Stochastic Hydrology and Hydraulics 10, 17-37 © Springer-Verlag 1996

# A diffusion approximation for a network of reservoirs with power law release rule

**J. E. Glynn**
Geological Survey of Canada, 601 Booth Street, Ottawa, Ontario, K1A 0E8, Canada

**P. W. Glynn**
Department of Operations Research, Stanford University, Stanford, CA 94305, USA

**Abstract**: A diffusion approximation for a network of continuous time reservoirs with power law release rules is examined. Under a mild assumption on the inflow processes, we show that for physically reasonable values of the power law constants, the system of processes converges to a multi-dimensional Gaussian diffusion process. We also illustrate how the limiting Gaussian process may be used to compute approximations to the original system of reservoirs. In addition, we study the quality of our approximations by comparing them to results obtained by simulations of the original watershed model. The simulations offer support for the use of the approximation developed here.

**Key words**. Diffusion, network, reservoir, power law.

## 1 Introduction

The linear reservoir plays a central role in hydrology (Nash, 1957, 1959). Although this model has been successfully used for many applications, a more general system of reservoirs with a power law storage-runoff relation can more accurately represent the physical characteristics of a natural storage system. The nonlinear reservoir has been examined by Laurenson (1964), Mein et al (1974), Klemeš (1978), Hughes and Murrel (1986), and Glynn (1989)) among others. The natural extension of this model to a nonlinear cascade of reservoirs has also been investigated (Klemeš et al. (1975,1985)). Unfortunately closed form formulae for the statistical properties of these reservoirs are only available for special cases (e.g. the linear reservoir, the fixed release reservoir, the power law release rule, for a single reservoir, with special inflow distribution and exponent less than one).

Two approaches have generally been used to model the continuous time reservoir. In the first case, the inflows are directly modeled as diffusion processes and the problem requires solving nonlinear stochastic differential equations (Unny (1984), Unny and Karmeshu (1984)). Such models have also been used to study optimal control problems associated with reservoirs (Harrison and Taylor (1978), Pliska (1975)). The second approach assumes that inflows follow a compound Poisson process. The

linear cascade with general additive homogeneous inflows has been investigated by Moran (1967). Exact solutions for the single reservoir with a power law release rule are available for the case of compound Poisson inflows where the power is less than one (Harrison and Resnick (1976)). Approximate solutions for general inflows have been obtained by Smith and Yeo (1981) using methods based on Hermite polynomial expansions.

In this paper, we adopt a different approach to the analysis of complex reservoir models. Based on a careful mathematical analysis of the underlying non-linear reservoir model, we show that our watershed model can be approximated by a certain diffusion process. The nature of the approximation is similar to that encountered in applying the central limit theorem (CLT) of statistics. The CLT basically asserts that when a certain parameter, namely the sample size, is large, then the corresponding sample mean has a distribution that can be approximated by that of a Gaussian random variable (r.v.) with parameters depending only on the population mean and variance of the sample. One consequence is that when the sample size is large, the influence of the third and higher moments on the distribution of the sample mean must necessarily diminish. Our approximation is similar in spirit. Namely, when a certain parameter, known as $\epsilon_*$, (see Section 3 for its precise definition), is small, we establish that the network of nonlinear reservoirs can be approximated by a certain Gaussian process having parameters that depend on the underlying inflow process only through their mean and (time-average) variance constants. As in the CLT, the impact of skewness, kurtosis, and higher order moments evidently decreases as the parameter $\epsilon_*$ gets smaller. The close correspondence of our results to those associated with the classical CLT arises, in part, because of the fact that we appeal to the CLT in one step of the derivation of the approximation. (See the Appendix for details.) In any case, such diffusion approximations have been successfully applied to biology, physics, computer science, and many other areas (see, for example, Karlin and Taylor (1981)). In particular, such approximations have been used by Yamada (1983, 1984) to study a single continuous time non-linear finite capacity reservoir with Poisson inflows. In a similar spirit, Harrison and Shepp (1984) develop a diffusion approximation for a cascade of two discrete time reservoirs.

Our diffusion approximation starts from a watershed network of continuous time nonlinear reservoirs with general (possibly correlated) inflows and power law release rules. This work differs from that of Yamada, Harrison, and Shepp both because we study an entire watershed of reservoirs (rather than a single reservoir) and because the power law release rule used here does not fit into their mathematical framework. (In particular, their diffusion limits apply to reservoirs in which the mean inflow rate is close to that of the maximal outflow rate, which is assumed to be finite.) Our approach leads to a diffusion limit that is especially tractable from a computational viewpoint, primarily because the approximating diffusion process turns out to be Gaussian. In addition, the mathematical condition under which our approximation is valid, namely that $\epsilon_*$ be small, appears to coincide with a physically reasonable subset of the parameter space that defines the family of power law release rules. Consequently, we believe that this approach offers a comprehensive means of developing approximations to large-scale watersheds.

This paper is organized as follows. In Section 2, we describe the basic watershed model, followed by a description of the diffusion approximation in Section 3. The diffusion approximation requires that the user supply some mean and covariance infor-

mation for the exogenous inflow processes to the reservoirs in the watershed. Section 4 provides some background on how to calculate these model parameters. In Section 5, we discuss the numerical computation of the transient and steady-state distributions of the approximating diffusion process. Because the process is Gaussian, this reduces to calculating the mean and covariance of the corresponding Gaussian random variables. We show how the transient version of these quantities can be calculated by solving a certain deterministic system of linear differential equations, the dimension of which equals the number, d, of reservoirs being studied. Approximating the steady-state distribution of the d reservoir watershed model is particularly easy; one merely needs to compute the solution to a d×d system of linear equations. Section 6 illustrates a different type of computation, namely that associated with developing an approximation for the expected time required for a reservoir to exceed a given level. Here, one needs to solve a (deterministic) linear partial differential equation in at most d+1 variables. In Section 7, we specialize our discussion to that of a watershed consisting of a single reservoir. In this simpler setting, closed-form formulae are available for several quantities of interest. Section 8 offers some numerical evidence which support the quality of our approximation. Section 9 provides a discussion of our results and Section 10 describes our conclusions.

## 2 Description of the model

The drainage network model consists of d reservoirs. In addition, the following notational conventions will be used:
P(j) - the collection of reservoirs flowing directly into reservoir j
$I_j(t)$ - the cumulative exogenous inflow into reservoir j ($1 \le j \le d$), and
$S_j(t)$ - the storage content of reservoir j at time t.

We will assume that the storage levels follow "power law" release rules. The continuity equation then states that

$$S_j(t) \;=\; s_j + I_j(t) + \sum_{k \in P(j)} \int_0^t a_k S_k(u)^{b_k} du - \int_0^t a_j S_j(u)^{b_j} du, \quad 1 \le j \le d\ , \tag{1}$$

where $s_1$, $a_1$, $b_1$, $s_2$ $a_2$, $b_2$, ..., $s_d$, $a_d$, $b_d$, are positive (deterministic) constants.

