POISSON'S EQUATION
FOR THE RECURRENT M/G/1 QUEUE

Peter W. Glynn
Department of Operations Research
Stanford University, Stanford, CA 94305-4022

ABSTRACT:

This paper shows how to calculate solutions to Poisson's equation for the waiting time sequence of the recurrent M/G/1 queue. The solutions are used to construct martingales that permit us to study additive functionals associated with the waiting time sequence. These martingales provide asymptotic expressions, for the mean of additive functionals, that reflect dependence on the initial state of the process. In addition, we show how to explicitly calculate the scaling constants that appear in the central limit theorems for additive functionals of the waiting time sequence.

KEYWORDS:

Additive functional, Poisson's equation, martingales, central limit theorems

---

Research was supported by the U.S. Army Research Office under Contract DAAL-03-88-K-0063, and a grant of the Natural Sciences of Engineering Research Council of Canada.

1. INTRODUCTION

For an M/G/1 queue, let W_n be the amount of time that the nth customer is required to wait until his service is initiated. It is well-known that the sequence $W = \{W_n : n \geq 0\}$ satisfies the recursion

$$W_{n+1} = [W_n + B_n - A_{n+1}]^+ \tag{1.1}$$

for $n \geq 0$ ($[x]^+ = \max(x, 0)$), where B_n is the service time of the nth customer and A_{n+1} is the inter-arrival time for the $(n+1)$st customer. (See, for example, p. 493 of Karlin and Taylor [1981]).

Of course, for the M/G/1 queue, it is assumed that $A = \{A_n : n \geq 1\}$ and $B = \{B_n : n \geq 0\}$ are independent sequences of i.i.d. r.v.'s, in which the A_n's are exponential with parameter $\lambda > 0$, and the B_n's are non-negative r.v.'s with finite positive mean. One important consequence of this assumption is that W is then a Markov chain taking values in the state space $\Re^+ = [0, \infty)$.

Our primary objective in this paper is to use Markov chain methods to analyze the behavior of time averages of the form

$$r_n(f) = \frac{1}{n} \sum_{k=0}^{n-1} f(W_k), \tag{1.2}$$

where f is a real-valued (Borel-measurable) function defined on $\Re^+$. When $\rho \equiv \lambda E B_1 < 1$, $r_n(f)$ has a limit $r(f)$ as $n \to \infty$, and the Pollaczek-Khintchine formula identifies this limit. In practice, this limit is used as an approximation to the distribution of $r_n(f)$, and, consequently, it is of some interest to obtain results giving the rate at which $r_n(f)$ converges to $r(f)$.

In this paper, we derive formula which describe the fluctuations of $r_n(f)$ about its limit. One way to do this is to obtain asymptotic relations of the form

$$E r_n(f) = r(f) + \frac{1}{n} c(f) + o(1/n). \tag{1.3}$$

A second possibility is to develop central limit theorems (CLT's) of the form

$$n^{1/2}(r_n(f) - r(f)) \Longrightarrow \sigma(f) N(0,1) \tag{1.4}$$

as $n \to \infty$; in (1.4), $\Longrightarrow$ denotes weak convergence, and $N(0,1)$ is a standard normal r.v.

Our approach is to express the process $S_n(f) = nr_n(f)$, in terms of a martingale sequence. The martingale is constructed by solving an integral equation known, in the Markov chain literature, as Poisson's equation. Using theory of random walk, we are able to solve Poisson's equation explicitly. The martingale structure of $S_n(f)$ then easily leads to limit theorems of the form (1.3) and (1.4).

This paper is organized as follows. In Section 2, we review the basic structure of the M/G/1 waiting time sequence. Section 3 constructs a probabilistic solution to Poisson's equation by using the regenerative structure of the recurrent M/G/1 queue, and analyzes the uniqueness properties of the solution. Section 4 continues the analysis of Poisson's equation, by using random walk theory to explicitly derive closed-form analytical expressions for the probabilistically-expressed solution kernel discussed in Section 3. In Section 5, the martingale structure obtained from Poisson's equation is used to derive asymptotic expressions of the form (1.3). This analysis generalizes, to arbitrary f, results due to Heathcote and Winer [1969] for the case $f(x) = x$. Section 6 concludes with a discussion of the central limit behavior of $S_n(f)$. The martingale central limit theorem permits us to obtain central limit theorems of the form (1.4), together with explicit expressions for $r(f)$ and $\sigma^2(f)$. The explicit expressions derived in this section generalize results due to Blomqvist [1967], Daley [1968], and Pagurek and Woodside [1979], in which these constants were calculated for $f(x) = x$. A more recent, related paper is that of Asmussen and Bladt [1993], in which Poisson's equation is solved for the virtual waiting time process of a certain class of queues having dependent input streams.

2. BASIC PROPERTIES OF THE M/G/1 WAITING TIME SEQUENCE

As discussed in Section 1, the sequence W is a Markov chain on $\Re^+$. To describe the transition kernel of W, set $X_{n+1} = B_n - A_{n+1}$ and observe that

$$P\{X_{n+1} \le x\} = \int_0^\infty P\{B_n \le x + y\}\lambda e^{-\lambda y}\ dy = \int_0^\infty \lambda e^{-\lambda(z-x)}G(z)\ dz$$

where $G(\cdot)$ is the distribution function of B_n. The r.v. X_{n+1} therefore has a Lebesgue density $k(\cdot)$ given by

$$k(x) = \int_0^\infty \lambda^2(G(z) - G(x))e^{-\lambda(z-x)}\ dz. \tag{2.1}$$

It follows from the recursion (1.1) that the transition kernel $P(x, A) \equiv P\{W_{n+1} \epsilon A | W_n = x\}$ has the form

$$P(x, A) = pe^{-\lambda x}\delta_0(A) + \int_A I_{(0,\infty)}(y)k(y - x)\ dy \tag{2.2}$$

where $\delta_z(\cdot)$ is a unit point mass measure at z, $I_A(y)$ is a function which is 1 or 0 depending on whether or not $y \in A$ and

$$p = \int_0^\infty \lambda e^{-\lambda z} G(z) \ dz.$$

The transition kernel P has a certain smoothness property which we shall later need to exploit.

(2.3) **LEMMA.** If $0 \le x \le y$, then

$$P(x, dz) \le \exp(\lambda(y - x)) P(y, dz).$$

PROOF. Let $h = y - x$ and note that for any u, (2.1) yields the inequality

$$\begin{aligned} k(u+h) &= \int_{u+h}^\infty \lambda^2 [G(z) - G(u+h)] e^{-\lambda(z-u-h)} dz \\ &\le \int_{u+h}^\infty \lambda^2 [G(z) - G(u)] e^{-\lambda(z-u-h)} dz \\ &\le e^{\lambda h} \int_u^\infty \lambda^2 [G(z) - G(u)] e^{-\lambda(z-u)} dz \\ &= e^{\lambda h} k(u). \end{aligned} \tag{2.4}$$

Relation (2.4), when substituted in (2.2), proves the result.

