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Uniform Cesaro limit theorems for synchronous processes with applications to queues

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Let $X=\{X(t): t\geq 0\}$ be a positive recurrent synchronous process (PRS), that is, a process for which there exists an increasing sequence of random times $\tau=\{\tau(k)\}$ such that for each k the distribution of $\theta_{\tau(k)}X \stackrel{\text{def}}{=} \{X(t+\tau(k)): t\geq 0\}$ is the same and the cycle lengths $T_n \stackrel{\text{def}}{=} \tau(n+1)-\tau(n)$ have finite first moment. Such processes (in general) do not converge to steady-state weakly (or in total variation) even when regularity conditions are placed on the cycles (such as non-lattice, spread-out, or mixing). Nonetheless, in the present paper we first show that the distributions of $\{\theta_s X: s>0\}$ are tight in the function space $\mathscr{D}(0,\infty)$. Then we investigate conditions under which the Cesaro averaged functionals $\bar{\mu}_t(f) \stackrel{\text{def}}{=} (1/t)\int_0^t E(f(\theta_s X))\,ds$ converge uniformly (over a class of functions) to $\pi(f)$, where π is the stationary distribution of X. We show that $\bar{\mu}_t(f)\to\pi(f)$ uniformly over f satisfying $\|f\|_\infty\leq 1$ (total variation convergence). We also show that to obtain uniform convergence over all f satisfying $|f|\leq g$ ($g\in L_1^+(\pi)$ fixed) requires placing further conditions on the PRS. This is in sharp contrast to both classical regenerative processes and discrete time Harris recurrent Markov chains (where renewal theory can be applied) where such uniform convergence holds without any further conditions. For continuous time positive Harris recurrent Markov processes (where renewal theory cannot be applied) we show that these further conditions are in fact automatically satisfied. In this context, applications to queueing models are given.

synchronous process * Cesaro convergence * limit theorems * point processes

1. Preliminaries and introduction

Throughout this paper, $X=\{X(t): t\geq 0\}$ will denote a stochastic process taking values in a complete separable metric state space (CSMS) $\mathscr{S}$ and having paths in the space $\mathscr{D}=\mathscr{D}_{\mathscr{S}}[0,\infty)$ of functions $f:\mathbb{R}_+\to\mathscr{S}$ that are right continuous and have left hand limits. $\mathscr{D}$ is endowed with the Skorohod topology (and is a complete

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separable metric space). $(\Omega, \mathscr{F}, P)$ will denote the underlying probability space and we view X as a random element of $\mathscr{D}$. Let Δ denote an arbitrary fixed element not in the set $\mathscr{S}$. We then endow $\mathscr{S} \stackrel{\text{def}}{=} \mathscr{S} \cup \{\Delta\}$ with the one-point compactification topology (in order to keep in the framework of CSMS).

Definition 1.1. X is said to be a *synchronous* process with respect to the random times $0 \leq \tau(0) < \tau(1) < \cdots$ (with $\lim_{n\to\infty} \tau(n) = \infty$ a.s.) if $\{X_n : n \geq 1\}$ forms a stationary sequence in the space $\mathscr{D}_{\mathscr{S}}^{\infty}$, where

$$X_n(t) = \begin{cases} X(\tau(n-1)+t), & \text{if } 0 \leq t < T_n, \\ \Delta, & \text{if } t \geq T_n. \end{cases}$$

$T_n \stackrel{\text{def}}{=} \tau(n) - \tau(n-1)$ is called the nth cycle length, X_n is called the nth cycle and we refer to $(\tau(n))$ as the synch-times for X with counting process $N(t) =$ the number of synch times that fall in the interval $[0, t]$.

Definition 1.2. A synchronous process X is called *non-delayed* if $\tau(0) = 0$ a.s., *delayed* otherwise. It is called *positive recurrent* if $E(T_1) < \infty$, *null recurrent* otherwise. It is called *ergodic* if it is positive recurrent and the invariant σ-field, $\mathscr{I}$, of $\{X_n, T_n\}$ is trivial. $\lambda \stackrel{\text{def}}{=} 1/E(T_1)$ is called the *rate* of the synch times. $\hat{\lambda} \stackrel{\text{def}}{=} 1/E(T_1 | \mathscr{I})$ is called the *conditional rate.*

From now on, PRS will be used to abbreviate *positive recurrent synchronous process.* To help the reader, an appendix is included at the end of this paper giving a brief introduction to PRS's.

$\theta_t : \mathscr{D} \to \mathscr{D}$ denotes the *shift operator* $(\theta_t x)(s) = x(t+s)$, P^0 denotes the probability measure under which X is non-delayed; $P^0(A) = P(\theta_{\tau(1)} X \in A)$ and P^* denotes the probability measure under which X has the stationary distribution

$$P^*(A) = \pi(A) = \lim_{t\to\infty} \frac{1}{t} \int_0^t P(\theta_s \circ X \in A)\, \mathrm{d}s$$

(see the Appendix).

$$\psi(A) \stackrel{\text{def}}{=} \lambda \int_0^\infty P^0(\theta_s \circ X \in A;\, \tau(1) > s)\, \mathrm{d}s$$

denotes a distribution on $\mathscr{D}$ which when X is ergodic is the same as π, but, ergodic or not, ψ defines a stationary (w.r.t. the shifts θ_t) distribution on $\mathscr{D}$ (see Proposition A.2 in the Appendix).

