

Smoothed Limit Theorems for Equilibrium Processes

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1. INTRODUCTION

Renewal ideas play a fundamental role in applied probability. In particular, renewal methods are powerful tools for dealing with the asymptotic behavior of the regenerative processes that are typical of queueing and operations research applications. In this paper we survey the main results of the theory and develop some "smoothed" versions of the convergence theorems. These smoothed versions typically have much weaker associated regularity hypotheses than the classical version. In fact, these smoothed limit theorems are established under minimal conditions that are essentially necessary.

This paper is organized as follows. Section 2 describes the classical limit theory for solutions to renewal equations; Section 3 discusses the extensions to equilibrium processes. In Section 4, a smoothed version of the renewal theorem is described. It is applied to equilibrium processes in Section 5.

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2. THE TWO BASIC FORMS OF THE RENEWAL THEOREM

In this section, we review the two basic forms of the renewal theorem. Let \mathcal{F} be the family of probability distribution functions F corresponding to non-negative r.v.'s which are positive with positive probability (i.e. $F(0-) = 0$ and $F(0) < 1$). Given $F \in \mathcal{F}$ and a (measurable) function $b(t)$ which is a.e. finite-valued (a.e. with respect to Lebesgue measure), the **renewal equation** takes the form

$$(2.1) \quad a(t) = b(t) + \int_{[0,t]} a(t-u)F(du)$$

for a.e. $t \geq 0$. The function $a(\cdot)$ appearing in (2.1) is termed the **solution** to the renewal equation (when it exists) and the goal here is to study the behavior of $a(t)$ as t approaches infinity.

As is usual in the literature, we adopt the notation $F * a$ to denote the convolution $\int_{[0,t]} a(t-u)F(du)$. The renewal equation (2.1) can then be re-written symbolically in the form $a = b + F * a$.

Before proceeding to the statements of the two basic forms of the renewal theorem, we will first give conditions under which unique solutions to the renewal equation exist. Define \mathcal{B} to be the class of Borel measurable functions $b : [0, \infty) \rightarrow \mathbb{R}$ such that $\sup\{|b(s)| : 0 \leq s \leq t\} < \infty$ for all $t \geq 0$. In other words, \mathcal{B} corresponds to the family of real-valued Borel measurable functions on $[0, \infty)$ that are bounded on bounded intervals. For $F \in \mathcal{F}$, we let $U_F(t)$ be the function on $[0, \infty)$ defined by

$$U_F(t) = \sum_{n=0}^{\infty} F^{(n)}(t),$$

where $F^{(n)}$ is the n -fold convolution of F . The function $U_F(t)$ is called the **renewal function** associated with F .

The following well-known theorem establishes a uniqueness-existence result for solutions to the renewal equation. (For a proof, see p. 184–186 of Karlin and Taylor (1975).)

(2.2) **THEOREM.** If $F \in \mathcal{F}$ and $b \in \mathcal{B}$, then there exists a solution $a \in \mathcal{B}$ to the renewal equation (2.1). The solution a is unique in the class \mathcal{B} and is

given by

$$(2.3) \quad a = U_F * b.$$

Furthermore, the solution a defined by (2.3) satisfies (2.1) at every $t \geq 0$.

In many applications, we can not assume that either a or b is in the class \mathcal{B} . For example, there may exist a non-empty Lebesgue set of measure zero on which $|b(t)|$ is infinite. To deal with this situation, we can often use the next result. (See Glynn (1988) for a proof.) Let \mathcal{L} be the class of functions $b : [0, \infty) \rightarrow \mathbb{R}$ such that $\int_0^t |b(s)|ds < \infty$ for all $t \geq 0$.

(2.4) **THEOREM.** If $F \in \mathcal{F}$ and $b \in \mathcal{L}$, then there exists a solution $a \in \mathcal{L}$ to the renewal equation (2.1). The solution a is unique in the class \mathcal{L} (i.e. if $a_1, a_2 \in \mathcal{L}$ satisfy (2.1), then $a_1 = a_2$ a.e.) and is given by $a = U_F * b$.