An example of such a network of reservoirs is given in Figure 1. In this case the system of equations given by Equation (1) takes the following form:

$$\begin{aligned}
S_1(t) &= s_1 + I_1(t) - \int_0^t a_1 S_1(u)^{b_1} du\ , \\
S_2(t) &= s_2 + I_2(t) - \int_0^t a_2 S_2(u)^{b_2} du\ , \text{ and} \\
S_3(t) &= s_3 + I_3(t) + \int_0^t a_1 S_1(u)^{b_1} du + \int_0^t a_2 S_2(u)^{b_2} du - \int_0^t a_3 S_3(u)^{b_3} du\ .
\end{aligned} \tag{2}$$

We pointed out, in the Introduction, that our approximation depends on the inflow processes only through their means and variances. Specifically, the approximation depends on the vector $\mu = (\mu_j : 1 \le j \le d)$ and covariance matrix $C = (C_{jk} : 1 \le j, k \le d)$ defined below:

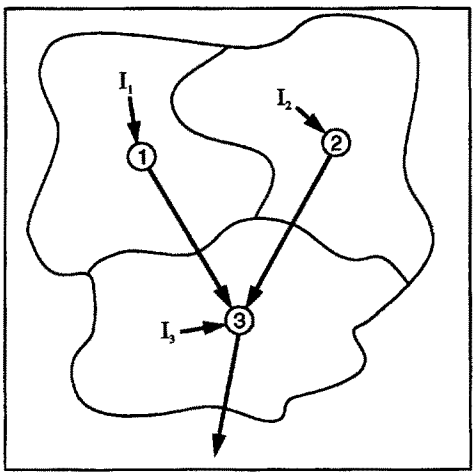


**Figure 1**. A storage system consisting of three reservoirs where reservoirs one and two feed into reservoir three.

$$\mu_j = \lim_{t\to\infty} \frac{1}{t} EI_j(t) \qquad (1 \le j \le d), \text{ and}$$

$$C_{jk} = \lim_{t\to\infty} \frac{1}{t} \operatorname{cov}(I_j(t), I_k(t)) \qquad (1 \le j,\ k \le d)\ . \tag{3}$$

## 3 The diffusion approximation

The application of the diffusion approximation is quite straightforward and follows a sequence of steps which will be described in this section. A basic assumption of the approximation is that the $a_i (1 \le i \le d)$ appearing in (1) are small in a certain sense. To calculate the approximations then requires applying the following steps:

**A-1**. Calculate $\epsilon_*$ where

$$\epsilon_* \triangleq \max_{1\le k\le d} a_k^{1/b_k}$$

Our approximation requires that $\epsilon_*$ should be "small". Storage-runoff models assuming a small value of a (and hence small $\epsilon_*$) have been cited in the literature (see, for example Laurenson, (1964, p. 145) where the release rule, r(x), is given by $r(x) = 0.0445\ x^{1.37}$ and $0.128\ x^{1.37}$) indicating that this model may be a reasonable approximation in some realistic situations. In any case, the storage continuity equations (1) can then be written in the form

$$S_j(t) = s_j + I_j(t) + \sum_{k\in P(j)} \int_0^t \alpha_k \epsilon_*^{b_k} S_k(u)^{b_k} du - \int_0^t \alpha_j \epsilon_*^{b_j} S_j(u)^{b_j} du\ , \tag{4}$$

for $1 \le j \le d$, where $\alpha_k \triangleq a_k \epsilon_*^{-b_k}$, $(\alpha_k \le 1,\ 1 \le k \le d)$.

**A-2.** Let $x_j \stackrel{\Delta}{=} \epsilon_* s_j$, $1 \leq j \leq d$. Solve the following system of ordinary differential equations:

$$\dot{x}_j(t) = \mu_j + \sum_{k \in P(j)} \alpha_k x_k(t)^{b_k} - \alpha_j x_j(t)^{b_j}, \qquad 1 \leq j \leq d,$$

and such that $x_j(0) = x_j, \quad 1 \leq j \leq d.$ (5)

**A-3.** Solve the following system of stochastic differential equations:

$$dZ_j(t) = \sum_{k \in P(j)} \alpha_k b_k x_k(t)^{b_k - 1} Z_k(t) dt - \alpha_j b_j x_j(t)^{b_j - 1} Z_j(t) dt + dB_j(t), \ (1 \leq j \leq d) \quad (6)$$

(where $(B_1(t), B_2(t) ..., B_d(t))$ is a d-dimensional Brownian motion process with zero drift and covariance matrix $C = (C_{ij} : 1 \leq i, j \leq d)$).

We note that since (6) comprises a linear system of stochastic differential equations, it follows that the solution $(Z_1(t), ..., Z_d(t))$ is a Gaussian process. Such processes possess the convenient property that they are completely characterized by their (time-dependent) mean and covariance functions.

**A-4.** The approximation is then given by the formula

$$S_j(t) \stackrel{D}{\approx} \frac{1}{\epsilon_*} x_j(\epsilon_* t) + \frac{1}{\sqrt{\epsilon_*}} Z_j(\epsilon_* t) \quad (7)$$

($\stackrel{D}{\approx}$ denotes "has approximately the same distribution as the process"). This approximation is mathematically justified in the Appendix.

It is clear that the right hand side of (7) inherits the Gaussian structure of the $Z_i(t)$'s. Furthermore the approximating process appearing in (7) must necessarily be a Markovian diffusion process, since the solution to the stochastic differential equation (6) is Markovian.

## 4 Calculating the mean and covariance structure of the exogenous inflow process

As mentioned in the Introduction, our approximation depends on the exogenous inflow processes only through their mean vector $\mu$ and time-average covariance matrix C. We will first discuss how these parameters are computed for several theoretical inflow models, and then discuss how they can be estimated from actual measured inflow data. It should be noted that the specific examples described in our discussion of the relevant computation for theoretical inflow models are intended primarily to illustrate the nature of the calculation. In any concrete application of our approximation, the user is free, of course, to supply an inflow model that reflects the user's own modelling preferences.

We start with a discussion of inflow modelling in the setting of a single reservoir.

Model 1: Let $T_k$ be the time at which the k'th storm group begins. We assume a "shot-noise" formulation (see, for example, Papoulis (1965)) in which the rate at which rainfall (attributable to the k'th storm group) occurs at time $T_k + t$ is given

by $h_k(t)$, where $h_k(\cdot)$ is itself (possibly) a stochastic process. If $N(t)$ is the number of storm groups to arrive by time t, then

$$N(t) = \max\{n \geq 0 : \ T_n \leq t\} .$$

Furthermore, the total inflow up to time t is

$$\begin{aligned} I(t) &= \sum_{k=1}^{N(t)} \int_0^t h_k(s - T_k)ds \\ &= \sum_{k=1}^{N(t)} \int_0^{t-T_k} h_k(u)du . \end{aligned}$$

Set $T_0 = 0$, $\tau_k = T_k - T_{k-1}$, and $\lambda = 1/E\tau_1$, and assume that $((\tau_k, h_k(\cdot)) : \ k \geq 1)$ is an ergodic stationary sequence in k. (In particular, we do not require that the $(\tau_k, h_k)$'s be independent in k). Then, $T_n/n \to E\tau_1$ as $t \to \infty$, so $T_{N(t)}/N(t) \to E\tau_1$ as $t \to \infty$. Since $T_{N(t)} \sim t$, it follows that $N(t)/t \to \lambda$ as $t \to \infty$. Hence,

$$\begin{aligned} I(t) &\approx \sum_{k=1}^{|\lambda t|} \int_0^{t-T_k} h_k(u)du \\ &\approx \sum_{k=1}^{|\lambda t|} \int_0^{\infty} h_k(u)du \\ &= \sum_{k=1}^{|\lambda t|} X_k \end{aligned}$$

where

$$X_k = \int_0^{\infty} h_k(u)du$$

is the total rainfall attributable to the k'th storm group. Clearly, $I(t)/t \to \lambda E(X_1)$ as $t \to \infty$, so we may conclude that $\mu = \lambda E(X_1)$. Note that

$$\begin{aligned} I(t) - \mu t &\approx \sum_{k=1}^{N(t)} (X_k - \mu\tau_k) \\ &\approx \sum_{k=1}^{|\lambda t|} (X_k - \mu\tau_k) . \end{aligned}$$

It is therefore evident that the time-average variance constant is given by

$$\begin{aligned} t^{-1}\mathrm{var}(I(t) &= t^{-1}\mathrm{var}(I(t) - \mu t) \\ &\approx t^{-1}\mathrm{var}\left(\sum_{k=1}^{|\lambda t|} (X_k - \mu\tau_k)\right) \\ &\to \lambda\left[\mathrm{var}(X_1 - \mu\tau_1) + 2\sum_{j=1}^{\infty} \mathrm{cov}(X_1 - \mu\tau_1, X_{j+1} - \mu\tau_{j+1})\right] . \end{aligned}$$

In particular, if the $(\tau_k, h_k)$'s are independent and $h_k$ is independent of $\tau_k$, then the time-average variance is just $\lambda(\mathrm{var}(X_1) + \mu^2\mathrm{var}(\tau_1))$.