The important point here is that at the random times $\tau(k)$, $X(t)$ and its future probabilistically start over. However, in contrast to classical regenerative processes (CRP's), or the regenerative structure found in Harris recurrent Markov chains (HRMC's), the future is not necessarily independent of any of the past $\{\tau(1), \ldots, \tau(k);\, X(s)\colon 0 \leq s \leq \tau(k)\}$. *In particular τ does not (in general) form a renewal process and hence renewal theory does not apply to synchronous processes.*

Natural questions arise, however, as to which of known limit theorems etc. that hold for CRP's and HRMC's actually do not depend upon renewal theory and can in fact be extended to cover PRS's. We first show that for a PRS, the distributions of $\{\theta_s \circ X\}$ are in fact tight in the function space $\mathscr{D}(0,\infty)$. Since it is known that weak or total variation convergence does not follow even when placing regularity conditions on the cycle distributions (such as non-lattice cycle length distribution or mixing cycles) as is the case for CRP's and HRMC's, we next attack problems concerning Cesaro limit theorems for PRS's. For example, although $N(t)/t \to \hat{\lambda}$, P-a.s. as $t\to\infty$, what can be said about

$$E(N(t))/t \to E\hat{\lambda}, \tag{1.1}$$

which *does* hold true for a renewal process? Similarly, for an ergodic PRS, although $(1/t)\int_0^t f(\theta_s \circ X)\,\mathrm{d}s \to \pi(f)$, P^0-a.s. for any $f \in L_1(\pi)$, what can be said about

$$\bar{\mu}_t(f) \stackrel{\text{def}}{=} \frac{1}{t}\int_0^t E^0 f(\theta_s \circ X)\,\mathrm{d}s \to \pi(f), \quad f \in L_1(\pi), \tag{1.2}$$

or

$$\sup_{\|f\|_\infty \le 1} |\bar{\mu}_t(f) - \pi(f)| \to 0, \tag{1.3}$$

or (more generally)

$$\sup_{|f| \le g} |\bar{\mu}_t(f) - \pi(f)| \to 0 \quad \text{for each } g \in L_1^+(\pi), \tag{1.4}$$

all three of which holds true for CRP's and HRMC's?

We show that (1.3) is always true for a PRS (ergodic or not), whereas (1.1), (1.2) and (1.4) require extra conditions (even in the ergodic case). These extra conditions turn out to be automatically satisfied for continuous time Harris recurrent Markov processes (HRMP's). In this context we give some applications to queueing models.

Although in the present paper, we are mainly concerned with tightness and Cesaro type limit theorems for a PRS X, we mention the book of Berbee [3] who considered some related problems. In order to generalize renewal process type results to other types of point processes, Berbee deeply analyzed point processes (with counting measure $N(B)$) that are constructed from a stationary ergodic sequence of interevent times. He is mainly concerned with obtaining total variation limit theorems (as $t\to\infty$) of the point process shifts, $N_t(B) \stackrel{\text{def}}{=} N(B-t)$.

In the following theorem when for each t we consider the shift, $\tilde{X}(t) \stackrel{\text{def}}{=} \theta_t \circ X$, as an element in function space, we are considering $\mathscr{D}(0,\infty)$ as our function space, that is, we are leaving out the origin. This is only a technical detail that implies that $\{\tilde{X}(t)\colon t>0\}$ indeed defines a synchronous process with paths in $\mathscr{D}_{\mathscr{D}}$. The problem is that otherwise, the sample paths of $\tilde{X}(t)$ will not have left hand limits (in the Skorohod topology) at the jump times of X. On the other hand, when we consider the marginal distributions (i.e. the distribution of $X(t)$ for fixed t) we are viewing X as a random element of $\mathscr{D}[0,\infty)$ as originally assumed.

Theorem 1.1. *A PRS has tight marginal distributions, that is, for each* $\varepsilon > 0$ *there exists a compact set* $K(\varepsilon) \subset \mathcal{S}$ *such that* $P(X(t) \in K(\varepsilon)) > 1 - \varepsilon$ *for all* $t \geqslant 0$. *In fact the distributions of the shifts* $\{\theta_t \circ X: t > 0\}$ *are tight in the function space* $\mathcal{D}(0, \infty)$, *that is, for each* $\varepsilon > 0$ *there exists a compact set* $C(\varepsilon) \subset \mathcal{D}(0, \infty)$ *such that* $P(\tilde{X}(t) \in C(\varepsilon)) > 1 - \varepsilon$ *for all* $t > 0$.

Proof. Let $F(x)$ denote the cdf for $\tau(1)$. Fix $\varepsilon > 0$ and then choose an $a = a(\varepsilon) > 0$ such that $F(a) \leqslant \frac{1}{2}\varepsilon\lambda$. From equation (A.4) it follows that

$$\psi(A) \geqslant \lambda \int_0^a P^0(\theta_s \circ X \in A;\ \tau(1) > s)\,\mathrm{d}s$$

$$\geqslant \lambda \int_0^a P^0(\theta_s \circ X \in A;\ \tau(1) > a)\,\mathrm{d}s. \tag{1.5}$$

Observe that

$$P^0(\theta_s \circ X \in A) = P^0(\theta_s \circ X \in A;\ \tau(1) > a) + P^0(\theta_s \circ X \in A;\ \tau(1) \leqslant a)$$

$$\leqslant P^0(\theta_s \circ X \in A;\ \tau(1) > a) + F(a).$$

Substituting the above into (1.5) we obtain

$$\psi(A) \geqslant \lambda \int_0^a P^0(\theta_s \circ X \in A)\,\mathrm{d}s - aF(a),$$

and hence

$$\int_0^a P^0(\theta_s \circ X \in A)\,\mathrm{d}s \leqslant \lambda^{-1}\psi(A) + \tfrac{1}{2}a\varepsilon. \tag{1.6}$$

For each $u \geqslant 0$ and each compact set B of $\mathcal{S}$ define $A(B, u) = \{x \in \mathcal{D}: x(t) \in B;\ t \in [u, u+a]\}$. By the compact containment condition (see Ethier and Kurtz [4, Remark 7.3, p. 129]) there exists a compact set $K_1 = K_1(\varepsilon, a)$ in $\mathcal{S}$ such that

$$\psi(A(K_1, u)) > 1 - \tfrac{1}{2}\lambda a\varepsilon. \tag{1.7}$$

Moreover, by stationarity of X under ψ, K_1 doesn't depend upon u. For any set A let $\bar{A}$ denote the complement of the set. From (1.6) and (1.7) we obtain

$$\int_0^a P^0(\theta_s \circ X \in \bar{A}(K_1, u))\,\mathrm{d}s \leqslant \lambda^{-1}\psi(\bar{A}(K_1, u)) + \tfrac{1}{2}a\varepsilon \leqslant a\varepsilon. \tag{1.8}$$