If $\rho \le 1$, the process W visits the state 0 infinitely often (see Karlin and Taylor [1981], p. 496), and it follows immediately that W is then a Harris recurrent Markov chain (see Athreya and Ney [1978]). For a measure μ on $\Re^+$ and a function $f : \Re^+ \to \Re$, set

$$\begin{aligned} \mu f &= \int_{[0,\infty)} f(y) \mu(dy) \\ (\mu P^n)(\cdot) &= \int_{[0,\infty)} \mu(dy) P^n(y, \cdot) \\ (P^n f)(\cdot) &= \int_{[0,\infty)} f(y) P^n(\cdot, dy) \end{aligned}$$

where $P^n(x, A) \equiv P\{W_n \in A \mid W_0 = x\}$. Finally, let $P_x(\cdot)$ and $E_x(\cdot)$ denote, respectively, probability and expectation taken with respect to the measure $P\{W \in \cdot \mid W_0 = x\}$.

The theory of Harris chains dictates the existence of a non-trivial σ-infinite measure ν, which is unique up to a multiplicative constant, such that ν is invariant for P in the sense that $\nu = \nu P$ (see Athreya and Ney [1978]and Orey [1971]). Furthermore, ν has the representation

$$\nu(\cdot) = E_0\left\{\sum_{k=0}^{T-1} I(W_k \epsilon \cdot)\right\} \tag{2.5}$$

where $T = \inf\{n \geq 1 : W_n = 0\}$, and $I(A)$ is the indicator r.v. of the event A. If $S(0) = W_0$ and $S(k) = X_1 + \ldots + X_k$ for $k \geq 1$, observe that $S(k) = W_k$ for $k < T$, so

$$\nu(\cdot) = \delta_0(\cdot) + \sum_{k=1}^{\infty} P_0\{S(k)\epsilon \cdot; \;\; S(1) > 0, \ldots, S(k) > 0\}. \tag{2.6}$$

As is well-known in random walk theory, the exchangeability of the X_k's implies that (2.6) can be re-written as

$$\begin{aligned} \nu(\cdot) &= \delta_0(\cdot) + \sum_{k=1}^{\infty} P_0\{S(k)\epsilon \cdots; \;\; S(k) > S(k-1), \ldots, S(k) > 0\} \\ &= \sum_{k=0}^{\infty} P_0\{S(N_k)\epsilon \cdot\} \end{aligned} \tag{2.7}$$

where the N_k's are the ascending ladder epochs for $S(\cdot)$ defined by $N_0 = 0$, $N_{k+1} = \inf\{n > N_k : S(n) > S(N_k)\}$. Since X_n has an exponential tail for $x < 0$ (see (2.1)), the occupation measure (2.7) for the process $S(N_k)$ can be calculated, using Wiener-Hopf theory, and one finds that

$$\nu(dx) = \sum_{k=0}^{\infty} \rho^k H^{(k)}(dx), \tag{2.8}$$

where $H^{(k)}(\cdot)$ is the kth convolution of the probability distribution $H(\cdot)$ with density $h(\cdot)$ defined by $h(\cdot) = \bar{G}(\cdot)/EB_1$, and $\bar{G}(\cdot) = 1 - G(\cdot)$ (see Feller [1971], p. 405). Note that by (2.5), $\nu(\Re^+) = E_0 T$, so (2.8) shows that $E_0 T = (1-\rho)^{-1}$ for $\rho < 1$, whereas $E_0 T = \infty$ for $\rho = 1$.

When $\nu(\cdot)$ is finite, it is more convenient to work with the normalized version of ν. Thus, if $\rho < 1$, we let $\pi(\cdot) = (1-\rho)\nu(\cdot)$, and if $\rho = 1$, we set $\pi(\cdot) = \nu(\cdot)$. Equation (2.8) is just a statement that the Pollaczek-Khintchine formula, which is well-known for $\rho < 1$, also defines an invariant measure for P when $\rho = 1$.

The following result is a statement of the strong law of large numbers for Harris chains (see Theorem 4.3 of Revuz [1984]). It can also be proved by standard regenerative process arguments.

(2.9) **THEOREM.** If $\rho \leq 1$ and $\pi|f| < \infty$, then $r_n(f) \to \pi f \;\; P_x$ a.s. as $n \to \infty$.

Thus, if $\rho \leq 1$, the limit $\alpha(f)$ discussed in Section 1 is given by πf. We will later need the following result.

(2.10) **LEMMA.** For $\rho \leq 1$, the measure ν has the form

$$\nu(dx) = \delta_0(dx) + I_{[0,\infty)}(x) r(x) \;\; dx \tag{2.11}$$

where $r(\cdot)$ is bounded away from zero on compact sets.

PROOF. Since H has a density h, the convolution $H^{(n)}$ has a density $h^{(n)}$, so (2.8) yields all the assertions of the lemma with the exception of the positivity of r. Since h is decreasing and $G(0) < 1$, there exists $\epsilon > 0$ such that $h(x) \geq h(\epsilon) > 0$ for $0 \leq x \leq \epsilon$. We shall show inductively that $h^{(n)}(x)$ is bounded away from zero on $2^{-1}[\epsilon n, \epsilon(n+1)]$. This is immediate for $n = 1$ and for $n \geq 1$,

$$\begin{aligned} h^{(n)}(x) &= \int_0^x h^{(n-1)}(x-y) h(y) dy \\ &\geq h(\epsilon) \int_0^\epsilon h^{(n-1)}(x-y) dy \\ &\geq h(\epsilon) \int_{2^{-1}\epsilon[n-1,n-1]} h^{(n-1)}(z) dz > 0; \end{aligned}$$

this provides the required positivity of r.