Model 2: Our second model is a generalization of the Neyman-Scott process that has been widely referenced in the hydrology literature; see, for example, Puente et al. (1993) and Rodriguez-Iturbe et al. (1987). Let $T_k$ be the initiation epoch for the k'th storm origin. The k'th storm consists of $M_k$ secondary rainfall cells; the j'th cell associated with the k'th storm has a duration $\beta_{kj}$ with an intensity of $Y_{kj}$ per unit time; and it is initiated at time $T_k + \Gamma_{kj}$. Set $T_0 = 0$, $\tau_k = T_k - T_{k-1}$, $\lambda = 1/E\tau_1$, and assume that

$$\{V_k \triangleq (\tau_k, M_k, \beta_{k1}, ..., \beta_{kM_k}, \Gamma_{k1}, ..., \Gamma_{kM_k}, Y_{k1}, ..., Y_{kM_k}) : \ k \geq 1\}$$

is an ergodic, stationary sequence of random variables. (In particular, we permit the $V_k$'s to be dependent random variables.) Note that the total rainfall $X_k$ associated with the k'th storm group is given by

$$X_k \ = \ \sum_{j=1}^{M_k} \beta_{kj} Y_{kj} \ .$$

We can now appeal to the derivation of Model 1 to conclude that $\mu = \lambda E(X_1)$ and the time-average variance constant is given by

$$C \ = \ \lambda \left[ \mathrm{var}(X_1 - \mu\tau_1) + 2 \sum_{j=1}^{\infty} \mathrm{cov}(X_1 - \mu\tau_1, \ X_{j+1} - \mu\tau_{j+1}) \right] \ .$$

In the Neyman-Scott process, it is typically assumed that the random variables $\tau_k$, $M_k$, $\beta_{k1}, ..., \beta_{kj}, \Gamma_{k1}, ..., \Gamma_{kj}, Y_{k1}, ..., Y_{kj}, ...$ are independent in k and j and that the $\beta_{kj}$'s, $\Gamma_{kj}$'s and $Y_{kj}$'s are identically distributed families of random variables. (In fact, it is also usually assumed that $\tau_k$, $\beta_k$, and $\Gamma_k$ are exponential random variables.) In this case, $\mu = \lambda E(M_{kj})E(Y_{kj})E(\beta_{kj})$ and the time-average variance constant is given by $\lambda(\mathrm{var}(X_k) + \mu^2\mathrm{var}(\tau_k))$. But

$$\mathrm{var}(X_k) \ = \ E(M_k)\mathrm{var}(Y_{kj}\beta_{kj}) + \mathrm{var}(M_k)(E(Y_{kj}\beta_{kj}))^2$$

and

$$\mathrm{var}(Y_{kj}\beta_{kj}) \ = \ \mathrm{var}(Y_{kj})\mathrm{var}(\beta_{kj}) + (E(\beta_{kj}))^2\mathrm{var}(Y_{kj}))^2\mathrm{var}(\beta_{kj})$$

and thus the time-average variance constant is given by

$$\begin{aligned} C = \ \lambda \{ & E(M_k) \left[ \mathrm{var}(Y_{kj})\mathrm{var}(\beta_{kj}) + (E(\beta_{kj}))^2\mathrm{var}(Y_{kj}) \right. \\ & \left. + (E(Y_{kj}))^2\mathrm{var}(\beta_{kj}) \right] + \mathrm{var}(M_k)(E(Y_{kj}))^2(E(\beta_{kj}))^2 \\ & + \mu^2\mathrm{var}(\tau_k) \} \ . \end{aligned} \tag{8}$$

Model 3: Our third model assumes that the inflow process is given by an autoregressive moving-average (ARMA) model. Specifically, we assume that over the interval

[nh, (n+1)h], the inflow rate is equal to $\mu + Y_n$, where the $Y_n$'s are stationary and satisfy

$$\sum_{j=0}^{p} a_j Y_{n-j} = \sum_{k=0}^{q} b_k \epsilon_{n-k}$$

with the $\epsilon_n$'s independent and identically distributed (i.i.d.) with mean zero. Note that

$$t^{-1}\mathrm{var}(I(t)) \approx t^{-1}h^2\mathrm{var}\left(\sum_{k=0}^{\lfloor t/h \rfloor} Y_k\right)$$

$$\rightarrow h\left[\mathrm{var}(Y_n) + 2\sum_{j=0}^{\infty} \mathrm{cov}(Y_i, Y_j)\right]$$

as $\tau \rightarrow \infty$. The latter quantity is just $2\pi h f(0)$, where

$$f(\lambda) = \frac{1}{2\pi} \sum_{j=-\infty}^{+\infty} e^{i\lambda j}\, \mathrm{cov}(Y_0, Y_j)$$

is the spectral density of the $Y_n$'s. It follows that

$$f(0) = \frac{1}{2\pi}\left(\sum_{j=0}^{q} b_j\right)^2 \left(\sum_{j=0}^{p} a_j\right)^{-2}$$

(see Anderson, (1971), p. 409), and hence the time-average variance constant is given by

$$C = h\left(\sum_{j=0}^{q} b_j\right)^2 \left(\sum_{j=0}^{p} a_j\right)^{-2} . \tag{9}$$

We turn now to a discussion of inflow modelling for multiple reservoir systems. This is a more challenging exercise, since the spatial, as well as the temporal, variation of the inflows must be captured. Again, we provide our examples primarily as illustrations of the calculation.

Model 4: We first consider a d-reservoir version of the Neyman-Scott process. The k'th storm system produces $M_{ik}$ secondary cells over the i'th reservoir's watershed; the j'th such secondary event has duration $\beta_{ikj}$, has a rainfall intensity of $Y_{ikj}$ per unit time, and begins at time $T_k + \Gamma_{ikj}$. As in the single reservoir case, it is evident that when t is large, the total exogenous inflow $I_i(t)$ into reservoir i satisfies

$$I_i(t) \approx \sum_{k=1}^{N(t)} X_{ik}$$

where

$$X_{ik} = \sum_{j=1}^{M_{ik}} \beta_{ikj} X_{ikj}$$

and $N(t) = \max\{n \geq 0 : T_n \leq t\}$. Set $\tau_k = T_k - T_{k-1}$ and assume that the $\tau_k$'s and $Y_{ik}$'s are stationary in k and ergodic. Let $\lambda = 1/E\tau_1$, $X_k = (X_{1k}, ..., X_{dk})'$, and observe that $\mu = \lambda E(X_k)$. Also, the time-average covariance matrix C of the exogenous inflows is given by

$$C = E[X_1 - \mu\tau_1)(X_1 - \mu\tau_1)']$$

$$+ \sum_{j=1}^{\infty} E[X_1 - \mu\tau_1)(X_{j+1} - \mu\tau_{j+1})' + (X_{j+1} - \mu\tau_{j+1})(X_1 - \mu\tau_1)'] \ .$$