But $u + a \in s + [u, u+a]$ for each $s \in [0, a]$ and hence for each $s \in [0, a]$, $P^0(X(u+a) \in \bar{K}_1) \leqslant P^0(\theta_s \circ X \in \bar{A}(K_1, u))$. Substituting into (1.8) yields

$$P^0(X(u+a) \in \bar{K}_1) \leqslant \varepsilon, \quad u \geqslant 0,$$

or equivalently

$$P^0(X(t) \in K_1) > 1 - \varepsilon, \quad t \geqslant a. \tag{1.9}$$

Using the compact containment condition again there exists a compact set $K_2 = K_2(\varepsilon, a)$ such that $P^0(X \in A(K_2, 0)) > 1-\varepsilon$; in particular $P^0(X(t) \in K_2) > 1-\varepsilon$ for all $t \in [0, a]$ so that for $K_\varepsilon = K_1 \cup K_2$ we obtain $P^0(X(t) \in K_\varepsilon) > 1-\varepsilon$ for all $t \geq 0$. We have just shown that the marginal distributions of a non-delayed PRS are tight. Since $\tilde{X}$ is also synchronous (with the same embedded synch times), our theorem is proved in the non-delayed case. To handle the delayed case, suppose X is delayed, fix $\varepsilon > 0$ and choose an M large enough so that $P(\tau(0) > M) \leq \varepsilon$. For any compact set K if $t > M$ then

$$
\begin{aligned}
P(X(t) \in \bar{K}) &\leq P(X(t) \in \bar{K};\ \tau(0) \leq M) + \varepsilon \\
&\leq P(X(\tau(0) + t - s) \in \bar{K}, \text{ some } s \in [0, M]) + \varepsilon \\
&= P^0(X(t - M + s) \in \bar{K}, \text{ some } s \in [0, M]) + \varepsilon \\
&= P^0(X(u + s) \in \bar{K}, \text{ some } s \in [0, M]) + \varepsilon,
\end{aligned}
\tag{1.10}
$$

where $u = t - M$. But now we are dealing with the non-delayed version of $\bar{X}$ which we just showed was tight; thus, for any $\delta > 0$ we can choose a compact set of paths $C(\varepsilon) \subset \mathcal{D}$ such that for all $t > 0$, $P^0(\theta_t \circ X \in C(\varepsilon)) > 1 - \delta$. Using this fact together with the compact containment condition, it follows that the last probability in (1.10) can be made arbitrarily small (uniformly over $u \geq 0$) for appropriate compact sets $K \subset \mathcal{S}$.

For $t \leq M$ we can use the compact containment condition on X over the time interval $[0, M]$ to obtain a compact set K such that $P(X(t) \in K) > 1 - \varepsilon$ for all $t \in [0, M]$. The proof is now complete. □

2. Limit theorems for $N(t)$

In this section we present counterexamples showing that (1.1) is false in general. In fact we show that even in the ergodic case it is possible that $E^0 N(t) = \infty$.

Let $\tau = \{\tau(n)\}$ be the synch times of a non-delayed PRS X. Let $N(t)$ denote the corresponding counting process. Under P^0, X is non-delayed and the point process τ is called a *Palm* version in which case $\{T_n\}$ forms a stationary sequence. Under P^*, X is stationary as is the point process τ (see for example, [9]).

Example 1. Let Z be a r.v. such that $P(Z \geq 1) = 1$ and $E(Z) = \infty$. Define $T_n = 1/Z$ $(n \geq 1)$. Then $\tau(n) = n/Z$ and $E(T_n) \leq 1 < \infty$. Observe that $P(N(t) > n) = P(\tau(n) < t) = P(tZ > n)$ so that indeed $E(N(t)) = \infty$ for all $t > 0$. Observe, however, that $\{T_n\}$ is not ergodic; its invariant σ-field is precisely $\sigma(Z)$.

Whereas Example 1 is not ergodic, our next example is.

Example 2. Consider a discrete time renewal process with cycle length distribution $\mathcal{P} = \{p_k : k \geq 1\}$ having finite and non-zero first moment, $1/\mu$, but infinite second moment. Let $B(n)$ denote the corresponding descrete time forward recurrence time process. B is a positive recurrent Markov chain with invariant probability distribution $\alpha_n \overset{\text{def}}{=} \mu \sum_{k\geq n} p_k$ (the *equilibrium* distribution of $\mathscr{P}$). α has infinite first moment; $\sum_{k=1}^{\infty} k\alpha_k = \infty$. Let $h(k) = 1/(k^2+1)$ and define a point process by $T_n = h(B(n))$; $\tau(n) = T_1 + T_2 + \cdots + T_n$. Observe that $0 < T_n \leq 1$ for all n. Under α, $\{T_n\}$ is a stationary ergodic sequence and hence corresponds to a Palm version with $0 < \lambda \overset{\text{def}}{=} \{E_\alpha h(B(0))\}^{-1} < \infty$. It is also positive recurrent regenerative; it regenerates whenever $B(n) = 0$. Let $\gamma = \min\{n \geq 0: B(n) = 0\}$, and observe that $\gamma = B(0)$. Let $M = \sum_{i=0}^{\infty} h(i)$. Consider the random time $\mathscr{T} = \tau(\gamma) = T_1 + \cdots + T_\gamma$. Observe that $\mathscr{T} < M \leq \infty$ and that $N(\mathscr{T} \mid X_0 = k) = k$; thus, for $t \geq M$ we obtain

$$E^0 N(t) = \sum_{k=1}^{\infty} E^0\{N(t) \mid B(0) = k\}\alpha_k$$

$$\geq \sum_{k=1}^{\infty} E^0\{N(\mathscr{T}) \mid B(0) = k\}\alpha_k = \sum_{k=1}^{\infty} k\alpha_k = \infty.$$

The important point here is that in general, $\{N(t)/t \colon t \geq 0\}$ is not uniformly integrable (UI). In the following Proposition, we provide some sufficient conditions ensuring UI; we point out that the k-dependent case has already been proved in Janson [7, Theorem 2.2] using Martingales. We provide here our own proof.

