E^*|X(0)|;\tau_1 > t\}$, it follows that $|b_c(t)|$ is dominated by an integrable function which tends to zero at infinity. Hence, by the renewal theorem (Theorem 1 of ARJAS, NUMMELIN, and TWEEDIE (1978) for part i), the discrete renewal theorem for ii)), $EX_c(T(0)+t) \to \lambda \int_0^\infty b_c(s)ds = 0$ as $t \to \infty$. Hence, $EX_c(T(0)+u)$ is bounded and converges to zero as $u \to \infty$. ||

Proof of Theorem 4.7. Noting that $Y_1(X(0)^2) \le V_1\tau_1^2$, it follows from Hölder's inequality that $EY_1(x(0)^2) < \infty$, so that Theorem 4.2 is in force. Analyzing the first term on the right-hand side of (4.3), we use (4.4) to bound it ($\tilde{V}_i^{1/2} = V_i^{1/2} + E^*|X_c(0)|$):

$$\begin{aligned}|E^*\{X_c(0)X_c(t);\tau_0 > t\}| &\le \lambda E\{\tilde{V}_1[\tau_1 - t]^+\}\\ &\le \lambda E\{\tilde{V}_1\tau_1;\tau_1 > t\}\\ &\le \lambda E^{\frac{1}{1+\epsilon}}\tilde{V}_1^{1+\epsilon} \cdot E^{\frac{\epsilon}{1+\epsilon}}\left\{\tau_1^{\frac{1+\epsilon}{\epsilon}};\tau_1 > t\right\} = O(e^{-ct})\end{aligned}$$

for $0 < c < \alpha$. Thus, the theorem will be proved if we show that the second term in (4.3) is $O(e^{-dt})$ for some $d > 0$. This, in turn, will follow from the bounded convergence theorem if we can show that $e^{ds}E^*\{X_c(0);\tau_0 \in ds\}$ is a finite (signed) measure and $e^{du}EX_c(T(0)+u)$ is a bounded function which goes to zero at infinity. From (4.5), it is evident that

$$\begin{aligned}|E^*\{X_c(0);\tau_0 \in ds\}| &\le \lambda E\{\tilde{V}_1^{1/2};\tau_1 > s\}ds\\ &\le \lambda E^{1/2}\tilde{V}_1 \cdot P^{1/2}\{\tau_1 > s\}ds = O\left(e^{\frac{-\alpha s}{2}}\right)ds.\end{aligned}$$

To prove that $EX_c(T(0)+u)$ decreases exponentially fast, recall that $EX_c(T(0)+u) = (U*b_c)(u)$ (see the proof of (4.6)). Since

$$|b_c(t)| \le E\{\tilde{V}_1^{1/2};\tau_1 > t\} \le E^{1/2}\tilde{V}_1 \cdot P^{1/2}\{\tau_1 > t\},$$

we can apply Theorem 4.1 of NUMMELIN and TUOMINEN (1982) to conclude that $(U*b_c)(t) = O(e^{-\eta t})$ for some $\eta > 0$ under the hypothesis of i); to make the same conclusion for ii), we use Theorem 6.6 of NUMMELIN (1984) ||

Proof of Theorem 4.8. We start by showing that $\int_0^\infty |EX_c(T(0)+t)|dt < \infty$. Since $E^*X(0)^2 = \lambda EY_1(X(0)^2)$, it follows that $E^*|X(0)| < \infty$. Thus, a standard argument (see Proposition 3.1 i)) shows that $EX_c(T(0)+t) = (U*b_c)(t)$ for a.e. t, where $b_c(t) = E\{X_c(T(0)+t);\tau_1 > t\}$.

Hence, observing that $\int_0^\infty b_c(t)dt = 0$, we get

$$\begin{aligned}
\int_0^\infty |EX_c(T(0)+t)|dt &= \int_0^\infty |(U * b_c)(t)|dt \\
&\le \int_0^\infty |\int_0^t b_c(t_s)\lambda ds|dt + \int_0^\infty \int_{[0,t]} |b_c(t-s)| \cdot |U(ds) - \lambda ds|dt \\
&= \int_0^\infty |\int_t^\infty b_c(s)ds|\lambda dt + \int_{[0,\infty)} \int_s^\infty |b_c(t-s)|dt \cdot |U(ds) - \lambda ds| \\
&\le \lambda \int_0^\infty \int_0^s |b_c(s)|dtds + \int_0^\infty |b_c(t)dt \cdot \int_{[0,\infty)} |U(ds) - \lambda ds| \\
&\le \lambda E\left\{\int_0^{\tau_1} s|X_c(T(0)+s)|ds\right\} + E\left\{\int_0^{\tau_1} |X_c(T(0)+s)|ds\right\} \cdot \int_{[0,\infty)} |U(ds) - \lambda ds|.
\end{aligned}$$