3. A SOLUTION KERNEL FOR POISSON'S EQUATION

Our goal here is to find a solution kernel Γ to the equation

$$(I - P)g = f. \tag{3.1}$$

In other words, we shall construct a family of σ-finite measures $\{\Gamma(x, \cdot) : x \in \Re^+\}$ by which we may represent solutions g to Poisson's equation in the form

$$g(x) = (\Gamma f)(x) \equiv \int_{[0,\infty)} f(y) \Gamma(x, dy).$$

Before stating our result, it is worth noting that if g is a π-integrable solution to Poisson's equation, then f is π-integrable, and $\pi f = \pi(I - P)g = 0$; in our theorem, we will therefore consider the question of solvability of Poisson's equation only when $\pi f = 0$. Our candidate solution kernel is

$$\Gamma(x, \cdot) = E_x \left\{ \sum_{k=0}^{T-1} I(W_k \epsilon \ \cdot) \right\}. \tag{3.2}$$

While most of the conclusions of our next theorem follow from the general theory for Poisson's equation for Markov chains (see Neveu [1972] and Nummelin [1985]), we offer a direct proof in the belief that readers will find it instructive.

(3.3)**THEOREM.** The set function $G(x, \cdot)$ defined by (3.2) is a σ-finite measure; for $\rho < 1$, $\Gamma(x, \cdot)$ is a finite measure. Furthermore, if $\pi|f| < \infty$, then $(\Gamma|f|)(x) < \infty$ for all $x \geq 0$. If $\pi f = 0$, then $g(x) = (\Gamma f)(x)$ solves Poisson's equation $(I - P)g = f$.

PROOF. Suppose $\pi(A) < \infty$. Then, by (2.5),

$$\nu(A) = E_0 \left\{ \sum_{k=0}^{T-1} I(W_k \epsilon A) \right\} < \infty. \tag{3.4}$$

Fix $x \geq 0$ and set $T(x) = \inf\{n \geq 0 : W_n > x\}$. By Lemma 2.10, $\pi((x, \infty)) > 0$, so that (2.5) proves that $P_0\{T(x) < T\} > 0$. By (3.4) and the strong Markov property at $T(x)$,

$$\begin{aligned} \infty > \nu(A) &\geq E_0 \left\{ \sum_{k=T(x)}^{T-1} I(W_k \epsilon A) \right\} \\ &= \int_{(x,\infty)} \Gamma(y, A) P_0\{W_{T(x)} \epsilon \ dy; \ \ T(x) < T\}. \end{aligned} \tag{3.5}$$

Hence, there exists $y > x$ such that

$$\infty > \Gamma(y, A) \geq E_y \left\{ \sum_{k=1}^{T-1} I(W_k \epsilon A) \right\} = \int_{(0,\infty)} P(y, dz) \Gamma(z, A).$$

Lemma 2.3 therefore implies that

$$\Gamma(x, A) \leq 1 + e^{\lambda(y-x)} \int_{(0,\infty)} P(y, dz) \Gamma(z, A) < \infty. \tag{3.6}$$

Since π is σ-finite, it follows that there exists $A_k \nearrow \Re^+$ such that $\Gamma(x, A_k) < \infty$ for all k. A straightforward verification then proves that $\Gamma(x, \cdot)$ is a σ-finite measure. Of course, if $\rho < 1$, then $\nu(\Re^+) < \infty$ so the measure $\Gamma(x, \cdot)$ is then finite.

Since $\Gamma(x,\cdot)$ is a σ-finite measure, a standard argument, based on approximating f by simple functions, shows that Γf is representable as

$$(\Gamma f)(x) = E_x\left\{\sum_{k=0}^{T-1} f(W_k)\right\} \tag{3.7}$$

provided $(\Gamma|f|)(x) < \infty$. Setting $x = 0$ and using (2.5), we conclude that if $\pi|f| < \infty$, then

$$E_0\left\{\sum_{k=0}^{T-1} |f(W_k)|\right\} < \infty. \tag{3.8}$$

Based on (3.8), an argument similar to that yielding (3.6) shows that $(\Gamma|f|)(x) < \infty$, for all $x \geq 0$.

Suppose now that $\pi f = 0$. If $g(x) = (\Gamma f)(x)$, observe that

$$\begin{aligned} g(x) &= E_x\left\{\sum_{k=0}^{T-1} f(W_k)\right\} \\ &= f(x) + E_x\{(\Gamma f)(W_1);\ T > 1\} \\ &= f(x) + E_x g(W_1) - E_x\{g(W_1);\ T = 1\} \\ &= f(x) + (Pg)(x) - (\Gamma f)(0)\cdot P_x\{T = 1\} \\ &= f(x) + (Pg)(x) - \nu f \cdot P_x\{T = 1\} \\ &= f(x) + (Pg)(x) \end{aligned}$$

so that g solves Poisson's equation.

We have therefore shown that if $\pi f = 0$, then Poisson's equation is solvable. We now turn to the uniqueness problem for Poisson's equation. First, observe that if g solves Poisson's equation, then so does $\hat{g}(\cdot) = g(\cdot) + c$ where c is an arbitrary constant. Thus, the most that one can hope for is that the solution g to Poisson's equation is unique, up to an additive constant.

(3.9) **THEOREM.** Let g be a π-integrable solution to Poisson's equation. Then $\pi f = 0$, and $g(\cdot) = (\Gamma f)(\cdot) + c$, for some constant c.

The proof of this theorem will depend on the martingale structure for $S_n(f)$, to which we alluded in Section 1. Note that $(I - P)g = f$, then

$$\begin{aligned} S_n(f) &= \sum_{k=0}^{n} g(W_k) - (Pg)(W_k) \\ &= \sum_{k=1}^{n+1} D_k + g(W_0) - g(W_{n+1}) \end{aligned}$$

where $D_k = g(W_k) - (Pg)(W_{k-1})$. Thus, if the $g(W_k)$'s are P_x-integrable, $E_x\{D_{k+1}|\mathcal{F}_k\} = 0$ P_x a.s. where $\mathcal{F}_k$ is the σ-field generated by $W_0, \ldots, W_k$. In this case,

$$M_n = \sum_{k=1}^{n} D_k + g(W_0) \tag{3.11}$$

will be a P_x-martingale; alternatively, (3.10) allows us to write M_{n+1} as

$$M_{n+1} = S_n(f) + g(W_{n+1}) \tag{3.12}$$

so that $S_n(f)$ is "almost" a martingale.