Proposition 2.1. *Suppose* $0 < \lambda^{-1} = E^0(T_1) < \infty$. *If either there exists an* $\varepsilon > 0$ *such that* $P^0(T_1 > \varepsilon) = 1$, *or the interevent times* $\{T_n\}$ *form a k-dependent process, then* (1.1) *holds.*

Proof. Suppose $P^0(T_1 > \varepsilon) = 1$. Then for all $t \geq 0$, $N(t)/t \leq 1/\varepsilon + 1$, P-a.s. (even in the delayed case) and hence is UI. Suppose now that the T_n are k-dependent, that is, for each n, $\{T_{n+j} \colon j \geq k\}$ is independent of $\{T_m \colon m \leq n\}$. It follows that for each i $(0 \leq i \leq k-1)$, $T(i) = \{T_{kn+i} \colon n \geq 1\}$ defines a (possibly delayed) renewal process. Let $N^{(i)}(t)$ denote the corresponding ith counting process. Clearly $N(t) \leq N^{(0)}(t) + N^{(1)}(t) + \cdots + N^{(k-1)}(t)$. Since $N^{(i)}(t)/t$ is UI for each i (because, for each i, it is from a renewal process), so is $N(t)/t$. □

Remark 2.1. By changing our Example 2 slightly, we can actually obtain a null recurrent version: Let $\{T_n\}$ and $N(t)$ be from Example 2 (under α), and let H_1 denote the distribution of T_n. Let $\{L_k\}$ be non-negative i.i.d. $\sim H_2$, with H_2 having infinite first moment. Define $\gamma_0 = 0$, $\gamma_{k+1} = \min\{n > \gamma_k \colon B(n) = 0\}$. Between $T_{\gamma_k - 1}$ and T_{γ_k}, insert L_k. The idea here is to start off the kth regenerative cycle with L_k and then proceed as before. This gives rise to a new sequence of interevent times $\tilde{T}_n$. Taking a Palm version of this new point process yields a stationary ergodic sequence $\tilde{T}_n^0$ such that $E(\tilde{T}_n^0) = \infty$; $\tilde{T}_n^0 \sim (1-p)H_1 + pH_2$ where $p = 1/(1 + 1/\mu)$. Letting $\tilde{N}(t)$ denote the associated counting process, we obtain $E(\tilde{N}(t)) \geq (1-p)E(N(t)) = \infty$ for $t \geq M$.

Remark 2.2. In our Example 2, it is true, however (as is well known more generally in the point process literature), that $E^*(N(t))=\lambda t$ for all $t\geq 0$ and hence that the *intensity* $\stackrel{\text{def}}{=} E^*(N(1))$ is finite and is equal to λ. It is only the Palm version that can blow up.

3. Uniform limit theorems for X

We first present an example of an ergodic PRS together with an $f\in L_1(\pi)$, such that $\bar{\mu}_t(f)=\infty$. In particular, (1.2) does not hold.

Example 3. Consider T_n and $B(n)$ from example (2). Form a semi-Markov process $X(t)$ by using T_n as the holding time for $B(n)$, that is, $X(t)\stackrel{\text{def}}{=} B(n)$; $\tau(n-1)\leq t<\tau(n)$. Then for $B(0)\sim\alpha$, X is an ergodic PRS with synch times $\tau(n)$. Now choose an $f\geq 0$ such that $fh\in L_1(\alpha)$ but

$$\sum_i f(i)h(i)\sum_{k\geq i}\alpha_k=\infty.$$

Then

$$\begin{aligned}E^0\int_0^{\tau(1)} f(X(s))\,\mathrm{d}s &=\sum_k E^0\left\{\int_0^{\tau(1)} f(X(s))\,\mathrm{d}s;\,X(0)=k\right\}\\ &=\sum_k E^0\{f(k)h(k);\,X(0)=k\}=\sum_k f(k)h(k)\alpha_k<\infty.\end{aligned}$$

Hence $f\in L_1(\pi)$. On the other hand, for $t\geq M$,

$$\begin{aligned}t\bar{\mu}_t(f)=E^0\int_0^t f(X(s))\,\mathrm{d}s\geq E^0\int_0^{\mathcal{T}} f(X(s))\,\mathrm{d}s\\ &=\sum_k E^0\left\{\int_0^{\mathcal{T}} f(X(s))\,\mathrm{d}s;\,X(0)=k\right\}\\ &=\sum_k E^0\left\{\sum_{j=0}^{k} f(k-j)h(k-j);\,X(0)=k\right\}\\ &=\sum_k \alpha_k\sum_{j=0}^{k} f(j)h(j)\\ &=\sum_{j=0}^{\infty} f(j)h(j)\sum_{k\geq j}\alpha_k=\infty.\end{aligned}$$

We do, however, have the following:

Theorem 3.1. *If X is PRS and $g\in L_1^+(\pi)$ such that $(1/t)E\int_0^{t\wedge\tau(0)} g(\theta_s X)\,\mathrm{d}s\to 0$ and $\{(1/t)\int_0^t g(\theta_s\circ X)\,\mathrm{d}s\colon t\geq 0\}$ is UI under P^0, then*

$$\sup_{|f|\leq g}|\bar{\mu}_t(f)-\pi(f)|\to 0. \tag{3.1}$$

Before proving Theorem 3.1 we state an important corollary obtained immediately by using the function $g \equiv 1$.

