Let $\bar{\varepsilon} = 2\varepsilon$ where ε is given in (14). By the Law of Iterated Logarithm for Brownian motion and (14) it follows that

$$\limsup_{t\to\infty}\left|\frac{1}{t^{(1/2)+\bar{\varepsilon}}}\int_0^t \langle e_i, \sigma dW(s)\rangle \frac{\int_0^t \langle e_i, U(s)\rangle ds}{t^{(1/2)-\bar{\varepsilon}}\int_0^t \langle U(s), E_{ii}U(s)\rangle ds}\right|$$

$$\leq \limsup_{t\to\infty}\left[\frac{t^{\varepsilon}}{\int_0^t \langle U(s), E_{ii}U(s)\rangle ds}\right]^{1/2} \lim_{t\to\infty}\left|\frac{1}{t^{(1/2)+\bar{\varepsilon}}}\int_0^t \langle e_i, \sigma dW(s)\rangle\right|$$

$$= 0 \quad \text{a.s.}$$

The third term on the right hand side of (19) tends to 0 a.s. as $t \to \infty$ by the Strong Law of Large Numbers for Brownian motion. Using (13) it follows that

$$\lim_{t\to\infty} b_i(t) = b_i \quad \text{a.s.} \tag{20}$$

□

An asymptotic optimality of a certainty equivalence adaptive control based on $(B(t), z(t))$ corresponding to (7) can be verified to obtain an analogue of the Theorem.

References

[1] H. F. Chen, T. E. Duncan, and B. Pasik-Duncan, *Stochastic adaptive control for continuous time linear systems with quadratic cost,* J. Appl. Math. Optim. **34** (1996), 113–138.

[2] T. E. Duncan, B. Pasik-Duncan, and Q. Zhang, *Adaptive control of stochastic manufacturing systems with hidden Markovian demands and small noise,* preprint.

[3] S. P. Sethi and Q. Zhang, *Hierarchical Decision Making in Stochastic Manufacturing Systems,* Birkhäuser, Boston, 1994.

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Stochastic Optimization via Grid Search

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ABSTRACT. This paper is concerned with the use of grid search as a means of optimizing an objective function that can be evaluated only through simulation. We study the question of how rapidly the number of replications per grid point must grow relative to the number of grid points, in order to reduce the "noise" in the function evaluations and guarantee consistency. This question is studied in the context of Gaussian noise, stable noise, and noise having a finite moment generating function. We particularly focus on the limit behavior in the "critical case".

1. INTRODUCTION

A common problem that arises in the analysis of manufacturing systems is the need to optimize the performance of such a system with respect to a given set of decision parameters. For example, in a "just–in–time" manufacturing environment, the specification of the inventory levels at which to re–order from suppliers can have a significant impact on the efficiency of the operation. Given the complexity of such systems, and the cost of experimentation with the physical facility itself, simulation is a widely used computational tool for studying such manufacturing systems. In this paper, our focus will be on the use of simulation to optimize the complex stochastic models that arise in connection with such problems. Specifically, we will be concerned with the behavior of the most naive of all such optimization approaches, namely "grid search".

To be precise, suppose that $\Lambda \subseteq \mathbb{R}^d$ is the decision parameter space over which we wish to optimize. Let $\alpha : \Lambda \to \mathbb{R}$ be a real–valued function that, for each $\theta \in \Lambda$, measures the performance of the system. Our goal, then, is to maximize α over Λ. To numerically optimize α over Λ, we approximate Λ by some finite set of m points $\Lambda_m = \{\theta_1, \ldots, \theta_m\} \subseteq \Lambda$, and then compute α over Λ_m. The maximum of α over Λ_m is then taken as an approximation to the maximum of α over Λ. Since Λ_m is frequently taken to be a discrete grid (when Λ is a hyper–rectangle), we refer to this approach as a "grid search" for the maximum.

Since our concern is with situations in which $\alpha(\theta)$ can only be computed via simulation, our function evaluations at the points $\theta \in \Lambda_{m_n}$ contain random "noise". In order to reduce the impact of the noise, one simulates multiple independent

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replicates at each "grid point" and averages over the replicates in order to obtain an estimator of $\alpha(\theta)$. The key question that this paper considers is the amount of sampling per grid point that must be done in order to guarantee that the grid search converges to the correct solution. Note that if the sampling per grid point is too low, then anomalous maxima will appear in our approximation to $\alpha(\cdot)$, because the random noise will create "false maxima" in suboptimal regions of Λ.

While grid search is clearly a naive algorithm, it has the advantage that it requires only information of a "function evaluation" type (and no gradients or Hessians). In addition, it is easy to apply in conjunction with a typical discrete-event simulation package, and intuitively natural.

Several papers have identified the critical rate at which the number of replications per grid point must grow relative to the number of grid points; see for example [**Dev76**], [**Dev78**] (for related results see [**Dev77**], [**YF73**] and [**YL90**]). Below the critical rate, grid search fails to converge to a correct solution; above the critical rate, it does. Our main contribution in this paper is to study the precise behavior of grid search in the critical regime, and to identify the appropriate limit laws. We also provide some new insight into the behavior of grid search when the number of replications grows more slowly than in the critical regime (so that the algorithm is inconsistent).

It should be noted that such stochastic optimization problems arise in many non–manufacturing contexts. One particularly important area of application is in parameter estimation for stochastic process, see [**EG96**] for details. [**EG96**] study an adaptive grid search algorithm in which the grid search refines itself iteratively, so as to concentrate most of the sampling effort in a neighborhood of the maximizer of α; the paper also considers the interaction between the random error introduced by simulation versus that error produced by the noise that is present in the underlying statistical data set.