PROOF OF THEOREM 3.9. Since g is π-integrable, so is Pg, and it follows from Theorem 3.3 that $(\Gamma|g|)(x) < \infty$ and $(\Gamma|Pg|)(x) < \infty$. Hence,

$$E_x|g(W_{T\wedge n})| \leq (\Gamma|g|)(x) + |g(0)| < \infty$$

for all $n \geq 0$, where $a \wedge b \equiv \min(a, b)$. Thus, $M_{T\wedge n}$ is a P_x-martingale (see Proposition 5.26 of Breiman [1968]), so

$$E_x M_{T\wedge n} + E_x M_0 = g(x). \tag{3.13}$$

Of course, as $n \to \infty$, $M_{T\wedge n} \to M_T$ P_x a.s. Also,

$$|M_{T\wedge n}| \leq \sum_{k=0}^{T-1} |g(W_k)| + (|Pg|)(W_k) + |g(0)|$$

and the dominating r.v. is P_x-integrable since $(\Gamma|g|)(x) + (\Gamma|Pg|)(x) < \infty$. By the dominated convergence theorem, (3.13) then yields $E_x M_T = g(x)$, which can be rewritten by (3.12) as

$$E_x\left\{\sum_{k=0}^{T-1} f(W_k)\right\} + g(0) = g(x);$$

setting $c = g(0)$ we obtain the theorem.

This uniqueness theorem will be used later to obtain certain necessity conditions. Note also that if g_1, g_2 are any two solutions to Poisson's equation, then $u = g_1 - g_2$ solves

$$u = Pu; \tag{3.14}$$

a function u satisfying (3.14) is said to be harmonic. Conversely, if g is a solution to Poisson's equation, then so is $\hat{g} = g + u$, where u is harmonic. The study of harmonic functions is central to the potential theory for W.

Since W is a Harris chain, a classical result states that all bounded harmonic functions are constants (see Revuz [1984]). Setting $f = 0$ in Theorem 3.9, we obtain the stronger result (stronger when $\rho < 1$) that all π-integrable harmonic functions are constants. In the Appendix, we further strengthen the result for the M/M/1 waiting time sequence.

(3.15) **THEOREM.** If $G(x) = 1 - \exp(-\mu x)$ for $x \geq 0$, with $\lambda < \mu$, then all finite-valued harmonic functions are constants.

4. EXPLICIT CALCULATION OF THE SOLUTION KERNEL

In this section, we use the special structure of W to explicitly calculate

$$\Gamma(x, \cdot) = E_x \left\{ \sum_{k=0}^{T-1} I(W_k \epsilon \ \cdot) \right\}.$$

Let $\eta_0 = 0$, $\eta_{n+1} = \inf\{k > \eta_n : S(k) < S(\eta_n)\}$ be the sequence of strict descending ladder epochs for the random walk $S(\cdot)$, and put $N = \inf\{n \geq 0 : S(\eta_n) \leq 0\}$, so that $T = \eta_N$.

(4.1) **PROPOSITION.** The measure $\Gamma(x, \cdot)$ has the representation

$$\Gamma(x, A) \int_{(0,\infty)} \nu(A - y) U_x(dy)$$

where

$$U_x(\cdot) = E_x \left\{ \sum_{k=0}^{\infty} I(S(\eta_k) \epsilon \ \cdot) \right\}.$$

PROOF. Note that

$$\begin{aligned} \Gamma(x, \cdot) &= E_x \left\{ \sum_{k=0}^{N-1} \sum_{j=\eta_k}^{\eta_{k+1}-1} I(S(j) \epsilon \ \cdot) \right\} \\ &= \sum_{k=0}^{\infty} E_x \left\{ \sum_{j=\eta_k}^{\eta_{k+1}-1} I(S(j) \epsilon \ \cdot); \ \ N > k \right\}. \end{aligned} \tag{4.2}$$

But by the strong Markov property applied at time η_k, and the spatial homogeneity of random walk,

$$\begin{aligned} &E_x \left\{ \sum_{j=\eta_k}^{\eta_{k+1}-1} I(S(j) \epsilon A); \ \ N > k \right\} \\ &= E_x\{\Gamma(0, A - S(\eta_k)); \ \ N > k\} \end{aligned}$$

so (4.2) yields

$$\Gamma(x,\cdot) = \sum_{k=0}^{\infty} E_x\{\Gamma(0, A - S(\eta_k)); \quad S(\eta_k) \geq 0\} \tag{4.3}$$
$$= \int_{[0,\infty)} \nu(A-y) U_x(dy).$$

We now turn to calculation of $U_x(\cdot)$; for $x < 0$, (2.1) shows that

$$P\{X_n \epsilon \; dx\} = p\lambda e^{\lambda x} \; dx.$$

As a consequence, we obtain the following lemma.

(4.4) **LEMMA.** The ladder increments $\{S(\eta_{k+1}) - S(\eta_k) : k \geq 0\}$ form an i.i.d. sequence with common distribution given by

$$P_y\{S(\eta_{k+1}) - S(\eta_k) \epsilon \; dx\} = I_{(-\infty,0]}(x) \lambda e^{\lambda x} \; dx. \tag{4.5}$$

The i.i.d. r.v. structure of the descending ladder increments is, of course, a well-known property of random walk with negative drift (see, for example, Feller [1971], p. 193). The exponential distribution (4.5) can be obtained from Wiener-Hopf theory (see Feller [1971] p. 403). As an alternative, one can determine (4.5) by straightforward calculation, using the "memoryless" property of the exponential left tail of X_n; for a similar argument, see Theorem 24.4 of Billingsley [1979].

(4.6) **PROPOSITION.** For $x \geq 0$,

$$U_x(dy) = I_{(-\infty,x)}(y)\lambda \; dy + \delta_x(dy).$$

PROOF. By spatial homogeneity of the random walk $S(\eta_k)$,

$$U_x(A) = \left\{ \sum_{k=0}^{\infty} I(S(\eta_k) \epsilon A) \right\} \tag{4.7}$$
$$= E_0 \left\{ \sum_{k=0}^{\infty} I(S(\eta_k) \epsilon A - x) \right\}$$
$$= E_0 \left\{ \sum_{k=0}^{\infty} I(-S(\eta_k) \epsilon x - A) \right\}.$$

Under P_0, Lemma 4.4 implies that $-S(\eta_k)$ is the sum of k i.i.d. exponentials with parameter λ so the last line of (4.7) is the Poisson renewal measure of the set $x - A$. Thus,

$$\begin{aligned} U_x(A) &= \delta_0(x-A) + \int_{x-A} I_{(0,\infty)}(y)\lambda \ dy \\ &= \delta_x(A) + \int_A I_{(-\infty,x)}(y)\lambda \ dy. \end{aligned}$$

Propositions 4.1 and 4.6 are now combined to obtain our formula for Γ.

(4.8) **THEOREM.** Let r be defined in (2.11). Then,

$$\begin{aligned} \Gamma(x,\cdot) = \delta_x(A) &+ \lambda \int_A I_{[0,x]}(y)dy + \lambda \int_A I_{(0,\infty)}(y)R(y)dy \\ &+ \int_A I_{(x,\infty)}(y)[r(y-x) - \lambda R(y-x)]dy \end{aligned} \tag{4.9}$$

where $R(y) = \int_0^y r(x)dx$.