A case of special interest is when all the random variables associated with a given storm system are independent of those associated with other storm systems. Then, the time-average covariance matrix C reduces to

$$C = E[X_k - \mu\tau_1)(X_k - \mu\tau_1)']$$

If we further assume that the $\beta_{ikj}$'s, $Y_{ikj}$'s, and $\Gamma_{ikj}$'s are independent random variables, with distributions depending only on i, then

$$C_{ii} = \lambda\{E(M_{ik})[var(Y_{ikj})var(\beta_{ikj}) + (E(\beta_{ikj}))^2 var(Y_{ikj})$$

$$+ E(Y_{ikj}))^2 var(\beta_{ikj})] + varM_{ik}(E(Y_{ikj}))^2(E(\beta_{ikj}))^2$$

$$+ \mu_i^2 \ var(\tau_k)\}$$

and

$$C_{ij} = \lambda\{cov(M_{ik}, M_{lk})E(\beta_{ikj})E(\beta_{lkj}), E(Y_{ikj})E(Y_{lkj})$$

$$+ \mu_i\mu_j \ var(\tau_k)\}$$

for $i \neq j$.

Model 5: Our final model assumes that the inflow process is given by a d-dimensional vector ARMA model. Specifically, assume that over the interval [nh, (n+1)h], the inflow rate into the i'th reservoir is equal to $\mu_i + Y_{in}$, where the vector sequence $Y_n = (Y_{1n}, ..., Y_{dn})'$ is stationary and satisfies

$$\sum_{j=0}^{p} A_j Y_{n-j} = \sum_{k=0}^{q} B_k \epsilon_{n-q}$$

with the $\epsilon_n$'s independent and identically distributed with mean zero. Here, the time-average covariance matrix is

$$C = \left(\sum_{j=0}^{p} A_j\right)^{-1} \left(\sum_{k=0}^{q} B_k\right) \tilde{C} \left(\sum_{k=0}^{q} B_k'\right) \left(\sum_{j=0}^{p} A_j'\right)^{-1}$$

where $\tilde{C}$ is the covariance matrix of the random vector $\epsilon_n$.

We conclude this section with a brief discussion of how the mean vector $\mu$ and the covariance matrix C may be statistically estimated directly from actual measured

exogenous inflow data. The statistical discussion assumes no model structure such as that associated with Models 1 through 5. (If we want to exploit such a model structure a more efficient statistical estimator for C could be developed.) Suppose that $(I_j(t) : 0 \le t \le T,\ 1 \le j \le d)$ is observed. Then, for $1 \le j \le d, \hat{\mu}_j = I_j(T)/T$ is an appropriate estimator for $\mu_j$. To estimate the covariance matrix C, let $\dot{I}_j(t)$ be the measured rate at which the exogenous inflow into the j'th reservoir occurs at time t. It is often reasonable to assume that the d-dimensional process $(((\dot{I}_1, ..., \dot{I}_d(t)) : \ t \ge 0)$ is stationary. In this case,

$$C_{jk} = \int_0^\infty \mathrm{cov}(\dot{I}_j(0), \dot{I}_k(t))dt + \int_0^\infty \mathrm{cov}(\dot{I}_k(0), \dot{I}_j(t))dt .$$

We note that $C_{jk}$ is related to the cross-spectral density of the two processes $(\dot{I}_j(t) : \ t \ge 0)$ and $(\dot{I}_k(t) : \ t \ge 0)$ evaluated at the frequency 0. Consequently, estimators from the theory of spectral density estimation can be used to calculate $\hat{C}_{jk}$ for $1 \le j,\ k \le d$. (See Brillinger (1981) for details.) One class of such estimators takes the form

$$\hat{C}_{jk} = \int_0^T w_T(t)\hat{\mathrm{cov}}(\dot{I}_j(0), \dot{I}_k(t))dt + \int_0^T w_T(t)\hat{\mathrm{cov}}(\dot{I}_k(0), \dot{I}_j(t))dt , \qquad (10)$$

where $w_T(t)$ is an appropriately chosen weighting function having the properties that $w_T(t) \to 0$ as $t \to \infty$ and $w_T(t) \to 1$ as $T \to \infty$, and where

$$\hat{\mathrm{cov}}(\dot{I}_j(0), \dot{I}_k(t)) = \frac{1}{T-t} \int_0^{T-t} \left(\dot{I}_j(s) - \hat{\mu}_j\right)\left(\dot{I}_k(s+t) - \hat{\mu}_k\right) ds .$$

## 5 Distributional characterization of the approximation

As noted in Section 3, the diffusion approximation for the vector storage process $(S(t): \ t \ge 0)$ is a Gaussian process, and so it follows that

$$S(t) \overset{D}{\approx} N(m(t), K(t))$$

where m(t) and K(t) are (respectively) the mean and the covariance of the d-dimensional normal random variable that approximates S(t). We will now describe, in a step-by-step fashion, how to calculate the functions m(t) and K(t), thereby completing the characterization of the approximation. (For additional detail, we refer the reader to Chapter 8 of Arnold (1974).)

Let $A(t) = (A_{ij}(t) : \ 1 \le i,\ j \le d)$ be the dxd matrix with elements given by

$$A_{kj}(t) = \begin{cases} \alpha_k b_k x_k(t)^{b_k - 1}, & \text{if } k \in P(j), \\ -\alpha_j b_j x_j(t)^{b_j - 1}, & \text{if } k = j, \text{and} \\ 0, & \text{else,} \end{cases}$$

where $(x_1(t)..., x_d(t))'$ is the solution of the system (5) of ordinary differential equations.

The rows of the matrix A are numbered in such a way that the further upstream the reservoir, the lower its number; in other words we require that $k \in P(j) \Rightarrow k < j$

for $1 \leq j \leq d$. This numbering convention will be used throughout the remainder of the paper.

Our procedure is to:

**B-1**. Solve the following deterministic matrix-valued differential equation for $\Phi(t)$:

$$\dot{\Phi}(t) = A(t)\Phi(t)$$

such that $\Phi(0) = I$.

**B-2**. Calculate

$$\Gamma(t) = \Phi(t) \int_0^t \Phi(u)^{-1} C (\Phi(u)^{-1})' du \Phi(t)'$$

where $A'$ denotes the transpose of $A$.

The matrix $\Gamma(t)$ can alternatively be calculated by solving the matrix-valued differential equation

$$\dot{\Gamma}(t) = A(t)\Gamma(t) + \Gamma(t)A(t)' + C$$

such that $\Gamma(0) = 0$. (11)

**B-3**. Then,

$$m(t) = \frac{1}{\epsilon_*} x(\epsilon_* t)$$

$$K(t) = \frac{1}{\epsilon_*} \Gamma(\epsilon_* t) . \tag{12}$$

Of particular interest is the steady-state distribution of the storage. It is easily seen that $m(t) \to m$ as $t \to \infty$, where $m = (m_j : 1 \leq j \leq d)$ is the stable solution of the differential equation (5) and is given by

$$m_j = \left( \frac{\mu_j + \sum_{k \in P(j)} \alpha_k m_k^{b_k}}{\alpha_j} \right)^{1/b_j} . \tag{13}$$

We note that the above system can be solved recursively in closed-form by solving first for the furthest upstream reservoirs and then working one's way downstream.