Corollary 3.1. *If X is a PRS then $\bar{\mu}_t$ converges to π in total variation.* □

Proof of Theorem 3.1. Assume at first that $0 \leqslant f \leqslant g$ and that X is non-delayed. For $\varepsilon > 0$ let $\mathscr{A}(\varepsilon, t)$ denote the event $\{N(t) \geqslant (\hat{\lambda}+\varepsilon)t\}$. Let $J_n = J_n(f) \stackrel{\text{def}}{=} \int_{\tau(n-1)}^{\tau(n)} f(\theta_t \circ X)\,\mathrm{d}t$. Let $E_{\mathscr{I}}$ denote E^0 conditional on the invariant σ-field $\mathscr{I}$. Then

$$\begin{aligned}
&\frac{1}{t} E_{\mathscr{I}} \int_0^t f(\theta_s X)\,\mathrm{d}s \\
&= \frac{1}{t} E_{\mathscr{I}}\left\{\int_0^t f(\theta_s X)\,\mathrm{d}s;\ \mathscr{A}(\varepsilon, t)^{\mathrm{c}}\right\} + \frac{1}{t} E_{\mathscr{I}}\left\{\int_0^t f(\theta_s X)\,\mathrm{d}s;\ \mathscr{A}(\varepsilon, t)\right\} \\
&\leqslant \left[\frac{(\hat{\lambda}+\varepsilon)t+1}{t}\right] E_{\mathscr{I}} J_1 + \frac{1}{t} E_{\mathscr{I}}\left\{\int_0^t g(\theta_s X)\,\mathrm{d}s;\ \mathscr{A}(\varepsilon, t)\right\} \\
&\leqslant \hat{\lambda} E_{\mathscr{I}} J_1 + \left(\varepsilon + \frac{1}{t}\right) E_{\mathscr{I}} J_1 + \frac{1}{t} E_{\mathscr{I}}\left\{\int_0^t g(\theta_s X)\,\mathrm{d}s;\ \mathscr{A}(\varepsilon, t)\right\}.
\end{aligned} \tag{3.2}$$

Taking expectations in (3.2) with respect to E^0 yields

$$\begin{aligned}
&\frac{1}{t} E^0 \int_0^t f(\theta_s X)\,\mathrm{d}s \\
&\leqslant \pi(f) + \left(\varepsilon + \frac{1}{t}\right) E^0 J_1 + \frac{1}{t} E^0\left\{\int_0^t g(\theta_s X)\,\mathrm{d}s;\ \mathscr{A}(\varepsilon, t)\right\}.
\end{aligned} \tag{3.3}$$

By the uniform integrability hypothesis, the last term in (3.3) tends to zero. Moreover, ε was arbitrary. We thus obtain

$$\limsup_{t\to\infty} \sup_{f \leqslant g} \{\bar{\mu}_t(f) - \pi(f)\} \leqslant 0. \tag{3.4}$$

In a similar manner we obtain a lower bound: For $\varepsilon > 0$,

$$\begin{aligned}
&\frac{1}{t} E_{\mathscr{I}} \int_0^t f(\theta_s X)\,\mathrm{d}s \\
&\geqslant \frac{1}{t} E_{\mathscr{I}}\left\{\sum_{k=1}^{\lceil(\hat{\lambda}-\varepsilon)t\rceil} J_k;\ \mathscr{A}(-\varepsilon, t)\right\} \\
&\geqslant \left(\frac{(\hat{\lambda}-\varepsilon)t-1}{t}\right) E_{\mathscr{I}} J_1 - \frac{1}{t} E_{\mathscr{I}}\left\{\sum_{k=1}^{\lceil(\hat{\lambda}-\varepsilon)t\rceil} J_k(g);\ \mathscr{A}(-\varepsilon, t)^{\mathrm{c}}\right\} \\
&\geqslant \hat{\lambda} E_{\mathscr{I}} J_1 - \left(\varepsilon + \frac{1}{t}\right) E_{\mathscr{I}} J_1 - \frac{1}{t} E_{\mathscr{I}}\left\{\sum_{k=1}^{\lceil(\hat{\lambda}-\varepsilon)t\rceil} J_k(g);\ \mathscr{A}(-\varepsilon, t)^{\mathrm{c}}\right\},
\end{aligned}$$

which after taking expectations yields

$$\bar{\mu}_t(f)-\pi(f) \geq -\left(\varepsilon+\frac{1}{t}\right)E^0 E_{\mathscr{J}} J_1 - \frac{1}{t}E^0\left\{\sum_{k=1}^{\lceil\hat{\lambda}t\rceil} J_k(g);\ \mathscr{A}(-\varepsilon,t)^c\right\}. \tag{3.5}$$

Since $g \in L_1(\pi)$, $(1/t)\sum_{k=1}^{\lceil\hat{\lambda}t\rceil} J_j(g)$ is UI since it converges a.s. to $\hat{\lambda}E_{\mathscr{J}}J_1(g)$ and has mean, $E^0\{(\lceil\hat{\lambda}r\rceil/t)E_{\mathscr{J}}J_1(g)\}$, for each t; thus, the last term in (3.5) tends to zero. Consequently

$$\limsup_{t\to\infty}\ \sup_{f\leq g}\{\pi(f)-\bar{\mu}_t(f)\} \leq 0, \tag{3.6}$$

and we obtain (3.1). The case of f with arbitrary sign can be handled similarly; we leave out the details. In the delayed case, we have on the one hand that

$$\frac{1}{t}E\int_0^t f(\theta_s\circ X)\,ds \leq \frac{1}{t}E\int_0^{t\wedge\tau(0)} g(\theta_s X)\,ds + \frac{1}{t}E^0\int_0^t f(\theta_s\circ X)\,ds. \tag{3.7}$$

The first term on their right-hand side tends to zero by assumption giving the necessary upper bound. On the other hand, for $t \geq M \geq 0$,

$$\begin{aligned}
E\int_0^t f(\theta_s\circ X)\,ds &= E\left\{\int_0^t f(\theta_s\circ X)\,ds;\ \tau(0)\leq M\right\}\\
&\quad + E^0\left\{\int_0^t f(\theta_s\circ X)\,ds;\ \tau(0)>M\right\}\\
&\geq E\left\{\int_{\tau(0)}^t f(\theta_s\circ X)\,ds;\ \tau(0)\leq M\right\}\\
&\geq E^0\left\{\int_0^{t-M} f(\theta_s\circ X)\,ds;\ \tau(0)\leq M\right\}\\
&\geq E^0\left\{\int_0^{t-M} f(\theta_s\circ X)\,ds\right\}\\
&\quad - E^0\left\{\int_0^t g(\theta_s\circ X)\,ds;\ \tau(0)>M\right\}\\
&\geq E^0\left\{\int_0^t f(\theta_s\circ X)\,ds\right\} - E^0\left\{\int_{t-M}^t g(\theta_s\circ X)\,ds\right\}\\
&\quad - E^0\left\{\int_0^t g(\theta_s\circ X)\,ds;\ \tau(0)>M\right\}.
\end{aligned} \tag{3.8}$$