This paper is organized as follows. Section 2 discusses grid search when the noise is Gaussian. In an effort to gain insight into the behavior of grid search when the noise has tails (much) heavier than Gaussian, we consider stable noise in Section 3. Finally, Section 4 is concerned with development of general asymptotics that cover the case in which it is only assumed that the noise has a finite moment generating function.

2. GRID SEARCH WITH GAUSSIAN NOISE

In this section, we study some of the asymptotic properties of grid search in the setting of Gaussian noise. As we shall see later in Section 4, the behavior of grid search in the Gaussian setting is quite representative of that obtained when the noise is non–Gaussian with a finite moment generating function. We choose to study the Gaussian case separately, because the proofs are particularly transparent and the results obtained are especially explicit, in this context.

We assume here (and throughout the remainder of the paper) that Λ is the unit hypercube in $\mathbb{R}^d$. We further require that the objective function α be expressable, for each $\theta \in \Lambda$, as an expected value of the form

$$\alpha(\theta) = \mathrm{E}X(\theta),$$

where $X(\theta)$ is Gaussian with mean $\alpha(\theta)$ and standard deviation $\sigma(\theta)$. Assume that:

A1. $\alpha(\cdot)$ and $\sigma(\cdot)$ are continuous over Λ, with $\sigma(\theta) > 0$ for $\theta \in \Lambda$.

Let $(\theta_m : m \geq 1)$ be a (deterministic) sequence that suitably fills out the unit hypercube asymptotically, namely:

A2. For each set of the form $A = \times_{i=1}^{d}[a_i, b_i] \subseteq \Lambda$,

$$\liminf_{m\to\infty} \frac{1}{m}\sum_{i=1}^{m} I(\theta_i \in A) > 0.$$

The grid search proceeds by replicating $X(\theta_i)$ n independent times, thereby producing $X_1(\theta_i), X_2(\theta_i), \ldots, X_n(\theta_i)$, at each grid point $\theta_i \in \Lambda_m \triangleq \{\theta_1, \ldots, \theta_m\}$, and forms the sample means

$$\overline{X}_n(\theta_i) = \frac{1}{n}\sum_{j=1}^{n} X_j(\theta_i).$$

We further assume that the simulations at differing grid points $\theta_1, \theta_2, \ldots$, are performed independently of one another. In order to develop limit theory that permits us to analyze the appropriate growth rates for m and n, we shall (for convenience) view m as a function of n, namely $m = m_n$. Then,

$$M_n = \max_{\theta_i \in \Lambda_{m_n}} \overline{X}_n(\theta_i)$$

is the grid search approximation to $\max_{\theta\in\Lambda}\alpha(\theta)$. Our first limit theorem establishes the maximal rate at which the number of grid points m may grow as a function of sample size n, while maintaining consistency.

THEOREM 2.1. *Assume A1-A2. Then,*

i.) *If* $\log m_n/n \to \infty$,

$$\sqrt{\frac{n}{\log m_n}}M_n \Rightarrow \sqrt{2}\max_{\theta\in\Lambda}\sigma(\theta)$$

as $n \to \infty$;

ii.) *If* $\log m_n/n \to c \in (0, \infty)$,

$$M_n \Rightarrow \max_{\theta\in\Lambda}[\alpha(\theta) + \sqrt{2c}\sigma(\theta)]$$

as $n \to \infty$;

iii.) *If* $\log m_n/n \to 0$,

$$M_n \Rightarrow \max_{\theta\in\Lambda}\alpha(\theta)$$

as $n \to \infty$, *where* $\Rightarrow$ *denotes weak convergence.*

This result estabishes that the minimal rate at which the number of simulations per grid point must grow relative to the number of grid points is logarithmic. Equivalently, the maximal rate at which the number of grid points may grow relative to the number of simulations per grid point is exponential. Note, also, that by conditioning on the sequence $(\theta_t : t \geq 1)$, the case in which the sites $\theta_1, \theta_2, \ldots, \theta_t$ are generated via *i.i.d.* sampling may be reduced to that covered by the above theorem. (A sufficient condition for A2 is that the distribution of the θ_i's have a positive Lebesgue density on Λ, see for example, [**Bil95**].) It should be noted that the critical nature of the logarithmic rate in the case in which the θ_i's are determined via *i.i.d.* sampling can also be found in [**Dev78**]. However, Theorem 1 above supplies more explicit information about the behavior of M_n when it is inconsistent as an estimator of the maximum.

A glance at the proof shows that the maximizer of $\overline{X}_n(\cdot)$ over Λ_{m_n} converges to the set of maximizers of $\alpha(\cdot)$ when $\log m_n/n \to 0$ as $n \to \infty$. Luc Devroye has pointed out to us that this consistency also holds in setting ii.), provided that $\sigma(\theta)$ is independent of θ.