It is clear that $A(t) \to A$ as $t \to \infty$, where

$$A_{kj} = \begin{cases} \alpha_k b_k m_k(t)^{b_k - 1}, & \text{if } k \in P(j), \\ -\alpha_j b_j m_j(t)^{b_j - 1}, & \text{if } k = j, \text{ and} \\ 0, & \text{else.} \end{cases}$$

It is then reasonable (see (11)) to expect that $\Gamma(t) \to \Gamma$ as $t \to \infty$, where $\Gamma$ solves

$$0 = A\Gamma + \Gamma A' + C . \tag{14}$$

Since A is lower triangular (due to the numbering system adopted at the beginning of this section), and has negative diagonal elements, it therefore has strictly negative eigenvalues and is hence stable. Consequently a unique solution to the above matrix equation is guaranteed (see Arnold (1974)).

If $S(\infty)$ is the d-dimensional random variable representing the steady-state vector storage of the system, the above discussion then suggests using the approximation

$$S(\infty) \overset{D}{\approx} N(m/\epsilon_*,\ \Gamma/\epsilon_*) . \tag{15}$$

Equation (15) gives a mathematically valid approximation for the steady-state that is valid when $\epsilon_*$ is small, and is calculated directly from the model parameters, requiring only that one solve the dxd linear system of equations (14).

## 6 An approximation for the expected time for a reservoir to exceed a given level

In this section, we exploit the fact that our approximating process is a diffusion. (This is in contrast to Section 5, where our analysis relied on the Gaussian character of the approximation.) The Markov structure of a diffusion process permits one to develop (deterministic) partial differential equations (P.D.E.'s) that describe a variety of different quantitative characteristics of the process. We shall illustrate the power of this property by applying this idea to a specific calculation, namely the development of an approximation for the expected time for a reservoir to exceed a given level. Specifically, we shall be interested in calculating the expected time required for reservoir d's content to exceed level $\beta$. (We note that any other reservoir can be viewed as the furthest downstream reservoir to the reservoirs that feed into it. Consequently, if one is interested in reservoir j ($j \neq d$), one can replace the original system by the appropriate smaller system of reservoirs. Thus, our assumption is without loss of generality.)

If T is the time required for reservoir d's level to exceed $\beta$, then T can be defined as $T = \inf\{t \geq 0 : S_d(t) \geq \beta\}$. Using the approximation (7), we conclude that

$$T \overset{D}{\approx} \inf\ \{t \geq 0 :\ x_d(\epsilon_* t)/\epsilon_* + Z_d(\epsilon_* t)/\sqrt{\epsilon_*} \geq \beta\}$$

$$= \frac{1}{\epsilon_*} \inf\{u \geq 0 :\ x_d(u)/\epsilon_* + Z_d(u)/\sqrt{\epsilon_*} \geq \beta\}$$

$$= \frac{1}{\epsilon_*} \inf\{u \geq 0 :\ W_d(u)/\beta\}$$

where $W(u) = (W_1(u), \ldots, W_d(u)) :\ u \geq 0)$ is the diffusion process with i'th component given by

$$W_i(u) = \frac{1}{\epsilon_*} x_i(u) + \frac{1}{\sqrt{\epsilon_*}} Z_i(u) .$$

Let $T_t = \inf\{u \geq 0 :\ W_d(u + t) \geq \beta\}$ be the elapsed time for reservoir d's level to exceed $\beta$, taken relative to a re-defined time origin at t. Then, we can set $u(w,t) = E[T_t | W(t) = w]$ for $t \geq 0$, $w \in \mathcal{R}^d$. The diffusion $w(\cdot)$ has infinitesimal generator L given by the linear partial differential operator

$$L(\cdot) = \sum_{j=1}^{d} \left( \frac{\dot{x}_j(t)}{\epsilon_*} + \sum_{k \in P(j)} \frac{\alpha_k b_k x_k(t)^{b_k - 1} w_k}{\sqrt{\epsilon_*}} - \frac{\alpha_j b_j x_j(t)^{b_j - 1} w_j}{\sqrt{\epsilon_*}} \right) \frac{\partial \cdot}{\partial w_j}$$

$$+ \frac{1}{2} \sum_{ij=1}^{d} \frac{C_{ij}}{\epsilon_*} \frac{\partial^2 \cdot}{\partial w_i \partial w_j} .$$

The theory of diffusion processes (see Karlin and Taylor (1981)) establishes that u satisfies the P.D.E. given by

$$\frac{\partial}{\partial t} u + Lu = -1 \tag{16}$$

subject to $u(w,t) = 0$ for $w_d \geq \beta$. Hence, we can calculate an approximation to the expected time E(T) required for reservoir d's level to exceed $\beta$ by following the procedure:

**C-1.** Solve the (deterministic) P.D.E.

$$\frac{\partial u}{\partial t} + \sum_{j=1}^{d} \left( \frac{\dot{x}_j(t)}{\epsilon_*} + \sum_{k \in P(j)} \frac{\alpha_k b_k x_k(t)^{b_k - 1} w_k}{\sqrt{\epsilon_*}} - \frac{\alpha_j b_j x_j(t)^{b_j - 1} w_j}{\sqrt{\epsilon_*}} \right) \frac{\partial u}{\partial w_j}$$

$$+ \frac{1}{2} \sum_{ij=1}^{d} \frac{C_{ij}}{\epsilon_*} \frac{\partial^2 u}{\partial w_i \partial w_j} = -1$$

for $u(w,t)$ in the domain $w \in \mathcal{R}^d$, $t \geq 0$, subject to $u(w,t) = 0$ for $w_d \geq \beta$.

$$\mathbf{C-2.} \quad E(T) \approx \frac{1}{\epsilon_*} u(s_1, ..., s_d, 0) . \tag{17}$$

## 7 The single reservoir case

The mass balance equation for a single reservoir is given by the following relation

$$S(t) = s_0 + I(t) - \int_0^t aS(u)^b du$$

where $s_0$, $a$, $b > 0$, and $I(t)$ is a non-decreasing process that represents the cumulative input to time t. As required by assumption (3), we assume that the limits $\mu = \lim_{t \to \infty} EI(t)/t$ and $C = \lim_{t \to \infty} \text{var } I(t)/t$ exist.

Following the procedure outlined in Section 3, it is first necessary to solve the ordinary differential equation given by

$$\dot{x}(t) = \mu - \alpha x(t)^b , \qquad \text{with}$$

$$x(0) = x_0$$

where $x_0 = a^{1/b} s_0$ and $\alpha = 1$. The solution to the differential equation is then given by

$x(t) = \phi^{-1}(t)$, where

$$\phi(t) = \int_{x_0}^{x} \frac{du}{\mu - u^b} .$$

In particular, if b = 1, then

$$\phi(x) = \log\left(\frac{\mu - x_0}{\mu - x}\right)$$

and so the solution of the differential equation becomes $x(t) = \mu - e^{-t}(\mu - x_0)$. The next step in the procedure is to solve the stochastic differential equation for the diffusion approximation which is given by $dZ(t) = -bx(t)^{b-1}\, Z(t)\, dt + dB(t)$ where $B(\cdot)$ is a Brownian motion having a variance parameter C. Then, in the linear case, we can apply (11) to obtain the result $\Gamma(t) = C(1-\exp(-2t))/2$. Since $\epsilon_* = a$, the transient approximation (7) becomes

$$S(t) \overset{D}{\approx} N((\mu - e^{-at}(\mu - s_0 a))/a,\ C(1 - e^{-2at})/2a) . \tag{18}$$

In the case where b is not equal to one, we can easily supply a closed-form solution for the steady-state storage $S(\infty)$. In particular, (14) takes the form $\Gamma = C\mu^{(1-b)/b}/2b$. Then (15) yields

$$S(\infty) \overset{D}{\approx} N\left(\left(\frac{\mu}{a}\right)^{1/b},\ C\left(\frac{\mu}{a}\right)^{1/b} \frac{1}{2b\mu}\right) \tag{19}$$

## 8 Numerical examples

We now consider two numerical examples. In the first case we will compare the diffusion approximation to exact known results from Harrison and Resnick (1976), and in the second case we will compare the diffusion approximation to simulations for a cascade of two reservoirs.