PROOF. By Propositions 4.1 and 4.6,

$$\Gamma(x,A) = \int_{[0,\infty)} \nu(A-y)\delta_x(dy) + \lambda \int_{[0,x)} \nu(A-y)dy. \tag{4.10}$$

From (2.11), it is easy to see that

$$\int_{[0,\infty)} \nu(A-y)\delta_x(dy) = \delta_x(A) + \int_A I_{[x,\infty)}(y)r(y-x)dy. \tag{4.11}$$

For the second term in (4.10), observe that

$$\begin{aligned} \int_{[0,x)} \nu(A-y)dy &= \int_A I_{[0,x)}(y)dy \\ &\quad + \int_A \int_{[0,x)} I_{[z,\infty)}(y)r(y-z)dz \ dy \\ &= \int_A I_{[0,x]}(y)dy + \int_A I_{[0,x]}(y) \int_0^y r(y-z)dz \ dy \\ &\quad + \int_A I_{(x,\infty)}(y) \int_0^x r(y-z)dz \ dy \\ &= \int_A I_{[0,x]}(y)dy + \int_A I_{(0,\infty)}(y)R(y)dy \\ &\quad - \int_A I_{(x,\infty)}(y)R(y-x)dy. \end{aligned} \tag{4.12}$$

Substitution of (4.12) and (4.11) into (4.10) yields (4.9).

For the M/M/1 queue with $EB_1 = 1/\mu \leq 1/\lambda$ the density $r(\cdot)$ takes the form $\lambda \exp(-(\mu-\lambda)x)dx$ (as may be verified by calculating ν via (2.8)), so Theorem 4.8 yields (for $\rho < 1$)

$$
\begin{aligned}
\Gamma(x,\cdot) =& \delta_x(\cdot) + \lambda(1-\rho)^{-1} \int_A I_{[0,x]}(y)dy \\
&+ \lambda(1-\rho)^{-1} \int_A I_{[x,\infty)}(y) e^{-(\mu-\lambda)(y-x)} dy \\
&- \lambda\rho(1-\rho)^{-1} \int_A I_{[0,\infty)}(y) e^{-(\mu-\lambda)y} dy.
\end{aligned} \tag{4.13}
$$

The computation of solutions to Poisson's equation is particularly simple for the class of functions $\{f_k(\cdot) : k \geq 1\}$ where $f_k(x) = x^k$ is the "kth moment" functional. If $\pi|f| < \infty$, we use the notation $\hat{f}$ to represent the function $\hat{f} = f - \pi f$.

(4.14) **PROPOSITION.** Suppose $\rho < 1$ and $k \geq 1$. If $EB_1^{k+1} < \infty$, then $\pi|f_k| < \infty$ and a solution g_k to Poisson's equation $(I-P)g_k = \hat{f}_k$ exists; one particular solution if $g_k = \Gamma \hat{f}_k$, where

$$
\begin{aligned}
(\Gamma \hat{f}_k)(x) = & x^k + \lambda(1-\rho)^{-1} x^{k+1}(k+1)^{-1} \\
&+ (1-\rho)^{-1} \sum_{j=1}^{k} \binom{k}{j} x^j \; [\pi f_{k-j} + \lambda(k-j+1)^{-1} \pi f_{k-j+1}] \\
&+ (1-\rho)^{-1} k x \pi f_{k-1}.
\end{aligned} \tag{4.15}
$$

PROOF. Kiefer and Wolfowitz [1956] proved that $\pi|f_k| < \infty$ if $EB_1^{k+1} < \infty$. By Theorem 3.3, $g_k = \Gamma \hat{f}_k$ is a solution to the equation $(I-P)g_k = \hat{f}_k$. To obtain (4.15), note that if $\rho < 1$, then $\nu(\Re^+) = (1-\rho)^{-1}$ so

$$
\begin{aligned}
\nu((x,\infty)) &\equiv \nu(\Re^+) - \nu(\{0\}) - \nu((0,x]) \\
&= (1-\rho)^{-1} - 1 - R(x) \\
&= \rho(1-\rho)^{-1} - R(x) \\
&\equiv \bar{R}(x).
\end{aligned}
$$

Then, (4.9) can be re-written in terms of $\bar{R}$ as

$$
\begin{aligned}
\Gamma(x,A) = & \delta_x(A) + \lambda(1-\rho)^{-1} \int_A I_{[0,x]}(y)dy \\
&+ \int_A I_{(x,\infty)}(y)[r(y-x) + \lambda \bar{R}(y-x)]dy \\
&- \lambda \int_A I_{[0,\infty)}(y) \bar{R}(y) dy.
\end{aligned} \tag{4.16}
$$

Thus,

$$(4.17) \qquad (\Gamma f_k)(x) = x^k + \lambda(1-\rho)^{-1} \int_0^x y^k \; dy + \int_k^\infty y^k r(y-x)dy + \lambda \int_x^\infty y^k \bar{R}(y-x)dy - \lambda \int_0^\infty y^k \bar{R}(y)dy.$$

Because $\pi(dy) = (1-\rho)r(y) \; dy$ for $y > 0$, it follows that

$$(4.18) \qquad \int_x^\infty y^k r(y-x)dy = \sum_{j=0}^{k} \binom{k}{j} x^{k-j} \int_x^\infty (y-x)^j r(y-x)dy = (1-\rho)^{-1} \sum_{j=0}^{k} \binom{k}{j} x^{k-j} \pi f_j .$$

where $\pi f_0 = \pi((0,\infty)) = \rho$. Also,

$$\pi((0,\infty)) = (1-\rho)\bar{R}(x) \text{ for } x > 0,$$

so

$$(4.19) \qquad \int_0^\infty [(y+x)^k - y^k] \; \bar{R}(y)dy = \sum_{j=0}^{k-1} \binom{k}{j} x^{k-j} \int_0^\infty y^j \bar{R}(y)dy = (1-\rho)^{-1} \sum_{j=0}^{k-1} \binom{k}{j} x^{k-j} (j+1)^{-1} \pi f^{j+1}$$

(see (21.9) of Billingsley [1979] for the last step). Finally, setting $k = 0$, we find that

$$\Gamma(x, \Re^+) = (1-\rho)^{-1}(1+\lambda x),$$

so that $\Gamma \hat{f}_k = \Gamma f_k - (1-\rho)^{-1}(1+\lambda x)\pi f_k$. Combining (4.17), (4.18), (4.19), and (4.20), we get (4.15).