PROOF OF THEOREM 1. For each $\epsilon > 0$, we may use A1 to partition Λ into sub–hypercubes $H_1, H_2, \ldots, H_l$ ($l = k^d$) of equal volume such that

$$|\alpha(x) - \alpha(y)| \le \epsilon \tag{2.1}$$
$$|\sigma(x) - \sigma(y)| \le \epsilon$$

for $x, y \in H_i$. Then for each $n \ge 1$,

$$\begin{aligned} M_n &= \max_{1\le j\le l}\max_{\theta_i\in H_j} \overline{X}_n(\theta_i) \\ &\overset{\mathcal{D}}{=} \max_{1\le j\le l}\max_{\theta_i\in H_j} [\alpha(\theta_i) + \sigma(\theta_i)N_i(0,1)/\sqrt{n}], \end{aligned} \tag{2.2}$$

where $N_1(0,1), \ldots, N_{m_n}(0,1)$ are *i.i.d.* normal random variables with mean zero and variance one, and $\overset{\mathcal{D}}{=}$ denotes "equality in distribution." Now, for each j, it is known that

$$\max_{\theta_i\in H_j} N_i(0,1) - \sqrt{2\log\left(\sum_{k=1}^{m_n} I(\theta_k \in H_j)\right)} \Rightarrow 0 \tag{2.3}$$

as $n \to \infty$; see, for example, [**BP75**]. By A2, it is evident that

$$\sqrt{\log(\sum_{k=1}^{m_n} I(\theta_k \in H_j)} - \sqrt{\log m_n} \to 0 \tag{2.4}$$

as $n \to \infty$. Let $x_1, x_2, \ldots, x_l$ be representative points chosen from each of the l sub-hypercubes. Then, for each j and n sufficiently large (so that the maximum of the $N_i(0,1)$'s over H_j is positive), (2.1) implies that

$$\begin{aligned} &(\alpha(x_j) - \epsilon) + (\sigma(x_j) - \epsilon)\max_{\theta_i\in H_j} N_i(0,1)/\sqrt{n} \\ &\le \max_{\theta_i\in H_j}[\alpha(\theta_i) + \sigma(\theta_i)N_i(0,1)/\sqrt{n}\,] \\ &\le (\alpha(x_j) + \epsilon) + (\sigma(x_j) + \epsilon)\max_{\theta_i\in H_j} N_i(0,1)/\sqrt{n}. \end{aligned} \tag{2.5}$$

If we let $n \to \infty$ (and use (2.2) through (2.5)), followed by sending $\epsilon \to 0$, we arrive at conclusions i)-iii). □

3. GRID SEARCH WITH STABLE NOISE

To obtain some idea as to how the theory changes when the noise has heavier tails than in the Gaussian setting, we consider now the case in which the noise has a stable distribution. Specifically, we assume that $\alpha(\theta)$ can be expressed, for each $\theta \in \Lambda$, as

$$\alpha(\theta) = \mathrm{E}X(\theta),$$

where $X(\theta)$ is a symmetric stable random variable of index ν ($1 < \nu < 2$), having characteristic function

$$\mathrm{E}\exp(iuX(\theta)) = \exp(iu\alpha(\theta) - \sigma(\theta)^\nu|u|^\nu)$$

for $\sigma(\theta) > 0$. (We restrict ourselves to indices $\nu \in (1,2)$, because if $0 < \nu \le 1$, the expectation of the random variable $X(\theta)$ does not exist, and it therefore makes no sense to base our grid search on averaging independent replicates of $X(\theta)$.) We note that

$$X(\theta) \overset{\mathcal{D}}{=} \alpha(\theta) + \sigma(\theta)Z, \tag{3.1}$$

where Z is a symmetric mean zero stable random variable having characteristic function $E\exp(iuZ) = \exp(-|u|^\nu)$.

As in the Gaussian case, our grid search technique involves averaging *i.i.d.* replicates $X_1(\theta_i), X_2(\theta_i), \ldots, X_n(\theta_i)$ at each grid point $\theta_1, \theta_2, \ldots, \theta_{m_n}$ (with simulations across grid points performed independently), and setting

$$M_n = \max_{\theta_i\in\Lambda_{m_n}} \overline{X}_n(\theta_i).$$

In contrast to the Gaussian case, the critical rate at which m_n may grow with n is of order $n^{\nu-1}$. In order to simplify our analysis, we shall assume that:

A3. ($\theta_n : n \ge 1$) is a sequence of *i.i.d.* random variables.

Set $d = (1-\nu)/(2\Gamma(2-\nu)\cos(\pi\nu/2))$.

THEOREM 3.1. *Assume A1 and A3. Then,*

i.) *If* $m_n/n^{\nu-1} \to \infty$,

$$\left(\frac{n^{\nu-1}}{m_n}\right)^{1/\nu} M_n \Rightarrow \Gamma_1$$

as $n \to \infty$, *where for* $x > 0$,

$$\mathrm{P}(\Gamma_1 \le x) = \exp(-d\,\mathrm{E}\sigma(\theta)^\nu/x^\alpha);$$

ii.) *If* $m_n/n^{\nu-1} \to c \in (0,\infty)$,

$$M_n \Rightarrow \Gamma_2$$

as $n \to \infty$, *where for* $x > \sup\{y : \mathrm{P}(\alpha(\theta) \le y) < 1\}$,

$$\mathrm{P}(\Gamma_2 \le x) = \exp(-cd\,\mathrm{E}(\frac{\sigma(\theta)}{x-\alpha(\theta)})^\nu)\,;$$

iii.) *If* $m_n/n^{\nu-1} \to 0$ *and* $\sup\{y : \mathrm{P}(\alpha(\theta) \le y) < 1\} = \max_{\theta\in\Lambda}\alpha(\theta)$,

$$M_n \Rightarrow \max_{\theta\in\Lambda}\alpha(\theta)$$

as $n \to \infty$.