Example 1. The behavior of the system for a single reservoir with Poisson inflows and exponentially distributed jump sizes will first be examined. The statistics of the inflow process can then be calculated as in Section (4). This case has been solved in closed form by Harrison and Resnick in the special case where the storage system empties in finite time i.e. for the case where $b < 1$. In this case define two functions $\rho(x)$ and $K(x)$ by the formulae

$\rho(x) = \lambda/(\nu a x^b)$, and

$$K(x) = \nu\rho(x)\exp(-\nu \int_0^x (1 - \rho(t))dt) ,$$

where $\lambda$ and $\nu$ are defined as in Model 1 of Section 4.

The steady-state density, $f_s(x)$, of the storage process is then given by the formula

$$f_s(x) = K(x)/\left(1 + \int_0^\infty K(t)dt\right) . \tag{20}$$

Similarly the density of the runoff process $f_r(x)$ is given by the formula $f_r(x) = abx^{b-1}f_s(ax^b)$. It is clear from these formulae and from the fact that the storage is depleted in finite time that the distributions of the storage and runoff processes have atoms at 0. A comparison of the exceedence probabilities of runoff for the diffusion process and for the exact solution of the storage process for various values of a is shown in Figure 2. In this case the inflow parameters $\lambda$ and $\mu$ are each set equal to one.

Example 2. Consider a cascade of two storage systems fed by a single exogenous inflow process $I_1(\cdot)$ into the first storage system. In this case the matrix A (see Section 5) takes the form

$$A = \begin{pmatrix} -\alpha_1 b_1 m_1^{b_1-1} & 0 \\ \alpha_1 b_1 m_1^{b_1-1} & -\alpha_2 b_2 m_2^{b_2-1} \end{pmatrix} . \tag{21}$$

We can calculate the steady-state variance of the storage process by solving Equation (14) for $\Gamma$, which results in the following formula

$$\Gamma = \frac{-C}{2} \begin{pmatrix} \frac{1}{A_{11}} & \frac{1}{A_{11}+A_{22}} \\ \frac{1}{A_{11}+A_{22}} & \frac{A_{11}}{A_{22}(A_{11}+A_{22})} \end{pmatrix} \tag{22}$$

where C is determined using equation (8) and the $A_{ij}$'s are the corresponding elements of the matrix A defined in (21). The steady-state distribution of this nonlinear cascade can then be approximated as in (15).

The diffusion approximation for the cascade was compared with simulation runs for the process. The simulation model was based on approximating the continuous time model given by equation (1) with the following approximate discrete time model

$$\begin{aligned} S_1(n+1) - S_1(n) &= I_n(\Delta t) - a_1 \Delta t S_1(n+1)^{b_1} \\ S_2(n+1) - S_2(n) &= a_1 \Delta t S_1(n+1)^{b_1} - a_2 \Delta t S_2(n+1)^{b_2} . \end{aligned} \tag{23}$$

where $\{S_1(n) : n \geq 0\}$ and $\{S_2(n) : n \geq 0\}$ are the storages of the first and second system respectively. Such a discretization is necessary in order to solve the system of integral equations (1) numerically. The inflow process $\{I_n(\Delta t) : n \geq 0\}$ was simulated using the formula

$$I_n(\Delta t) = I((n+1)\Delta t) - I(n\Delta) \tag{24}$$

where I(t) is the Neyman-Scott process described by the second model in Section 4.

In our cascade example, the time of the k'th storm origin is a Poisson process with parameter $\lambda$ equal to one, the number, $M_k$, of cells per storm origin parameter is equal to $N_k + 1$ where $N_k$ is a sequence of i.i.d. Poisson random variables with mean equal to four. The $\Gamma_{kj}$, $\beta_{kj}$ and $Y_{kj}$ are all assumed to be i.i.d. exponential random variables with means 0.5, 0.1, and 20.0 respectively. The time increment, $\Delta t$, was set equal to 0.1, and the process was simulated up to time 100000, i.e. 1000000 time steps. Figure 3 compares the stationary exceedence probability for the simulation with the diffusion approximation as b goes to zero and Figure 4 compares the

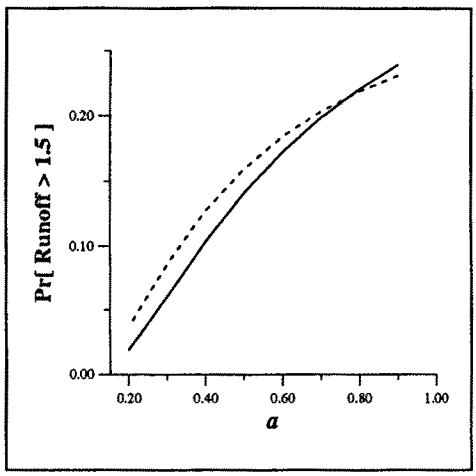


**Figure 2**. Exceedence probabilities for runoff from single reservoir with b = 0.7; solid line represents diffusion approximation and dashed line is exact solution.

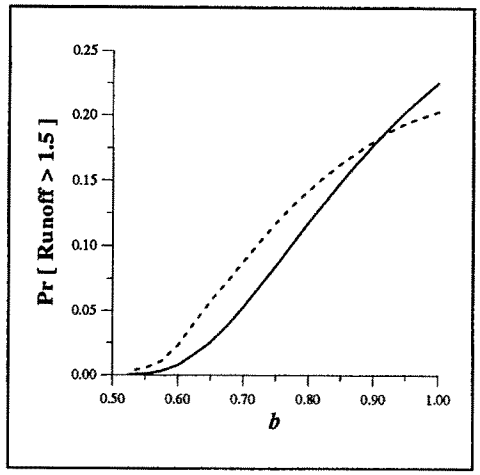


**Figure 3**. Exceedence probabilities for runoff from downstream reservoir with $a_1 = a_2 = 0.1$; solid line represents diffusion approximation and dashed line is simulation.

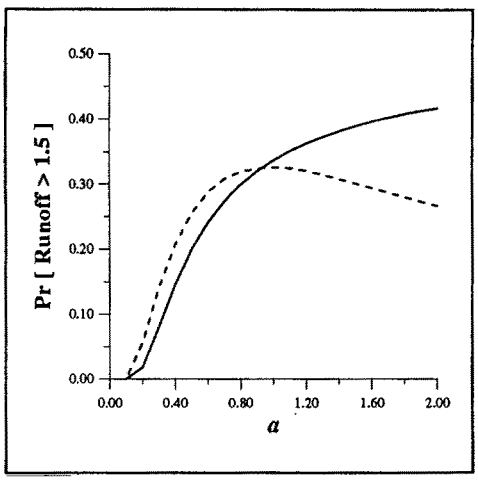


**Figure 4.** Exceedence probabilities for runoff from downstream reservoir with $b_1 = b_2 = 0.5$; solid line represents diffusion approximation and dashed line is simulation.

stationary exceedence probabilities as a goes to zero. The diffusion approximation falls within the 95% confidence interval for the simulation for all the cases shown. The confidence intervals for the estimated parameters were calculated using an assumed t distribution based on 10 replications of each simulation. The justification for this confidence interval methodology is given in Law and Kelton (1981).