Using in $M=\varepsilon t, 0<\varepsilon<1$, in (3.8) yields

$$\begin{aligned}
\frac{1}{t}E\int_0^t f(\theta_s\circ X)\,ds &\geq \frac{1}{t}E^0\left\{\int_0^t f(\theta_s\circ X)\,ds\right\} - \frac{1}{t}E^0\left\{\int_{t-t\varepsilon}^t g(\theta_s\circ X)\,ds\right\}\\
&\quad - \frac{1}{t}E^0\left\{\int_0^t \left\{g(\theta_s\circ X)\,ds;\ \tau(0)>\varepsilon t\right\}\right\}.
\end{aligned} \tag{3.9}$$

The last integral above tends to zero by the UI assumption under P^0. The middle integral converges to $\varepsilon E^0\{\hat{\lambda} E_{\mathscr{I}} J_1(g)\}$. Letting ε tend to zero yields (together with (3.7)) the desired result. □

Proposition 3.1. *For a PRS, if either there exists an $\varepsilon>0$ such that $P^0(T_1>\varepsilon)=1$, or the cycles $\{X_n\}$ form a k-dependent process, then* (3.1) *holds for all $g \in L_1^+(\pi)$ such that $E\int_0^{\tau(0)} g(\theta_s X)\,\mathrm{d}s<\infty$.*

Proof. From Theorem 3.1, it suffices to show that $\{(1/t)\int_0^t g(\theta_s \circ X)\,\mathrm{d}s: t\geqslant 0\}$ is uniformly integrable under P^0. If $P^0(T_1>\varepsilon)=1$ then $(1/t)\int_0^t g(\theta_s\circ X)\,\mathrm{d}s \leqslant (1/t)\sum_1^{\lceil t/\varepsilon\rceil} J_k(g)$ which is UI for $g\in L_1^+(\pi)$. Now suppose that the cycles are k-dependent (in particular, X is ergodic). Then

$$\int_0^t g(\theta_s\circ X)\,\mathrm{d}s \leqslant \sum_{j=1}^{N(t)+k} J_j(g)$$

$$= \sum_{j=1}^{k} J_j(g) + \sum_{j=k+1}^{\infty} J_j(g) I(N(t)\geqslant j-k). \tag{3.10}$$

By the assumption of k-dependency, the indicator $I(N(t)\geqslant j-k)$ is independent of $J_j(g)$ and hence taking expectations in (3.10) yields

$$E^0\int_0^t g(\theta_s\circ X)\,\mathrm{d}s \leqslant (E^0 N(t)+k)E^0 J_1(g). \tag{3.11}$$

By Proposition 2.1, $\{N(t)/t\}$ is UI and hence by (3.11) so is $\{(1/t)\int_0^t g(\theta_s\circ X)\,\mathrm{d}s: t\geqslant 0\}$. □

Remark 3.1. If X is null recurrent and non-ergodic, it is still possible that π as defined in (A.3) is a probability measure. In this case Theorem 3.1 remains valid. Take for example, a mixture of Poisson processes: Choose a r.v. Y such that $P(0<Y\leqslant 1)=1$ and $E(1/Y)=\infty$. Conditional on Y, let $\{\tau(k)\}$ be a (non-delayed) Poisson process at rate Y. Define $X(t)$ as the forward recurrence time of this point process. Then the invariant σ-field is precisely $\sigma(Y)$, $E(T_1\,|\,Y)=1/Y$ and hence $E(T_1)=\infty$. Moreover, conditional on Y, the (marginal) steady-state distribution of $X(t)$ is exponential at rate Y and thus the (unconditional) steady-state distribution is given by $F(x)=1-E(\mathrm{e}^{-Yx})$.

Remark 3.2. The condition $(1/t)E\int_0^{t\wedge\tau(0)} g(\theta_s X)\,\mathrm{d}s\to 0$ is equivalent to UI of $\{(1/t)\int_0^{t\wedge\tau(0)} g(\theta_s X)\,\mathrm{d}s\}$.

4. Continuous time Harris recurrent Markov processes

In this section we establish uniform limit theorems for continuous time Harris recurrent Markov processes (HRMP's) analogous to those already known (in the literature) to be true for discrete time Harris recurrent Markov processes, called Harris recurrent Markov chains (HRMC's). Although renewal theory can be used to analyze HRMC's, the same is not true for general HRMP's (as defined below).

Let $\{Z(t): t \geq 0\}$ denote a Markov process with Polish state space $\mathcal{S}$ and paths in $\mathcal{D}_{\mathcal{S}}$. We shall always assume that Z has the strong Markov property.

Z is called *Harris recurrent* if there exists a non-trivial σ-finite measure μ on the Borel sets of $\mathcal{S}$ such that for any Borel set $A \subset \mathcal{S}$,

$$\mu(A) > 0 \quad \Rightarrow \quad P_z\left(\int_0^\infty 1_A \circ Z(t)\,\mathrm{d}t = \infty\right) = 1 \quad \text{for all } z. \tag{4.1}$$

It is known that an HRMP has a unique invariant measure (up to multiplicative constant); see for example, [2] and [12]. If the invariant measure is finite then it is normalized to a probability measure in which case Z is called *positive* recurrent. In Theorem 2 of [12], it is proved that a Markov process Z is a positive HRMP if and only if it is a positive recurrent one-dependent regenerative (od-R) process, that is, an ergodic synchronous process with one dependent cycles. In particular, Corollary 3.1 and Proposition 3.1 both apply to positive HRMP's. So, for example, given any initial state $Z_0 = z$, it follows that the Cesaro averaged measures $\bar{\mu}_t^z(A) \overset{\text{def}}{=} (1/t)\int_0^t E_z I_A \circ (\theta_s Z)\,\mathrm{d}s$ converge to π in total variation as $t \to \infty$.