PROOF. Let $Z_1, Z_2, \ldots$ be *i.i.d.* copies of Z, and note that

$$n^{-1/\nu}(Z_1 + \ldots + Z_n) \overset{\mathcal{D}}{=} Z, \tag{3.2}$$

for $n \ge 1$ (see, for example, p. 13 of [**ST94**]). From (3.1) and (3.2), it follows that

$$M_n \overset{\mathcal{D}}{=} \max_{\theta\in\Lambda_{m_n}} [\alpha(\theta_i) + \sigma(\theta_i)\cdot n^{1/\nu-1}Z_i]. \tag{3.3}$$

Turning first to case i.), observe that A3 and (3.3) yield

$$\begin{aligned} P&\left(\left(\frac{n^{\nu-1}}{m_n}\right)^{1/\nu} M_n \le x\right) \\ &= \exp\left(m_n \log\left(1 - \mathrm{P}\left(Z_1 > m_n^{1/\nu}(x - \epsilon_n \alpha(\theta_1))/\sigma(\theta_1)\right)\right)\right) \\ &= \exp\left(m_n \log\left(1 - \mathrm{E}\overline{\mathrm{F}}\left(m_n^{1/\nu}(x - \epsilon_n \alpha(\theta_1))/\sigma(\theta_1)\right)\right)\right), \end{aligned}$$

where $\overline{\mathrm{F}}(x) = \mathrm{P}(Z_1 > x)$ and $\epsilon_n \stackrel{\triangle}{=} (n^{\nu-1}/m_n)^{1/\nu}$. According to p. 16 of [**ST94**],

$$\overline{\mathrm{F}}(x) \sim dx^{-\nu} \tag{3.4}$$

as $x \to \infty$. Since $\epsilon_n \downarrow 0$ as $n \to \infty$, it follows that

$$\begin{aligned} m_n \overline{\mathrm{F}}&\left(m_n^{1/\nu}(x - \epsilon_n \alpha(\theta_1))/\sigma(\theta_1)\right) \\ &\to d\left(\sigma(\theta_1)/x\right)^\nu \quad \text{a.s.} \end{aligned} \tag{3.5}$$

as $n \to \infty$. By A1, $|\alpha(\cdot)|$ and $|\sigma(\cdot)|$ are bounded away from zero and infinity, so (3.4), (3.5), and the Dominated Convergence Theorem imply

$$m_n \mathrm{E}\overline{\mathrm{F}}\left(m_n^{1/\nu}(x - \epsilon_n \alpha(\theta_1))/\sigma(\theta_1)\right) \to d\,\mathrm{E}\sigma(\theta_1)^\nu/x^\nu$$

as $n \to \infty$, from which i.) follows immediately.

For ii.), use (3.3) to conclude that

$$\begin{aligned} &\mathrm{P}(M_n \le x) \\ &= \exp\left(m_n \log\left(1 - \mathrm{E}\overline{\mathrm{F}}\left((x - \alpha(\theta_1))n^{1-1/\nu}/\sigma(\theta_1)\right)\right)\right). \end{aligned} \tag{3.6}$$

For $x > \sup\{y : \mathrm{P}(\alpha(\theta) \le y) < 1\}$, (3.4) implies that

$$\begin{aligned} m_n \overline{\mathrm{F}}&\left((x - \alpha(\theta_1))n^{1-1/\nu}/\sigma(\theta_1)\right) \\ &\to cd\sigma(\theta_1)^\nu/(x - \alpha(\theta_1))^\nu \quad \text{a.s.} \end{aligned}$$

as $n \to \infty$. The fact that $(x - \alpha(\theta_1))$ is a random variable having support bounded away from zero, in conjunction with A1 and (3.4), permits us to apply the Dominated Convergence Theorem to conclude that

$$m_n \mathrm{E}\overline{\mathrm{F}}\left((x - \alpha(\theta_1))n^{1-1/\nu}/\sigma(\theta_1)\right) \to cd\,\mathrm{E}\sigma(\theta_1)^\nu/(x - \alpha(\theta_1))^\nu \quad \text{a.s.}$$

as $n \to \infty$, proving ii.).

For iii.), an argument essentially identical to that for ii.) shows that for $x > \sup\{y : \mathrm{P}(\alpha(\theta_1) \le y) < 1\}$, $\mathrm{P}(M_n \le x) \to 1$ as $n \to \infty$. On the other hand, for x such that $\mathrm{P}(\alpha(\theta_1) > x) > 0$,

$$\begin{aligned} \mathrm{E}\overline{\mathrm{F}}&\left((x - \alpha(\theta_1))n^{1-1/\nu}/\sigma(\theta_1)\right) \\ &\ge \mathrm{E}\left\{\overline{\mathrm{F}}\left((x - \alpha(\theta_1))n^{1-1/\nu}/\sigma(\theta_1)\right); \alpha(\theta_1) > x\right\} \\ &\ge \overline{\mathrm{F}}(0)\mathrm{P}(\alpha(\theta_1) > x) > 0, \end{aligned}$$

so (3.6) implies that $\mathrm{P}(M_n \le x) \to 0$ as $n \to \infty$, proving the theorem. □

This theorem complements the results of [**Dev78**] by providing explicit limit laws (in the stable noise context) for the case in which $\liminf_{n\to\infty} m_n/n^{1-\nu} > 0$. It shows that consistency requires that the number of replications per grid point be large relative to $m^{1/(\nu-1)}$, or equivalently, the number of grid points be small relative to $n^{\nu-1}$.

4. GRID SEARCH WITH WELL-BEHAVED NOISE

We have seen in Section 3 how heavy tails in the noise can adversely affect grid search, in the sense that the number of grid points permitted, relative to the number of simulations per grid point, must be made somewhat smaller than in the Gaussian case in order to guarantee consistency.

In this section, we study grid search without making the strong parametric assumptions of Section 2 and 3. We start by showing that if $\log m_n$ grows relative to n, the grid search algorithm is typically inconsistent.