## 9 Discussion

The results of the simulation indicate that the stationary behavior of the diffusion approximation is close to the simulation for a small, and for b small when a is less than one. This is what one expects from the theory since the behavior of the system of stochastic integral equations (1) converges to that of the diffusion process (6) as $\epsilon_*$ goes to zero. Small values of a are physically plausible according to Laurensen (1964).

Of course, since our diffusion approximation is based on ideas related to the central limit theorem, similar problems occur in applying this result as in applying the CLT in statistics. In the statistical context, while the theory states that the "normalized" sample mean has a distribution which is asymptotically Gaussian as the sample size gets large, there is no guarantee, for any fixed and finite n, as to how good the approximation will be. The quality of the approximation, for a given finite sample size, depends on the skewness, kurtosis, and higher order moments of the underlying population from which the sample is being gathered. (In particular, the more non-Gaussian the higher order moments, the slower the rate of convergence in the CLT and the worse the Gaussian approximation, for any given finite sample size.) In a similar spirit, while the Appendix establishes that our nonlinear watershed can be approximated, as $\epsilon_*$ goes to zero, by a certain Gaussian process, there are no guarantees, for any fixed and positive value of $\epsilon_*$, as to the quality of the approximation. For example, one would expect that our Gaussian approximation degrades as one

moves to inflow processes for which the time-averaged inflow has large skewness and kurtosis. The magnitude of the degradation is, however, difficult to predict a priori.

Consider the following example from statistics, in which one uses the CLT to approximate the probability that the sample mean differs from the population mean by at least x. It is well known that the absolute error of the Gaussian approximation to this probability tends to decrease as the sample size gets large. On the other hand, the relative error will generally tend to decrease, but then increase at some point. The reason for this is that the CLT only gives good approximations so long as x is of the order of $1/\sqrt{n}$, where n is the sample size. As n gets (very) large, x becomes of a larger order than $1/\sqrt{n}$, and the relative error begins to degrade. (On the other hand, as the sample size gets large, both the CLT approximation and the probability itself are predicting that deviations larger than x are becoming successively rarer, and the absolute error of the approximation tends to zero.) A similar behavior is observed in our simulations of the watershed. Specifically, as $\epsilon_*$ gets smaller, the absolute error of our approximation gets smaller but the percentage relative error tends to decrease and then increase. As in the standard CLT, the underlying reason is that our Gaussian approximation only gives good (relative error) approximations when the random fluctuations of the runoff, under consideration, are of the order of $\sqrt{\epsilon_*}$.

Despite the above approximation problems (which are characteristic of any application of CLT methods), we believe that our diffusion approximation offers several distinct advantages over numerical simulations:

1. For determining the stationary distribution of runoff, the diffusion approximation is easier to apply, only requiring the solution of the system of linear equations (14). On the other hand, the simulation may be fairly difficult to implement, particularly for relatively complex inflow processes such as the Neyman-Scott process.

2. When running simulations to estimate steady-state characteristics, a common problem is the need to determine the duration of the "warm-up" period, during which the initial condition of the simulation may have a particularly large impact on the measured values of the process. This problem is completely avoided when using (14).

3. When simulation is used to study the watershed (1), one must perform certain numerical integrations. These integrations require that one discretize the integrals, thereby creating an unknown and systematic error in the answers computed. Again, these discretization issues are avoided with equations such as (14).

4. With an analytical approximation such as ours, certain qualitative behaviors can be predicted a priori. For example, our analysis suggests that the parameter $\epsilon_*$ is a key parameter when dealing with "power law" watersheds. Our theory suggests that the storage of such watersheds will be of order $1/\epsilon_*$, with random fluctuations that are substantially smaller, namely of order $1/\sqrt{\epsilon_*}$.

5. In reservoir management contexts, it is often of interest to solve certain optimization problems. The availability of a simple-to-evaluate steady-state solution such as (14) makes the numerical solution of such problems significantly more viable.

## 10 Conclusions

In this paper, we have developed a diffusion approximation for a watershed consisting of d reservoirs with non-linear power law release rules. Our simulation experiments suggest that it works reasonably well for some values of the release rule parameters which are physically plausible. Of course, the assumptions guaranteeing the validity of our approximation are not universally applicable, and further research is therefore necessary to more fully delineate its domain of applicability.

## Appendix - Justification for the diffusion approximation

We wish to show that when the parameter $\epsilon_*$ is small, then the diffusion approximation is mathematically justified. To formulate this problem mathematically, let

$$S_j(\epsilon,t) \;=\; S_j(\epsilon,0) + I_j(t) + \sum_{k\in P(j)} \int_0^t \alpha_k \epsilon^{b_k} S_k(\epsilon,u)^{b_k} du - \int_0^t \alpha_j \epsilon^{b_j} S_j(\epsilon,u)^{b_j} du, \quad 1 \le j \le d\ , \tag{25}$$

be the storage contents processes associated with the parameter $\epsilon$. We need to show that $S_j(\epsilon,t)$ can be approximated as in (A-4) when $\epsilon \downarrow 0$.

To study the behavior of $\{S_j(\epsilon,t):\ 1 \le j \le d,\ t \ge 0\}$ as $\epsilon \downarrow 0$, we need to strengthen the hypothesis (3) on the inflow processes somewhat. (In particular, we need to say something about the distribution of the inflow processes.) In addition, since our approximation requires a rescaling of the spatial coordinates, our initial condition for $S_j(\epsilon,0)$ ought to reflect this. Our precise mathematical requirement is:

$$\epsilon^{-1/2}\left[\epsilon S_1(\epsilon,0) - x_1, ..., \epsilon S_d(\epsilon,0) - x_d, \epsilon I_1(t/\epsilon) - \mu_1 t, ..., \epsilon I_d(t/\epsilon) - \mu_d t\right]$$

$$\Rightarrow\ \left[Z_1(0), ..., Z_d(0), B_1(t), ..., B_d(t)\right]\ , \tag{26}$$

as $\epsilon \downarrow 0$, where $\Rightarrow$ denotes convergence in distribution and the process $(B_1(t), ..., B_d(t))$ is assumed to be a Brownian motion with zero mean and covariance matrix C. From a practical standpoint, the strengthening of (3) to the above distributional requirement is quite mild. Virtually all stochastic processes that exhibit the behavior (3) also exhibit the Brownian behavior assumed above (see, for example, Ethier and Kurtz (1986)).

Under the assumption (26), we can now proceed to analyze $S_j(\epsilon,\cdot)$ as $\epsilon \downarrow 0$. Note that (25) can be rewritten as

$$\epsilon S_j(\epsilon,t/\epsilon) \;=\; \epsilon S_j(\epsilon,0) + I_j(t/\epsilon) + \sum_{k\in P(j)} \int_0^t \alpha_k (\epsilon S_k(\epsilon,u/\epsilon))^{b_k} du - \int_0^t \alpha_j (\epsilon S_j(\epsilon,u/\epsilon))^{b_j} du\ .$$

By assumption, we have $\epsilon I_j(t/\epsilon) \Rightarrow \mu_j t$ as $\epsilon \downarrow 0$, and so assuming in addition that the limit process

$$x_j(t) \;=\; \lim_{\epsilon \downarrow 0} \epsilon S_j(\epsilon,t/\epsilon) \tag{27}$$

exists for $1 \le j \le d$, the limit must satisfy the integral equation

$$x_j(t) \;=\; x_j + \mu_j t \;+\; \sum_{k\in P(j)} \int_0^t \alpha_k x_k(u)^{b_k} du - \int_0^t \alpha_j x_j(u)^{b_j} du, \quad 1 \le j \le d \; .$$