(4.21) **COROLLARY.** Suppose $\rho < 1$. A necessary and sufficient condition for existence of a π-integrable solution g_k to $(I-P)g_k = \hat{f}_k$ is that $EB_1^{k+2} < \infty$.

PROOF. By Theorem 3.9, the only π-integrable solution to Poisson's equation, up to an additive constant, is $g_k = \Gamma \hat{f}_k$. By (4.15), $\Gamma \hat{f}_k$ is π-integrable if and only if $\pi f_{k+1} < \infty$.

But $EB_1^{k+2} < \infty$ is known to be necessary and sufficient for $\pi f_{k+1} < \infty$ (see Kiefer and Wolfowitz [1956]).

The expression (4.15) gives $\Gamma \hat{f}_k$ as a polynomial of order $k+1$ having coefficients involving the moments πf_j for $j \leq k+1$. The πf_j's can be computed in a straightforward way. For $\alpha \geq 0$, let

$$\tilde{\pi}(\alpha) = \int_0^\infty e^{-\alpha x} \pi(dx)$$
$$\tilde{G}(\alpha) = \int_0^\infty e^{-\alpha x} G(dx).$$

Taking Laplace transforms in the equation (2.8) we get

$$\tilde{\pi}(\alpha) = \frac{(1-\rho)\alpha}{\alpha - \lambda + \lambda \tilde{G}(\alpha)}; \tag{4.22}$$

for an alternative derivation, see Heyman and Sobel [1982], p. 251. The moment πf_k can then be found by differentiating (4.22) k times.

5. COMPUTATION OF EXPECTATIONS

Our goal, in this section, is to compute $E_x S_n(f)$, when $\pi|f| < \infty$. The idea is to use the martingale structure of $S_n(f)$, which was introduced in Section 3.

(5.1) **THEOREM.** If $\pi f = 0$ and $g = \Gamma f$, then

$$E_x S_n(f) = g(x) - (P^{n+1} g)(x). \tag{5.2}$$

PROOF. The result follows immediately from (3.12), provided that we are able to prove the integrability of $g(W_n)$ required for M_n to be a martingale. Let $T_n = \inf\{m > n : W_m = 0\}$ and observe that the Markov property yields

$$\begin{aligned} E_x|g(W_n)| &\leq E_x(\Gamma|f|)(W_n) \\ &= E_x\left(\sum_{k=n}^{T_n - 1} |f(W_k)|\right). \end{aligned} \tag{5.3}$$

we will now show inductively that the last expectation in (5.3) is finite for all $n \geq 0$. Of course, for $n = 0$, the expectation is just $(\Gamma|f|)(0) = \nu|f| < \infty$. For $n \geq 1$,

$$\begin{aligned} E_x\left\{\sum_{k=n}^{T_n - 1} |f(W_k)|\right\} &= E_x\left\{\sum_{k=n}^{T_n - 1} |f(W_k)|;\; T_{n-1} > n\right\} + (\Gamma|f|)(0) P_x\{T_{n-1} = n\} \\ &\leq E_x\left\{\sum_{k=n-1}^{T_{n-1} - 1} |f(W_k)|\right\} + \nu|f| \cdot P_x\{T_{n-1} = n\} \end{aligned}$$

which is finite by the inductive hypothesis.

If $\pi|f| < \infty$ and $\rho < 1$, then $\hat{f} = f - \pi f$ satisfies the hypotheses of Theorem 5.1, and (5.2) shows that

$$E_x r_n(f) = \pi f + n^{-1}(g(x) - (P^{n+1}g)(x)).$$

Note that if $\rho = 1$, then $\hat{f} = f - \pi f$ is not, in general, π-integrable so the above centering may not work.

(5.4) **PROPOSITION.** If $\pi f = 0$ and $g = \Gamma f$ is π-integrable, then

$$E_x S_n(f) = g(x) - \pi g + o(1). \tag{5.5}$$

PROOF. Since $P_0\{T = 1\} > 0$, it follows that W is an aperiodic Harris chain. Corollary 6.7 of Nummelin [1985] allows one to conclude that

$$\int_{[0,\infty)} \pi(dy) \ |(P^n g)(y) - \pi g| \to 0$$

as $n \to \infty$. Since $\pi = \pi P$,

$$\int_{[0,\infty)} \pi(dy) \int_{[0,\infty)} P(y, dz) \ |(P^n g)(z) - \pi g| \to 0. \tag{5.6}$$

Fix $x \geq 0$. By Lemma 2.10, (5.6) implies that there exists $y \geq x$ such that

$$\int_{[0,\infty)} P(y, dz) \ |(P^n g)(z) - \pi g| \to 0. \tag{5.7}$$

Lemma 2.3 then proves that

$$\begin{aligned} |(P^{n+1}g)(x) - \pi g| &\leq \int_{[0,\infty)} P(x, dz) \ |(P^n g)(z) - \pi g| \\ &\leq e^{\lambda(y-x)} \int_{[0,\infty)} P(y, dx) \ |(P^n g)(z) - \pi g| \to 0. \end{aligned} \tag{5.8}$$

as $n \to \infty$. Substitution of (5.8) into (5.2) yields (5.5).

Given (5.5), we can easily calculate asymptotic relations for the expectation $E_x r_n(f)$. For example, if $f_1(x) = x$ and $\rho < 1$, Proposition 4.14 shows that when $EB_1^2 < \infty$,

$$(\Gamma \hat{f}_1)(x) = (1 - \rho)^{-1}(x + \lambda x^2/2).$$

Thus, if $EB_1^3 < \infty$, it follows from Corollary 4.21 and Proposition 5.4 that

$$\frac{1}{n+1}\sum_{k=0}^{n} E_x W_k = \pi f_1 + \frac{(1-\rho)^{-1}}{n+1}\left(x - \pi f_1 + \lambda/2(x^2 - \pi f_2)\right) + o(n^{-1}).$$

Proposition 5.4 improves upon the previous result of Heathcote and Winer [1969] in several ways. Their formulae requires that X_n possess a moment generating function; furthermore, their methods are specific to the particular function $f(x) = x$ and initial condition $W_0 = 0$.

We conclude this section with a rate-of-convergence result. If $\rho < 1$ and $E\exp(\alpha B_1) < \infty$ for some $\alpha > 0$, then $\pi|f_k| < \infty$ for all "moment" functionals f_k. Hence, there exists π-integrable solutions g_k to Poisson's equation $(I-P)g_k = \hat{f}_k$.

(5.9) **THEOREM.** If $\rho < 1$ and $E\exp(\alpha B_1) < \infty$ for some $\alpha > 0$, then there exists $0 \le \beta < 1$ such that for all $k \ge 1$,

$$E_x r_n(f_k) = \pi f_k + n^{-1}(g_k(x) - \pi g_k) + O(\beta^n)$$

as $n \to \infty$.