As in Sections 2 and 3, we assume the existence of a family of real-valued random variables $(X(\theta) : \theta \in \Lambda)$ such that

$$\alpha(\theta) = \mathrm{E}X(\theta)$$

for $\theta \in \Lambda$. For each $\theta \in \Lambda_{m_n} = \{\theta_1, \dots, \theta_{m_n}\}$, we independently run n *i.i.d.* replications $X_1(\theta), \dots, X_n(\theta)$ of the random variable $X(\theta)$, and average them, thereby producing $\overline{X}_n(\theta)$. Our estimator for the maximum is then

$$M_n = \max_{\theta \in \Lambda_{m_n}} \overline{X}_n(\theta).$$

For $\theta \in \Lambda$, set $s(\theta) = \sup\{y : \mathrm{P}(X(\theta) \le y) < 1\}$ as the right endpoint of the support of $X(\theta)$. Put $s = \sup\{s(\theta) : \theta \in \Lambda\}$.

Consider the assumptions:

A4.1 For each $\epsilon > 0$, there exists $\theta_0 \in \Lambda$ and $\delta > 0$ such that

$$\inf_{||\theta - \theta_0|| < \delta} \mathrm{P}(X(\theta) > s - \epsilon) > 0.$$

A4.2 For each $x > 0$, there exists $\theta_0 \in \Lambda$ and $\delta > 0$ such that

$$\inf_{||\theta - \theta_0|| < \delta} \mathrm{P}(X(\theta) > x) > 0.$$

Then, we have the following result.

PROPOSITION 4.1. Suppose A2 holds and $\log m_n/n \to \infty$ as $n \to \infty$. Then,

i.) If $s < \infty$ and A4.1 is in force,

$$M_n \Rightarrow \sup_{\theta \in \Lambda} s(\theta)$$

as $n \to \infty$;

ii.) If $s = +\infty$ and A4.2 is in force,

$$M_n \Rightarrow \infty$$

as $n \to \infty$.

PROOF. For i.), it is clear that it is sufficient to establish that $\mathrm{P}(M_n > s - \epsilon) \to 1$ as $n \to \infty$ for each $\epsilon > 0$. So, fix $\epsilon > 0$. Then, as in the proof of Theorem 1, partition Λ into l sub–hypercubes of equal volume, with l chosen so large that one of the l sub–hypercubes, say H_j, lies entirely within $\{\theta \in \Lambda : ||\theta - \theta_0|| < \delta\}$ (with θ_0, δ as in A4.1). Clearly,

$$M_n \geq \max_{\theta_i \in H_j} \overline{X}_n(\theta_i).$$

But

$$P\left(\max_{\theta_i \in H_j} \overline{X}_n(\theta_i) \leq s - \epsilon\right)$$
$$= \exp\left(\sum_{k=1}^{m_n} I(\theta_k \in H_j) \log\left(1 - \mathrm{P}(\overline{X}_n(\theta_k) > s - \epsilon)\right)\right).$$

Clearly,

$$\begin{aligned} P\left(\overline{X}_n(\theta_k) > s - \epsilon\right) &\geq P\left(X_1(\theta_k) > s - \epsilon, \ldots, X_n(\theta_k) > s - \epsilon\right) \\ &\geq \left(\inf_{||\theta - \theta_0|| < \delta} \mathrm{P}(X(\theta) > s - \epsilon)\right)^n. \end{aligned}$$

So,

$$P\left(\max_{\theta_i \in H_j} \overline{X}_n(\theta_i) \leq s - \epsilon\right)$$
$$\leq \exp\left(\sum_{k=1}^{m_n} I(\theta_k \in H_j) \log(1 - (\inf_{||\theta - \theta_0|| < \delta} \mathrm{P}(X(\theta) > s - \epsilon))^n)\right).$$

By A2 and the fact the $\log m_n / n \to \infty$, it follows that

$$\sum_{k=1}^{m_n} I(\theta_k \in H_j) \cdot (\inf_{||\theta - \theta_0|| < \delta} \mathrm{P}(X(\theta) > s - \epsilon))^n \to \infty$$

as $n \to \infty$, thereby proving that $\mathrm{P}(\max_{\theta_i \in H_j} \overline{X}_n(\theta_i) \leq s - \epsilon) \to 0$ as $n \to \infty$, establishing i.).

Noting that ii.) requires proving that for each $x > 0$, $\mathrm{P}(M_n \leq x) \to 0$ as $n \to \infty$, we observe that an identical style of proof can be followed there, substituting A4.2 for A4.1. □

We now turn to the issue of consistency.

A5. Suppose there exists $\gamma > 0$ such that

$$\sup_{\theta \in \Lambda} \mathrm{E} \exp(\gamma |X(\theta)|) < \infty.$$

This assumption forces the tails of the noise distributions to go to zero at least exponentially fast, uniformly in $\theta \in \Lambda$. It is clearly satisfied, in the Gaussian case, when the mean and variance are uniformly bounded over $\theta \in \Lambda$.

PROPOSITION 4.2. Suppose A5 holds and $\alpha(\cdot)$ is continuous over Λ. Then, if $\log m_n / n \to 0$ as $n \to \infty$,

$$M_n \Rightarrow \max_{\theta \in \Lambda} \alpha(\theta)$$

as $n \to \infty$.