Expressed as a differential equation, the above result becomes

$$\dot{x}_j(t) \;=\; \mu_j + \sum_{k\in P(j)} \alpha_k x_k(t)^{b_k} - \alpha_j x_j(t)^{b_j}, \quad 1 \le j \le d \; , \quad \text{and}$$

$$x_j(0) \;=\; x_j \quad 1 \le j \le d \; .$$

In order to establish the existence of the limit, one needs to prove the relative compactness of the family of processes $\{\epsilon S_j(\epsilon, t/\epsilon) : \epsilon \ge 0\}$. This can be done by appealing to standard tools from, for example, Ethier and Kurtz (1986). We can view (27) as a law of large numbers for $S_j(\epsilon, \cdot)$. To develop a "central limit theorem" we rewrite (25) as

$$\epsilon^{-1/2}\left[\epsilon S_j(\epsilon, t/\epsilon) - x_j(t)\right] = \; \epsilon^{-1/2}\left[\epsilon S_j(\epsilon, 0) - x_j\right] \; + \; \epsilon^{-1/2}\left[\epsilon I_j(t/\epsilon) - \mu_j t\right]$$

$$+ \sum_{k\in P(j)} \int_0^t \alpha_k \epsilon^{-1/2}\left[(\epsilon S_k(\epsilon, u/\epsilon))^{b_j} - x_k(u)^{b_k}\right] du$$

$$- \int_0^t \alpha_j \epsilon^{-1/2}\left[(\epsilon S_j(\epsilon, u/\epsilon))^{b_j} - x_j(u)^{b_j}\right] du \; .$$

A Taylor series expansion to one term then establishes that

$$\epsilon^{-1/2}\left[(\epsilon S_k(\epsilon, u/\epsilon))^{b_k} - x_k(u)^{b_k}\right] \;=\; b_k \zeta_k(\epsilon, u)^{b_k - 1}\left[\epsilon^{-1/2}(\epsilon S_k(\epsilon, u/\epsilon) - x_k(u))\right]$$

where $\zeta_k(\epsilon, u)$ lies between $x_k(u)$ and $\epsilon\, S_k(\epsilon, u/\epsilon)$ and hence tends to $x_k(u)$ as $\epsilon \downarrow 0$.

Assume, for the moment, that we can establish the existence of the limit

$$\epsilon^{-1/2}(\epsilon S_1(\epsilon, t/\epsilon)) - x_1(t), ..., \epsilon S_d(\epsilon, t/\epsilon) - x_d(t)) \Rightarrow (Z_1(t), ..., Z_d(t)) \; .$$

Since $\epsilon^{-1/2}[\epsilon S_1(\epsilon, 0) - x_1 ..., \epsilon S_d(\epsilon, 0) - x_d] \Rightarrow [Z_1(0), ..., Z_d(0)]$, the limit process $(Z_1(t), ..., Z_d(t)$ must then satisfy the following stochastic differential equation

$$Z_j(t) \;=\; Z_j(0) + B_j(t) + \sum_{k\in P(j)} \int_0^t \alpha_k b_k x_k(u)^{b_k - 1} Z_k(u) du - \int_0^t \alpha_j b_j x_j(u)^{b_j - 1} Z_j(u) du \; .$$

As before, the existence of the limit may be justified by using a relative compactness argument (Ethier and Kurtz (1986)).

## Acknowledgements

The research of the second author was supported by the Army Research Office under Contract No. DAAL-03-91-G-0101 and by the National Science Foundation under Grant NSF-DDM-9101580.

## References


Anderson, T.W. 1971: The Statistical Analysis of Time Series, John Wiley and Sons, Inc.

Arnold, L. 1974: Stochastic Differential Equations: Theory and Applications, John Wiley and Sons, Inc.

Bratley, P., Fox, B.L., Schrage, L. 1987: A Guide to Simulation, Springer, New York

Brillinger, D. 1981: Time Series: Data Analysis and Theory, Gregg-McGraw

Ethier, S.N.; Kurtz, T.G. 1986: Markov Processes: Characterization and Convergence, John Wiley and Sons, Inc.

Glynn, J.E. 1989: A discrete time storage system with a general release rule, J. Appl. Prob. 26, 566-583

Harrison, J.M.; Resnick, S.I. 1976: The stationary distribution and first exit probabilities of a storage process with a general release rule, Mathematics of Operations Research 3, 347-358

Harrison, J.M.; Taylor, A.J. 1978: Optimal control of a Brownian storage system, Stochastic Processes and their Applications 6, 179-194

Hardson, J.M.; Shepp, L.A. 1984: A tandem storage system and its diffusion limit, Stochastic Processes and their Applications 16, 257-274

Hughes, D.A.; Murrell, H.C. 1986: Nonlinear runoff routing - A comparison of solution methods, J. Hydrol. 85, 339-347

Karlin, S.; Taylor, H.C. 1975: A First Course in Stochastic Processes, Academic Press, Inc., New York

Kadin, S.; Taylor, H.C. 1981: A Second Course in Stochastic Processes, Academic Press, Inc., New York

Klemel, V. 1978: Physically based stochastic hydrologic analysis, Advances in Hydroscience 11

Klemeš, V.; Boruvka, L. 1975: Output from a cascade of discrete time linear reservoirs with stochastic control, J. Hydrol. 27, 1-13

Klemeš, V.; Klemeš, I.; Glynn, J.E. 1985: Discrete time linear cascade under time averaging, J. Hydrol. 77, 107-123

Laurenson, E.M. 1964: A catchment model for runoff routing, Journal of Hydrology 2, 141-163

Law, A.M.; Kelton, D. Simulation Modelling and Analysis, McGraw

Mein, R.G.; Laurensen, E.M.; McMahon, T.A. 1974: Simple nonlinear model for flood estimation, Journal of the Hydraulics Division 100, HY11, 1507-1518

Moran, P.A.P. 1967: Dams in series with continuous release, J. Appl. Prob. 4, 380-388

Nash, J. E., The form of the instantaneous unit hydrograph, International Association of Scientific Hydrology, Publication No. 45 Vol. 3, 14-121, 1957

Nash, J.E. 1959: A note on the Muskingum flood-routing method, J. Geophys. Res. 64(8), 1053-1056

Papoulis, A. 1965: Probability, Random Variables, and Stochastic Processes. McGraw-Hill

Pliska, S. R. 1975: A diffusion process model for the optimal operation of a reservoir system, J. Appl. Prob. 12, 859-863

Puente, C.E.; Bierkens, M.F.P.; Diaz-Granados, M.A.; Dik, P.E.; López, M.M. 1993: Practical use of analytically derived runoff models based on rainfall point processes, Water Resour. Res. 29(10), 3551-3560

Rodriguez-Iturbe, I.; Cox, D.R.; Isham V. 1987: Some models for rainfall based on stochastic point processes, Proceedings of the Royal Society of London, Series A 410, 269-288

Smith, N.M.H.; Yeo, G.F. 1981: On a general storage problem and its approximating solution, Advances in Applied Probability 13, 567-602

Unny, T.E. 1984: Numerical integration of stochastic differential equations in catchment modelling, Water Resour. Res. 20(3), 360-368

Unny, T.E.; Karmeshu 1984: Stochastic nature of outputs from conceptual reservoir model cascades, J. Hydrol. 68, 161-180

Yamada, K. 1983: Diffusion approximations for storage processes, Operations Res. Lett. 2(1), 22-26

Yamada, K. 1984: Diffusion approximations of storage processes with general release rules, Mathematics of Operations Res. 9, 459-470