But $E\{\int_0^{\tau_1} s|X_c(T(0)+s)|ds\} \le E\{\tau_1 Y_1(|X_c(0)|)\} \le E^{1/2}\tau_1^2 \cdot E^{1/2} Y_1(|X_c(0)|)^1 < \infty$. Also, $E\{\int_0^{\tau_1} |X_c(T(0)+s)|ds\} = EY_1(|X_c(0)|) < \infty$. Finally, we apply equation (1) of STONE (1967) to obtain finiteness of $\int_{[0,\infty)} |U(ds) - \lambda ds| < \infty$ in the spread-out case; for the discrete-time case, we use Theorem 6.4 of NUMMELIN (1984).

We now show that $\int_0^\infty |c^*(t)|dt < \infty$. Since $E^* X(0)^2 < \infty$, Theorem 4.2 applies. Integrating the right-hand side of (4.3), we see that

$$\begin{aligned}
\int_0^\infty |E^*\{X_c(0)X_c(t); \tau_0 > t\}|dt &\le E^*|X_c(0)| \cdot \int_0^{\tau_0} |X_c(t)|dt \\
&= \lambda E \int_{T(0)}^{T(1)} |X_c(s)| \int_s^{T(1)} |X_c(t)|dtds \\
&\le \lambda E Y_1(|X_c(0)|)^2 < \infty.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\int_0^\infty |\int_{[0,t]} E^* X_c(T(0)+t-s) E^*\{X_c(0); \tau_0 \in ds\}|dt \\
&\le \int_0^\infty |E^* X_c(T(0)+t)|dt \cdot \int_{[0,\infty]} |E^*\{X_c(0); \tau_0 \in ds\}| \\
&\le \int_0^\infty |EX_c(T(0)+t)|dt \cdot \lambda \cdot \int_0^\infty E\{|X_c(T(1)-s)|; \tau_1 > s\}ds \\
&= \int_0^\infty |EX_c(T(0)+t)|dt \cdot \lambda \cdot EY_1(|X_c(0)|) < \infty. \quad ||
\end{aligned}$$

Proof of Proposition 5.2. The following operations are justified by Fubini's theorem. Note that

$$\begin{aligned}
&\int_0^\infty \cos(\alpha t) E^*\{X_c(0)X_c(t); \tau_0 > t\}dt \\
&= E^*\{X_c(0) \int_0^{\tau_0} \cos(\alpha t) \cdot X_c(t)dt\} \\
&= \lambda E\left\{\int_0^{\tau_1} X_c(T(0)+s) \int_s^{\tau_1} \cos(\alpha(t-s)) X_c(T(0)+t)dtds\right\}.
\end{aligned}$$

But

$$\int_0^{T_1} X_c(T(0)+s)\int_s^{T_1}\cos(\alpha(t-s))X_c(T(0)+t)dtds$$
$$=\int_0^{T_1}\int_0^{t} X_c(T(0)+s)X_c(T(0)+t)\cos(\alpha(t-s))dsdt$$
$$=\int_0^{T_1} X_c(T(0)+s)\int_0^{s}\cos(\alpha(t-s))X_c(T(0)+t)dtds,$$

using the fact that $\cos(\cdot)$ is even. Hence,

$$E\left\{\int_0^{T_1} X_c(T(0)+s)\int_s^{T_1}\cos(\alpha(t-s))X_c(T(0)+t)dtds\right\}$$
$$=\frac{1}{2}E\left\{\int_0^{T_1}\int_0^{T_1}\cos(\alpha(t-s))X_c(T(0)+s)X_c(T(0)+t)dsdt\right\},$$

proving the result. ||

Proof of Proposition 5.3. The fact that $EX_c(T(0)+\cdot)\in L^1[0,\infty)$ was shown in the proof of Theorem 4.8. To prove (5.4), we first assume that $\alpha\notin\mathcal{D}(F)$. Using absolute integrabilty hypotheses established in the proof of (4.8), we find that for $\varepsilon>0$,

$$\int_0^\infty \exp(-\varepsilon t+i\alpha t)(U*b_c)(t)$$
$$=\int_{[0,\infty)}\exp(-\varepsilon t+i\alpha t)U(dt)\cdot\int_0^\infty \exp(-\varepsilon t+i\alpha t)b_c(t)dt.$$