PROOF. As in the proof of Theorem 1, we use the continuity of α to partition Λ into l sub–hypercubes $H_1, H_2, \ldots, H_l$ of equal volume such that

$$|\alpha(x) - \alpha(y)| < \epsilon$$

for $x, y, \in H_i$. Let $x_1, x_2, \ldots, x_l$ be representative points chosen from each of the l sub–hypercubes and note that for $\theta_i \in H_j$,

$$\begin{aligned} (4.1) \quad P\left(\max_{\theta_k \in H_j} \overline{X}_n(\theta_k) > \alpha(x_j) - 2\epsilon\right) & \\ \geq P\left(\overline{X}_n(\theta_i) \geq \alpha(x_j) - 2\epsilon\right) & \\ \geq P\left(\overline{X}_n(\theta_i) - \alpha(\theta_i) > -\epsilon\right) \to 1 & \end{aligned}$$

as $n \to \infty$, by the Law of Large Numbers. On the other hand,

$$P\left(\max_{\theta_k \in H_j} \overline{X}_n(\theta_k) > \alpha(x_j) + 2\epsilon\right)$$
$$= \exp\left(\sum_{k=1}^{m_n} I(\theta_k \in H_j) \log(1 - \mathrm{P}(\overline{X}_n(\theta_k) \geq \alpha(x_j) + 2\epsilon))\right)$$
$$(4.2) \quad \geq \exp\left(\sum_{k=1}^{m_n} I(\theta_k \in H_j) \log(1 - \mathrm{P}(\overline{X}_n(\theta_k) \geq \alpha(\theta_k) + \epsilon))\right)$$
$$\geq \exp\left(\sum_{k=1}^{m_n} I(\theta_k \in H_j) \log(1 - \sup_{\theta \in \Lambda} \mathrm{P}(\overline{X}_n(\theta) \geq \alpha(\theta) + \epsilon))\right).$$

Set $\psi(\theta, \eta) = \log E \exp(\eta X(\theta))$. Then, for $\theta \in \Lambda$,

$$P\left(\overline{X}_n(\theta) > \alpha(\theta) + \epsilon\right)$$
$$\leq \exp\left(-n(\eta(\alpha(\theta) + \epsilon) - \psi(\theta, \eta)\right).$$

Now, there exists an $\eta_0 > 0$ such that $0 < \eta < \eta_0$, $x^2 \exp(\eta x) \leq a \exp(\gamma |x|)$, so

$$\mathrm{E}X^2(\theta) \exp(\eta X(\sigma)) \leq a \,\mathrm{E} \exp(\gamma |X(\theta)|).$$

It follows that for $0 < \eta < \eta_0$,

$$\sup_{\theta \in \Lambda} \frac{\partial^2}{\partial \eta^2} \psi(\theta, \eta) < \infty.$$

Hence,

$$\begin{aligned} \psi(\theta, \eta) &= \psi(\theta, 0) + \eta \frac{\partial}{\partial \eta} \psi(\theta, 0) + \frac{\eta^2}{2} \frac{\partial^2}{\partial \eta^2} \psi(\theta, \xi) \\ &= \eta \alpha(\theta) + O(\eta^2), \end{aligned}$$

where $O(\eta^2)$ is uniform in $\theta \in \Lambda$ and ξ lies between zero and η. So, for $0 < \eta < \eta_0$,

$$\mathrm{P}(\overline{X}_n(\theta) > \alpha(\theta) + \epsilon) \leq \exp(-n(\eta\epsilon + O(\eta^2))).$$

By choosing η sufficiently small so that $\eta\epsilon + O(\eta^n) > 0$, we observe that

$$\sup_{\theta \in \Lambda} \mathrm{P}(\overline{X}_n(\theta) > \alpha(\theta) + \epsilon) = O(\rho^2)$$

for $\rho \in (0, 1)$. Since $\log m_n / n \to 0$, (4.2) therefore implies that

$$(4.3) \quad \mathrm{P}(\max_{\theta_k \in H_j} \overline{X}_n(\theta_k) < \alpha(x_j) + 2\epsilon) \to 1$$

as $n \to \infty$. Relations (4.1) and (4.3) together imply that

$$\mathrm{P}(|M_n - \max_{1 \leq j \leq l} \alpha(x_j)| < 2\epsilon) \to 1$$

as $n \to \infty$. Sending $\epsilon \downarrow 0$ completes the proof. □

We note that this proof also proves that the maximizer of $\overline{X}_n(\cdot)$ over Λ_{m_n} converges to the set of maximizers of $\alpha(\cdot)$, under the condition stated. This result generalizes that of **[Dev78]**, in which the noise distribution is assumed independent of θ.

As in our analyses of the Gaussian and stable noise cases, the most interesting behavior occurs in the "critical case", in which

$$\log m_n/n \to c \tag{4.4}$$

as $n \to \infty$, where $0 < c < 1$. Recall, from the proof of Proposition 2, that $\psi(\theta,\gamma) = \log \mathrm{E}\exp(\gamma X(\theta))$. Here, we assume that:

A6. i.) For each $\theta \in \Lambda$, there exists a root $\tilde{\gamma} = \gamma(\theta) > 0$ such that

$$\tilde{\gamma}\frac{\partial}{\partial\gamma}\psi(\theta,\tilde{\gamma}) - \psi(\theta,\tilde{\gamma}) = c.$$

ii.) $\psi(\cdot)$ is twice continuously differentiable on $\Lambda \times [0,\gamma_0]$, where $\gamma_0 > \sup_{\theta\in\Lambda}\gamma(\theta)$.

THEOREM 4.1. *Assume (4.4), A2, and A6. Then,*

$$M_n \Rightarrow \max_{\theta\in\Lambda}\frac{\partial}{\partial\gamma}\psi(\theta,\gamma(\theta))$$

as $n \to \infty$.