PROOF. By Theorem 5.1, it suffices to show that there exists $O \le \beta < 1$ such that

$$(P^n g_k)(x) = \pi g_k + O(\beta^n). \tag{5.10}$$

By Theorem 3 of Tweedie [1983], there exists $\gamma < 1$ such that for all $k \ge 1$,

$$a_n = \int_{[0,\infty)} \pi(dy)\ \ |(P^n g_k)(y) - \pi g_k| = O(\gamma^n).$$

Since the sequence $\{\gamma^{-n} a_n : n \ge 1\}$ is bounded, it follows that if $\gamma < \beta < 1$, then

$$\int_{[0,\infty)} \pi(dy) \sum_{n=0}^{\infty} \beta^{-1}\ |(P^n g_k)(y) - \pi g_k| = \sum_{n=0}^{\infty} (\gamma/\beta)^n \gamma^{-n} a_n < \infty. \tag{5.11}$$

For $x \ge 0$, an argument similar to that yielding (5.8) shows that (5.11) implies

$$\sum_{n=0}^{\infty} \beta^{-1}\ \ |(P^n g_k)(x) - \pi g_k| < \infty,$$

from which we immediately obtain (5.10).

6. CENTRAL LIMIT THEOREM

In this section, we explore central limit theory for the sequence $\{r_n(f) : n \geq 1\}$. In fact, we shall prove the stronger result that $r_n(f)$ obeys a functional central limit theorem, and we shall compute explicitly the variance $\sigma^2(f)$ of the limiting Brownian motion.

Given the martingale representation of Section 3, namely

$$S_n(f) = M_{n+1} - g(W_{n+1}), \tag{6.1}$$

the results mentioned above will basically follow by appealing to the invariance principle for martingales. Set

$$X_n(t, f) = n^{-1/2}(S_{[nt]}(f) - [nt]\pi f)$$

where $[x]$ denotes the greatest integer less than or equal to x. Also, $D[0, \infty)$ will represent the space of real-valued right continuous functions, with left limits, endowed with Skorohod topology. (See Chapter 3 of Billingsley [1968] for a complete description of weak convergence in this space.)

(6.2) **THEOREM.** Assume $\rho < 1$ and $\pi|f| < \infty$. If there exists a solution g to Poisson's equation $(I - P)g = \hat{f}$ satisfying $\pi g^2 < \infty$, then

$$X_n(\cdot, f) \Rightarrow \sigma(f)B(\cdot) \tag{6.3}$$

as elements in $D[0, \infty)$, where $\sigma^2(f) = 2\pi(\hat{f}g) - \pi\hat{f}^2$, and $B(\cdot)$ is a standard Brownian motion on [0, 1].

PROOF. Note that if g^2 is π-integrable, then a standard inequality for conditional expectations shows that

$$\pi(Pg)^2 \leq \pi g^2 < \infty$$

and it follows from Poisson's equation that $\pi\hat{f}^2 < \infty$. We can therefore apply a functional central limit theorem for Harris chains due to Maigret [1978]; more precisely, we use a slight variant to be found in Niemi and Nummelin [1982]. (These theorems, as indicated above, are obtained by applying martingale invariance principles to the representation (6.1)). We conclude that

$$X_n(\cdot, f) \Rightarrow \tilde{\sigma}(f)B(\cdot)$$

as $n \to \infty$, where $\tilde{\sigma}^2(f) = \pi\hat{f}^2 + 2\pi(\hat{f}Pg)$. But $Pg = g - f$, from which we immediately obtain $\sigma^2(f) = \tilde{\sigma}^2(f)$.

We emphasize that the limit theorem (6.3) should be understood to hold for **any** initial distribution μ for W_0. Also, by Theorem 4.9, we see that a square-integrable solution g to Poisson's equation exists if and only if $\pi(\Gamma\hat{f})^2 < \infty$. If f_k is a "moment" functional, then $g_k^2 = (\Gamma\hat{f}_k)^2$ is a polynomial of degree $2(k+1)$ by (4.15). Since $EB_1^{2k+3} < \infty$ is necessary and sufficient for $\pi f_{2k+2} < \infty$ (see Kiefer and Wolfowitz [1956]), it follows that g_k is square-integrable if and only if $EB_1^{2k+3} < \infty$.

For $k = 1$, we therefore require that $EB_1^5 < \infty$ in order to apply Theorem 6.2. This theorem provides the means to calculate closed-form expressions for $\hat{\sigma}^2(f)$ for a broad class of functionals f (in particular, for all "moment" functionals). This generalizes results due to Blomqvist [1967], Daley [1968], and Pagurek and Woodside [1979], in which $\hat{\sigma}^2(f)$ was calculated from $f(x) = x$.

While the above discussion of the functional central limit theorem for $S_n(f)$ involves assuming that g^2 is π-integrable, it is easily seen that what is really required for such a result is finiteness of $E_\pi D_1^2$, where $D_1 = g(X_1) - (Pg)(X_0)$ and g is the solution to Poisson's equation $(I - P)g = \hat{f}$. It turns out that if f_k is a "moment" functional, a certain "cancellation" occurs in the r.v. D_1, so that a necessary and sufficient condition for finiteness of $E_\pi D_1^2$ is EB_1^{2k+2}. This agrees with the analysis of Daley [1968], in which he shows that $\tilde{\sigma}^2(f_1)$ can be defined in terms of a sum of stationary covariances of $\{W_n : n \geq 0\}$ if and only if $EB_1^4 < \infty$. We omit the details of this analysis of $E_\pi D_1^2$.

APPENDIX

Proof of Theorem 3.15. We start by showing that any harmonic function h must be reasonably well-behaved. We note that in order for Ph to be well-defined, we (implicitly) require that $(P|h|)(x) < \infty$ for all $x \geq 0$. In particular, $(P|h|)(0) < \infty$, which implies that

$$\int_0^\infty |h(y)|e^{-\mu y}\,dy < \infty. \tag{A.1}$$

Furthermore, since h is harmonic, h necessarily satisfies the integral equation

$$\begin{aligned} h(x) = \frac{\mu}{\lambda+\mu}e^{-\lambda x}h(0) &+ \frac{\mu}{\lambda+\mu}\int_0^x h(y)\lambda e^{-\lambda(x-y)}\,dy \\ &+ \frac{\lambda}{\lambda+\mu}\int_x^\infty h(y)\mu e^{-\mu(y-x)}\,dy. \end{aligned} \tag{A.2}$$

By the finiteness guaranteed by (A.1) and the fact that $\exp((\mu-\lambda)y)$ is bounded on bounded intervals, both integrals appearing in (A.2) are necessarily continuous in x. Consequently, (A.2) implies that h is continuous in x. We therefore conclude, from the fundamental theorem of calculus, that h is continuously differentiable. By proceeding inductively, we conclude that any (measurable) harmonic function must necessarily be infinitely continuously differentiable.