Once the od-R points have been selected for an HRMP, a natural question arises as to whether or not, by placing some regularity conditions (non-lattice (or spread-out) cycle length distribution, etc.) on the cycles of an HRMP Z, the *unaveraged* distributions will converge weakly (or, even better, in total variation) to π, that is, if $\mu_t^z(f) \overset{\text{def}}{=} E_z(f(\theta_t Z)) \to \pi(f)$ for all bounded continuous f. The answer is no; a counterexample is given in Remark (3.2) of [12]. However, an immediate application of Theorem 1.1 yields:

Proposition 4.1. *If Z is a positive HRMP then the shifts $\{\theta_t \circ Z\}$ are tight in the function space $\mathcal{D}(0,\infty)$.* □

Continuing in the spirit of Cesaro convergence we have:

Proposition 4.2. *If Z is a positive HRMP with stationary distribution π then for each $g \in L_1^+(\pi)$,*

$$\sup_{|f| \leq g} |\bar{\mu}_t^z(f) - \pi(f)| \to 0 \quad \textit{for almost every } z \textit{ w.r.t. } \pi. \tag{4.2}$$

Proof. Let $r(z) \overset{\text{def}}{=} E_z \int_0^{\tau(0)} g(\theta_s X)\,\mathrm{d}s$ and $\varepsilon \overset{\text{def}}{=} \{z: r(z) < \infty\}$. From Proposition 3.1 it suffices to show that $\pi(\varepsilon) = 1$. Now,

$$\pi(\varepsilon) = \lambda E^0 \int_0^{\tau(1)} I(r(Z(s) < \infty)\,\mathrm{d}s. \tag{4.3}$$

Moreover,

$$E^0\{r(Z(s); \tau(1) > s\} = E^0\left\{ E^0_{Z(s)}\left\{ \int_0^{\tau(1)} (g(\theta_u Z)\, du \right\}; \tau(1) > s) \right\}$$
$$= E^0\left\{ \int_0^{\tau(1)} (g(\theta_u Z)\, du; \tau(1) > s) \right\}$$
$$\leqslant E^0\left\{ \int_0^{\tau(1)} (g(\theta_u Z)\, du \right\} < \infty.$$

Thus $r(Z(s)I(\tau(1) > s)) < \infty$, P^0-a.s., and hence $P^0\{r(Z(s)) < \infty; \tau(1) > s\} = 1$ for all $s \geqslant 0$. Integrating over s yields the result. □

Proposition 4.3. *If Z is a positive HRMP with stationary distribution π then*

$$\int \pi(dz) \sup_{|f| \leqslant g} |\bar{\mu}_t^z(f) - \pi(f)| \to 0, \tag{4.4}$$

for all $g \in L_1^+(\pi)$ such that $E^0\{\int_0^{\tau(1)} ug(\theta_u X)\, du\} < \infty$.

Proof. We must show that the error bounds for $|\bar{\mu}_t^z(d) - \pi(f)|$ can be integrated over z with respect to π. From the bounds obtained in (3.5)-(3.9) it suffices to show that $h(z) \overset{\text{def}}{=} E_z \int_0^{\tau(0)} g(\theta_s X)\, ds$ is in $L_1(\pi)$. An easy calculation yields

$$\int \pi(dz) h(z) = \lambda E^0 \int_0^{\tau(1)} \theta_s \circ \int_0^{\tau(1)} g(\theta_u X)\, du\, ds$$
$$= \lambda E^0 \int_0^{\tau(1)} \int_0^{\tau(1)} g(\theta_u X)\, du\, ds$$
$$= \lambda E^0 \int_0^{\tau(1)} ug(\theta_u X)\, du. \qquad \square$$

Remark 4.1. In the proof of Proposition 4.2, the assertion that $\pi(\varepsilon) = 1$ amounts, in the terminology of discrete time Markov chain theory, to showing that a.e. state z (with respect to π) is *g-regular* (see Proposition 5.13 of Nummelin [8]). In fact, Proposition 4.2 can be viewed as a continuous time Cesaro average analog of Corollary 6.7i in [8].

Remark 4.2. In Asmussen [1] the definition of HRMP is different than ours. Ours comes from Azema, Duflo and Revuz [2]. Asmussen's definition is more restrictive and in particular implies the existence of an embedded renewal process.

5. Applications to queues

In [11] the stability to open Jackson queueing networks with c nodes is established where service times at each node are i.i.d. with a general distribution, exogenous interarrival times are i.i.d. with a general distribution, and the routing is Markovian. We present here some immediate consequences of Section 4 in the context of the above stability result. The discrete time Markov process $X_n = (Q_n, Y_n)$ denotes the queue length vector and residual service time vector at the c nodes at time t_n- (the arrival time of the nth customer) and $Z(t) = (Q(t), Y(t), B(t))$ the associated continuous time Markov process (where $B(t)$ denotes the forward recurrence time of the exogenous renewal arrival process). ρ_i denotes the long run average rate at which work arrives (exogenously) to the system destined for node i. The reader is referred to [11] for the details of the model and the proofs of the following two theorems:

Theorem 5.1. *The Markov chain $X = \{X_n\}$ for JON is Harris ergodic if $\rho_i < 1$ for each i $(1 \leqslant i \leqslant c)$.* □

Theorem 5.2. *The Markov process Z for JON is a positive HRMP if $0 < \rho_i < 1$ for each i $(1 \leqslant i \leqslant c)$. In particular it is positive recurrent one-dependent regenerative (od-R) with a unique steady-state distribution π. In general, $\pi\{(X(t) = 0\} = 0$ and hence the regeneration points of Z are not described by consecutive visits of X to the empty state.* □