PROOF. Let $h(\gamma) = \gamma\frac{\partial}{\partial\gamma}\psi(\theta,\gamma) - \psi(\theta,\gamma)$, and note that

$$h'(\gamma) = \gamma\frac{\partial^2}{\partial\gamma^2}\psi(\theta,\gamma).$$

Consequently, $h'(\gamma) > 0$ for $\gamma > 0$, so $\tilde{\gamma}$ exists (by A6 i.)) and is unique. Furthermore, $\gamma(\cdot)$ is, because of A6 ii.) and the Implicit Function Theorem, twice continuously differentiable on Λ. Set $k(\theta) = \frac{\partial}{\partial\gamma}\psi(\theta,\gamma(\theta))$. For $\epsilon > 0$, partition Γ into l sub-hypercubes of equal volume, with l chosen large enough that

$$|k(x) - k(y)| < \epsilon$$

for $x, y \in H_i$, $1 \le i \le l$. Let $x_1, x_2, \ldots, x_l$ be l representative points chosen from $H_1, \ldots, H_l$. Then,

$$\begin{aligned}
&\mathrm{P}(\max_{\theta_j\in H_i}\overline{X}_n(\theta_j) > k(x_i) + 2\epsilon)\\
&\quad\le \sum_{j=1}^{m_n} I(\theta_j \in H_i)\mathrm{P}(\overline{X}_n(\theta_j) > k(\theta_j) + \epsilon)\\
&\quad\le m_n \cdot \sup_{\theta\in\Lambda}\mathrm{P}(\overline{X}_n(\theta) > k(\theta) + \epsilon).
\end{aligned}$$

But

$$\begin{aligned}
&\mathrm{P}(\overline{X}_n(\theta) > k(\theta) + \epsilon)\\
&\quad\le \exp(-n(\gamma(\theta)(k(\theta)+\epsilon) - \psi(\theta,\gamma(\theta)))\\
&\quad= \exp(-n(c + \epsilon\gamma(\theta))) \le \exp(-n(c + \epsilon\inf_{\theta\in\Lambda}\gamma(\theta))).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\mathrm{P}\left(\max_{\theta_j\in H_i}\overline{X}_n(\theta_j) > k(x_i) + 2\epsilon\right)\\
&\quad\le \exp\left(n(\frac{\log m_n}{n} - c - \epsilon\inf_{\theta\in\Lambda}\gamma(\theta))\right) \to 0
\end{aligned}\tag{4.5}$$

as $n \to \infty$. On the other hand,

$$\begin{aligned}
&\mathrm{P}(\max_{\theta_j\in H_i}\overline{X}_n(\theta_j) > k(x_i) - 2\epsilon)\\
&\quad= \exp\left(\sum_{j=1}^{m_n} I(\theta_j\in H_i)\log(1 - \mathrm{P}(\overline{X}_n(\theta_j) > k(x_i) - 2\epsilon))\right)\\
&\quad\le \exp\left(\sum_{j=1}^{m_n} I(\theta_j\in H_i)\log(1 - \inf_{\theta\in\Lambda}\mathrm{P}(\overline{X}_n(\theta) > k(\theta) - \epsilon))\right).
\end{aligned}\tag{4.6}$$

For $0 < \eta < \inf_{\theta\in\Lambda}\gamma(\theta)$, let $\widetilde{\mathrm{P}}$ be the probability measure under which the $X_i(\theta)$'s are *i.i.d.* with common distribution $\exp((\gamma(\theta)-\eta)x - \psi(\theta,\gamma(\theta)-\eta)))\cdot \mathrm{P}(X(\theta) \in dx)$, and let $\widetilde{\mathrm{E}}(\cdot)$ be the corresponding expectation operator. If $S_n = X_1(\theta) + \cdots + X_n(\theta)$, then

$$\begin{aligned}
&\mathrm{P}(\overline{X}_n(\theta) > k(\theta) - \epsilon)\\
&\quad= \widetilde{\mathrm{E}}[\exp(-(\gamma(\theta)-\eta)S_n + n\psi(\theta,\gamma(\theta)-\eta))); \overline{X}_n(\theta) > k(\theta) - \epsilon]\\
&\quad\ge \widetilde{\mathrm{E}}[\exp(-(\gamma(\theta)-\eta)S_n + n\psi(\theta,\gamma(\theta)-\eta))); k(\theta) > \overline{X}_n(\theta) > k(\theta) - \epsilon]\\
&\quad\ge \exp(-n((\gamma(\theta)k(\theta) - \eta k(\theta) - \psi(\theta,\gamma(\theta)-\eta)))\\
&\qquad\cdot\widetilde{\mathrm{P}}(k(\theta) > \overline{X}_n(\theta) > k(\theta) - \epsilon).
\end{aligned}\tag{4.7}$$

Observe that

$$\psi(\theta,\gamma(\theta)-\eta) \ge \psi(\theta,\gamma(\theta)) - \eta k(\theta) + \eta^2\zeta, \tag{4.8}$$

where

$$\zeta \triangleq 2\inf_{\substack{\theta\in\Lambda\\ 0\le\gamma\le\gamma_0}}\frac{\partial^2}{\partial\gamma^2}\psi(\theta,\gamma);$$

ζ is positive by A6. So, $\gamma(\theta)k(\theta) - \eta k(\theta) - \psi(\theta,\gamma(0)-\eta) \le c - \eta^2\zeta$. In addition, note that the mean of the $X_i(\theta)$'s under $\widetilde{\mathrm{P}}$ is $\frac{\partial}{\partial\gamma}\psi(\theta,\gamma(\theta)-\eta)$. Choose η sufficiently small so that

$$r(\theta) \triangleq \frac{\partial}{\partial\gamma}\psi(\theta,\gamma(\theta)-\eta) > k(\theta) - \epsilon$$

for $\theta \in \Lambda$. Set $\kappa = \inf_{\theta\in\Lambda}(k(\theta) - r(\theta))$, and note that

$$\begin{aligned}
&\widetilde{\mathrm{P}}(k(\theta) > \overline{X}_n(\theta) > k(\theta) - \epsilon)\\
&\quad\ge \widetilde{\mathrm{P}}(\kappa > \overline{X}_n(\theta) - r(\theta) > 0).
\end{aligned}$$