Also, since $U=\delta_0+F*U$ (δ_0 is a point mass at zero), we obtain

$$\int_{[0,\infty)}\exp(-\varepsilon t+i\alpha t)U(dt)=1+\int_{[0,\infty)}\exp(-\varepsilon t+i\alpha t)F(dt)\cdot\int_{[0,\infty)}\exp(-\varepsilon t+i\alpha t)U(dt).$$

Hence,

$$\int_{[0,\infty)}\exp(-\varepsilon t+i\alpha t)U(dt)=(1-\int_{[0,\infty)}\exp(-\varepsilon t+i\alpha t)F(dt))^{-1}.$$

It follows that for $\varepsilon>0$,

$$\int_0^\infty e^{-\varepsilon t+i\alpha t}EX_c(T(0)+t)dt=\frac{\int_0^\infty e^{-\varepsilon t+i\alpha t}b_c(t)dt}{1-\int_{[0,\infty)}e^{-\varepsilon t+i\alpha t}F(dt)}. \tag{7.8}$$

By dominated convergence and Fubini's theorem, we see that

$$\int_0^\infty \exp(-\varepsilon t+i\alpha t)EX_c(T(0)+t)dt\to\int_0^\infty EX_c(T(0)+t)dt$$
$$\int_0^\infty \exp(-\varepsilon t+i\alpha t)b_c(t)dt\to\Gamma(\alpha)$$
$$\int_{[0,\infty)}\exp(-\varepsilon t+i\alpha t)F(dt)\to\chi(\alpha)$$

as $\varepsilon\to 0$, so that if $\chi(\alpha)\neq 1$, (7.8) implies that

$$\int_0^\infty \exp(i\alpha t)EX_c(T(0)+t)dt=\frac{\Gamma(\alpha)}{1-\chi(\alpha)}. \tag{7.9}$$

But $\{\alpha : \chi(\alpha) = 1\} = \mathcal{D}(F)$ (see P. 174–175 of CHUNG (1974)). For $\alpha \in \mathcal{D}(F)$, let $\alpha_n \to \alpha$ through a sequence $\alpha_n \notin \mathcal{D}(F)$ and take limits of both sides of (7.9). By using dominated convergence on the left-hand side (recall that $EX_c(T(0)+\cdot) \in L^1[0,\infty)$) and l'Hopital's rule on the right-hand side, we get

$$\int_0^\infty \exp(i\alpha t)EX_c(T(0)+t)dt = -\frac{\Gamma'(\alpha)}{\chi'(\alpha)} \tag{7.10}$$

for $\alpha \in \mathcal{D}(F)$. The derivatives $\Gamma'(\alpha), \chi'(\alpha)$ may be calculated by interchanging derivative and expectation, as can be justified by dominated convergence and the moment conditions: $\Gamma'(\alpha) = E\{\int_0^{\tau_1} sX_c(T(0)+s)ds\}, \chi'(\alpha) = E\tau_1$. The proof follows from (7.9) and (7.10), by writing $\cos(\alpha t) = (\exp(i\alpha t) + \exp(-i\alpha t))/2$. ||

Proof of Theorem 5.5. We need to evaluate the integral of the second term on the right-hand side of (4.3):

$$\begin{aligned}\lambda \int_0^\infty \exp(i\alpha t) \int_0^t & EX_c(T(0)+t-s)E\{X(T(1)-s); \tau_1 > s\}dsdt \\ = \int_0^\infty & \exp(i\alpha t)EX_c(T(0)+t)dt \cdot \beta(\alpha)\end{aligned}$$

(the necessary absolute convergence necessary to justify (7.11) was established in the proof of (4.8)). Of course, the first integral on the right-hand side of (7.11) was evaluated in (7.9) and (7.10). We then obtain (5.6) by observing that $\cos(\alpha t) = (\exp(i\alpha t) - \exp(-i\alpha t))/2$.

To get the result for $\alpha \in \mathcal{D}(F)$, note that $\beta(\alpha) = \beta(0)$ for $\alpha \in \mathcal{D}(F)$. But $\beta(0) = \lambda E\int_0^{\tau_1} X_c(s)ds = 0$. ||

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