Let $\tau(y) = \min\{n \geq 0 : W_n \leq 0 \text{ or } W_n \geq y\}$. We note that $|h(W_{\tau(y)\wedge n})| \leq \|h\|_y \equiv \max\{|h(z)| : 0 \leq z \leq y\}$ on $\{\tau(y) > n\}$, whereas (using the "memoryless" structure of the exponential service times)

$$(A.3) \qquad E_x|h(W_{\tau(y)})| = |h(0)|P_x\{W_{\tau(y)} = 0\} + P_x\{W_{\tau(y)} \geq y\} \int_y^\infty |h(z)|\mu e^{-\mu(z-y)}\,dz$$

for $0 \leq x \leq y$. Since h is continuous, $\|h\|_y < \infty$ for all $y \geq 0$. Furthermore, the integral on the right hand side of (A.3) is finite by (A.1). Since h is harmonic, we therefore conclude that $\{h(W_{\tau(y)\wedge n}) : n \geq 0\}$ is a uniformly integrable martingale with respect to $(\mathcal{F}_n : n \geq 0)$. Hence, it is evident that

$$E_x h(W_{\tau(y)}) = h(x)$$

for $0 \leq x \leq y$ (see Breiman [1968], p. 98). Again, using the "memoryless" nature of the exponential service times, we obtain

$$(A.4) \qquad E_x h(W_{\tau(y)}) = h(0)P_x\{W_{\tau(y)} = 0\} + P_x\{W_{\tau(y)} \geq y\} \int_y^\infty h(x)\mu e^{-\mu(z-y)}\,dz$$

for $0 \leq x \leq y$. But $P_x\{W_{\tau(y)} = 0\} = P_x\{S(\tilde{\tau}(y)) \leq 0\}$ and $P_x\{W_{\tau(y)} \geq y\} = P_x\{S(\tilde{\tau}(y)) \geq y\}$, where $\tilde{\tau}(y)) = \min\{n \geq 0 : S(n) \leq 0 \text{ or } S(n) \geq y\}$. Using the "memoryless" nature of the exponential inter-arrival and service time distributions, we observe that $\{S(\tilde{\tau}(y) \wedge n) : n \geq 0\}$ is a uniformly integrable martingale with respect to $(\mathcal{F}_n : n \geq 0)$ so that

$$(A.5) \qquad E_x S(\tilde{\tau}(y)) = x.$$

But

$$(A.6) \qquad E_x S(\tilde{\tau}(y)) = -\lambda^{-1} P_x\{S(\tilde{\tau}(y)) \leq 0\} + (y + \mu^{-1})P_x\{S(\tilde{\tau}(y)) \geq y\}.$$

Hence, we may conclude from (A.5)-(A.6) that $P_x\{S(\tilde{\tau}(y)) \leq 0\} = a_y + b_y x$ for $0 \leq x \leq y$ (i.e. is affine in x). Relation (A.4) then guarantees that $h(x) = a'_y + b'_y x$ for $0 \leq x \leq y$. Since y is arbitrary, it is therefore evident that h must necesarily take the form $h(x) = a + bx$ for some (suitably defined) constants a and b. By returning to (A.2), it is easily verified that b must necessarily equal zero. Hence, h must be constant, proving the result.

ACKNOWLEDGEMENT

The writing of this paper was completed while the author was visiting the Department of Mathematics and Statistics at Carleton University, Ottawa, Canada. The author wishes to thank the Department for its hospitality.

REFERENCES

ASMUSSEN, S. and M. BLADT (1993). Poisson's equation for queues driven by a Markovian marked point process. Technical Report, Aalborg University, Denmark.

ATHREYA, K.B. and P. NEY (1978). A new approach to the limit theory of recurrent Markov chains. *Transactions of the American Mathematical Society 245*, 493-501.

BILLINGSLEY, P. (1979). *Probability and Measure*, John Wiley, New York.

BLOMQVIST, N. (1967). The covariance function of the M/G/1 queueing system. *Skand. Aktuarietidskrift 50*, 157-164.

BREIMAN, L. (1968). *Probability*, Addison-Wesley, Reading, MA.

DALEY, D. J. (1968). The serial correlation coefficients of waiting times in a stationary single server queue. *J. Australian Math. Soc. 8*, 683-699.

ETHIER, S. N. and T. G. KURTZ (1986). *Markov Processes: Characterization and Convergence*, John Wiley, New York.

FELLER, W. (1971). *An Introduction to Probability Theory and its Applications, Volume II*, John Wiley, New York.

HEATHCOTE, C.R. and P. WINER (1969). An approximation for hte moments of waiting times. *Operations Research 17*, 175-196.

HEYMAN, D.P. and M. SOBEL (1982). *Stochastic Models in Operations Research*, McGraw-Hill, New York.

KARLIN, S. and H.M. TAYLOR (1981). *A Second Course in Stochastic Processes*, Academic Press, New York.

MAIGRET, N. (1978). Théorèm de limite centrale functionnel pour une chaîne de Markov récurrente au sens de Harris et positive. *Annales de l'Institut Heni Poincaré, B, 14*, 425-440.

NEVEU, J. (1972). Potentiel Markovien récurrente des chaînes de Harris. *Ann. Inst. Fourier, Grenoble 22*, 7-130.

NIEMI, S. and E. NUMMELIN (1982). Central limit theorems for Markov random walks. *Commentationes Physico-Mathematicae, 54*, Societas Scientiarum Fennica, Helsinki.

NUMMELIN, E. (1985). On the Poisson eqaution in the potential theory of a single kernel. Technical Report, Department of Mathematics, University of Helsinki, Iceland.

OREY, S. (1971). *Lecture Notes on Limit Theorems for Markov Chain Transition Probabilities*, Van Nostrand Reinhold, London.

PAGUREK, G. and C.M. WOODSIDE (1979). The sum of serial correlations of waiting and system times in GI/G/1 queues. *Operations Research 27*, 755-766.

REVUZ, D. (1984). *Markov Chains*, North-Holland, New York.

TWEEDIE, R.L. (1983). The existence of moments for stationary Markov chains *Journal of Applied Probability 20*, 191-196.