Note that for $t > 0$,

$$\begin{aligned}
&\widetilde{\mathrm{P}}(|X_1(\theta)| > t) \le \exp(-tx)\widetilde{\mathrm{E}}\exp(t|X(\theta)|)\\
&\quad\le \exp(-t\kappa - \psi(\theta,\gamma(\theta)-\eta))\\
&\quad\cdot(\exp(\psi(\theta,\gamma(\theta)-\eta+t)) + \exp(\psi(\theta,\gamma(\theta)-\eta-t)).
\end{aligned}$$

By choosing t sufficiently small, it is evident from A6 ii.) that the tail of $X(\theta)$ under $\widetilde{\mathrm{P}}$ converges to zero exponentially fast, uniformly in θ. Thus, the $X_i(\theta)$'s have the first three moments (under $\widetilde{\mathrm{P}}$) uniformly bounded in θ. So, the central limit theorem and Berry-Esseen theorem together imply that

$$\liminf_{n\to\infty} \inf_{\theta\in\Lambda} \widetilde{\mathrm{P}}(\kappa > \overline{X}_n(\theta) - r(\theta) > 0) > 0. \tag{4.9}$$

Relations (4.7)–(4.9) and A2 imply that

$$\sum_{j=1}^{m_n} I(\theta_j \in H_i)\log(1 - \inf_{\theta\in\Lambda} \mathrm{P}(\overline{X}_n(\theta) > k(\theta) - \epsilon)) \to \infty$$

as $n \to \infty$, and thus (4.6) yields the conclusion

$$\mathrm{P}(\max_{\theta_j\in H_i} \overline{X}_n(\theta_j) < k(x_i) - 2\epsilon) \to 0 \tag{4.10}$$

as $n \to \infty$. The theorem then follows by letting $n \to \infty$, applying (4.9) and (4.10), and letting $\epsilon \downarrow 0$. □

The proof of this theorem combines large deviation results with extreme value theory. The proof implicitly contains large deviation estimates which are uniform in the parameter θ. For a general discussion of large deviations see [**Buc90**] or [**DZ93**]. It is easy to verify that the result ii.) of Theorem 1 which pertains to the Gaussian situation is a special case of the above theorem.

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References

[Bil95] Patrick Billingsley, *Probability and measure*, John Wiley & Sons, New York, 1995.

[BP75] R. E. Barlow and F. Proschan, *Statistical theory of reliability and life testing*, Holt, Rinehart, and Winston, New York, 1975.

[Buc90] James A. Bucklew, *Large deviation techniques in decision, simulation, and estimation*, John Wiley & Sons, New York, 1990.

[Dev76] Luc P. Devroye, *On the convergence of statistical search*, IEEE Transactions on Systems, Man and Cybernetics **SMC-6** (1976), 46–56.

[Dev77] Luc P. Devroye, *An expanding automation for use in stochastic optimization*, Journal of Cybernetics and Information Science **1** (1977), 82–94.

[Dev78] Luc P. Devroye, *The uniform convergence of nearest neighbor regression function estimators and their application in optimization*, IEEE Transactions on Information Theory **IT-24** (1978), 142–151.

[DZ93] Amir Dembo and Ofer Zeitouni, *Large deviation techniques, and applications*, A.K. Peters, Wellesley, MA, 1993.

[EG96] K. B. Ensor and P. W. Glynn, *Grid-based simulation and the method of conditional least squares*, Proc. of the 1996 Winter Simulation Conference (1996), 325–331.

[ST94] G. Samorodnitsky and M. S. Taqqu, *Stable non-gaussian random processes*, Chapman and Hall, New York, 1994.

[YF73] S. J. Yakowitz and Lloyd Fisher, *On sequential search for the maximum of an unknown function*, Journal of Mathematical Analysis and Applications **41** (1973), 234–259.

[YL90] S. Yakowitz and E. Lugosi, *Random search in the presence of noise, with application to machine learning*, SIAM J. Sci. Stat. Comput. **11** (1990), 702–712.

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Optimal Ergodic Control of Singularly Perturbed Hybrid Stochastic Systems *

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Abstract

This paper deals with a class of singularly perturbed stochastic systems composed of a fast mode, described as a deterministic or stochastic diffusion subsystem and a slow mode described as a jump process. The control acts on both modes. The system is controlled over an infinite time horizon with a limit-average cost criterion. Under appropriate ergodicity conditions we define a limit-control problem which takes the form of a controlled Markov chain and we establish the convergence of the optimal average cost of the perturbed system toward the optimal average cost of the limit-control problem. Finally we sketch a numerical approximation method that could provide an alternative approach to show the convergence toward the limit control problem. We also recall the link that can be established between this numerical scheme and the decomposition technique in linear programming.

1 Introduction

Hybrid stochastic control systems have provided an interesting paradigm for the study of manufacturing and economic production systems (see [10], [13], [17], [18], [19], [25] for a small sample of the abundant literature in this area). A very common feature of these stochastic production systems is the occurence of events at very different time scales. In [13], the hierarchical structure of the manufacturing flow control systems is clearly exposed. In [16], the time scale decomposition methods in manufacturing flow control models has been reviewed. The links with the theory of singularly perturbed stochastic systems have been explored in [24] and [25] in a context where the discrete stochastic events (e.g. the failures and repairs of the machines) occur at different time scales. The first attempt to study, in the context

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