From the above theorem we see that Z is an ergodic PRS with one-dependent cycles and therefore so is any continuous functional $f(Z(t))$ such as total queue length $Q_T(t)$ (sum of the c queue lengths). Moreover, *total work in system* $w(t)$ is also; $w(t)$ denotes the sum of all remaining service times of all customers in the system (including their feedback) at time t (see Section 4 of [11]). We thus obtain the following special cases of the results in Section 4:

Proposition 5.1. *For a JON with $\rho_i < 1$ for each i $(1 \leqslant i \leqslant c)$, the following hold:*

$$\{\theta_t \circ Q_T\} \text{ and } \{\theta_t \circ w\} \text{ are tight in function space}$$

for each fixed initial condition,

$$\frac{1}{t}\int_0^t P_z(w(s) \in \cdot)\,\mathrm{d}s \to P_\pi(w(0) \in \cdot) \text{ in total variation for each } z,$$

$$\frac{1}{t}\int_0^t P_z(Q_T(s) \in \cdot)\,\mathrm{d}s \to P_\pi(Q_T(0) \in \cdot) \text{ in total variation for each } z.$$

If $E_\pi(w(0)) < \infty$ then

$$\frac{1}{t}\int_0^t E_z(w(s))\,\mathrm{d}s \to E_\pi(w(0)) \text{ for almost every } z \text{ w.r.t. } \pi,$$

$$\frac{1}{t}\int_0^t E_z(Q_T(s))\,\mathrm{d}s \to E_\pi(Q_T(0)) \text{ for almost every } z \text{ w.r.t. } \pi. \qquad \square$$

Remark 5.1. Although the above queueing model has the special feature of i.i.d. input, this is not the key ingredient. The real importance of the above results is that they apply to any queue that can be modeled as a positive HRMP (and there are many, see [12]).

Appendix: A brief introduction to synchronous processes

Our use of the word *synchronous* is from [5]. Other names have been given to a synchronous process; for example Serfozo [10] refers to them as semi-stationary processes. In Rolski [9] they arise as *Palm* versions of stationary processes (associated with point processes). Closely related to this is the general theory of stationary marked point processes [5]. In any case, the ergodic properties of synchronous processes are well known in the literature. We state several such results the proofs of which can be found in, for example [6] and [9].

Let $\theta_t:\mathscr{D}\to\mathscr{D}$ denote the *shift operator* $(\theta_t x)(s)=x(t+s)$.

Theorem A.1. *Suppose X is a PRS and $f:\mathscr{D}_{\mathscr{S}}\to\mathbb{R}$ is measurable. Let $J_n=J_n(f)\stackrel{\text{def}}{=}\int_{\tau(n-1)}^{\tau(n)} f(\theta_t\circ X)\,\mathrm{d}t$. If $J_0(|f|)<\infty$ a.s. and if either $f\geqslant 0$ a.s. or $E\{J_1(|f|)\}<\infty$ then*

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t f(\theta_s\circ X)\,\mathrm{d}s=\frac{E\{J_1\mid\mathscr{I}\}}{E\{T_1\mid\mathscr{I}\}}\quad a.s. \tag{A.1}$$

where $\mathscr{I}$ denotes the invariant σ-field associated with $\{(X_n, T_n)\}$. □

Let P^0 denote the probability measure under which X is non-delayed, that is, $P^0(X\in A)=P(\theta_{\tau(1)}\circ X\in A)$.

Corollary A.1. *Under the conditions of Theorem* A.1, *if in addition $\mathscr{I}$ is trivial (every set has probability* 0 *or* 1) *then $\{J_n, T_n\colon n\geqslant 1\}$ is ergodic and hence a.s.*

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t f(\theta_s\circ X)\,\mathrm{d}s=\frac{E\{J_1\}}{E\{T_1\}}=\lambda\int_0^\infty P^0(\theta_s\circ X\in A;\ \tau(1)>s)\,\mathrm{d}s. \tag{A.2}$$

Under these circumstances, X is called ergodic. □

The following Corollary follows from (A.1) by an elementary application of Fubini's Theorem and the Bounded Convergence Theorem.

Corollary A.2. *Under the hypothesis of Theorem* A.1, *if in addition f is bounded then*

$$\bar{\mu}_t(f)\stackrel{\text{def}}{=}\frac{1}{t}\int_0^t Ef(\theta_s\circ X)\,\mathrm{d}s\to\pi(f)\stackrel{\text{def}}{=}E\left\{\frac{E\{J_1\mid\mathscr{I}\}}{E\{T_1\mid\mathscr{I}\}}\right\}. \tag{A.3}$$

π above defines a measure on $\mathscr{D}$ and (for reasons given below in Proposition A.1) *is called the stationary probability measure for X. In particular, by choosing $f=1_A$ (an indicator function), we have $\bar{\mu}_t(A)\to\pi(A)$ for each Borel set A of $\mathscr{D}$. In particular, the Cesaro averaged distributions converge weakly.* □

Proposition A.1. *Let π be the stationary measure of a PRS X. Then under π, $\theta = (\theta_s)$ is measure preserving on $\mathscr{D}$, that is, for each Borel set A, $\pi(A) = \pi(\theta_{-s}A)$ for all $s \geq 0$. In particular, if X has distribution π, then X is time stationary, that is, $\theta_t X$ has the same distribution for each $t \geq 0$.* □

Let P^* denote the probability measure under which X has distribution π, that is, $P^*(X \in A) = \pi(A)$. From (A.2) we obtain for an *ergodic* synchronous process that

$$P^*(X \in A) = \lambda \int_0^\infty P^0(\theta_s \circ X \in A;\ \tau(1) > s)\ \mathrm{d}s. \tag{A.4}$$

If X is positive recurrent but not ergodic then the right-hand side of (A.4) still defines a probability measure on $\mathscr{D}$ (but not necessarily the same as the π from (A.3)). In fact, more can be said:

Proposition A.2. *For a PRS the right-hand side of* (A.4) *defines a probability measure on $\mathscr{D}$ (in general, not the same as π) under which $\theta = (\theta_s)$ is measure preserving.* □

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