Typically, the solution $U_F * b$ to the renewal equation (2.1) can not be computed explicitly. However, the renewal theorem allows us to analyze the limiting behavior of the solution $(U_F * b)(t)$ as $t \rightarrow \infty$. Let $\lambda = (\int_{[0,\infty)} tF(dt))^{-1}$ and define λ to equal zero if F has infinite mean. Basically, the renewal theorem provides conditions under which

$$(2.5) \quad (U_F * b)(t) \rightarrow \lambda \int_0^{\infty} b(s)ds$$

as $t \rightarrow \infty$. The two versions of the renewal theorem give two different sets of conditions for establishing the validity of (2.5). The first version that we shall discuss is known as Smith's form of the renewal theorem; it imposes a fairly strong regularity hypothesis on F and, as a result, demands less structure on b .

(2.6) **THEOREM.** Suppose that $F \in \mathcal{F}$ and is spread-out (i.e. for some $n \geq 1$, $F^{(n)}$ has a non-trivial absolutely continuous component with respect to Lebesgue measure). If b is bounded, Lebesgue integrable on $[0, \infty)$, and $b(t) \rightarrow 0$ as $t \rightarrow \infty$, then $(U_F * b)(t) \rightarrow \lambda \int_0^{\infty} b(s)ds$.

For a proof, see Smith (1955, 1960). Looking at the conditions on b , the integrability hypothesis is natural in view of the fact that the integral $\int_0^{\infty} b(s)ds$ appears in the limit. As for the two other conditions, the following examples show that they are, in some sense, necessary.

(2.7) EXAMPLE. (necessity of boundedness): Suppose that b is non-negative, Lebesgue integrable on $[0, \infty)$, and $b(t) \rightarrow 0$ as $t \rightarrow \infty$, but that $\sup\{|b(s)| : 0 \leq s \leq 1\} = \infty$. We will show that there exists a spread-out $F \in \mathcal{F}$ such that $(U_F * b)(n) = \infty$ for $n \geq 1$.

There exists $\{t_n : n \geq 1\}$ such that $0 \leq t_n \leq 1$ and $b(t_n) \rightarrow \infty$. By picking a suitable subsequence, we may further assume that $b(t_n) \cdot 2^{-n} \rightarrow \infty$ as $n \rightarrow \infty$. Then, let $B(x) = \sum_{k=1}^{\infty} 2^{-k} I(x \geq 1 - t_k)$ and observe that

$$\int_{[0,1]} b(1-x)B(dx) = \sum_{k=1}^{\infty} 2^{-k} b(t_k) = \infty.$$

Now, let $F(x) = 2^{-1}(x \wedge 1) + \sum_{n=1}^{\infty} 2^{-n-1} B(x-n+1)$ for $x \geq 0$; note that F is spread-out. But

$$(U_F * b)(n) \geq \int_{[n-1,n]} b(n-x)F(dx) \geq 2^{-n-1} \int_{[0,1]} b(1-x)B(dx) = \infty.$$

(2.8) EXAMPLE. (necessity of $b(t) \rightarrow 0$ as $t \rightarrow \infty$): Suppose that b is bounded and Lebesgue integrable on $[0, \infty)$, but $b(t) \not\rightarrow 0$ as $t \rightarrow \infty$. We will show that there exists a spread-out $F \in \mathcal{F}$ such that $(U * b)(t) \not\rightarrow \lambda \int_0^{\infty} b(s)ds$. Choose $F(x) = 1 - \exp(-\lambda x)$ for $x \geq 0$. Then, $U_F(t) = 1 + \lambda t$ (as may be verified via Laplace transforms), so that

$$(U_F * b)(t) = b(t) + \lambda \int_0^t b(s)ds.$$

Since $\int_0^t b(s)ds \rightarrow \int_0^{\infty} b(s)ds$, it is evident that $b(t)$ must tend to zero, in order that $(U_F * b)(t)$ converge to the correct limit.

More recently, the following uniform version of Smith's theorem has been obtained by Arjas, Nummelin, and Tweedie (1978).

(2.9) THEOREM. Suppose that $F \in \mathcal{F}$ and is spread-out. If c is a non-negative bounded Lebesgue integrable function on $[0, \infty)$, and $c(t) \rightarrow 0$ as $t \rightarrow \infty$, then

$$\lim_{t \rightarrow \infty} \sup_{|b| \leq c} |(U_F * b)(t) - \lambda \int_0^{\infty} b(s)ds| = 0.$$

Given an operator H , the notation $\sup_{|b| \leq c} H(b)$ means $\sup\{H(b) : b \text{ is a Borel measurable function on } [0, \infty) \text{ such that } |b(t)| \leq c(t) \text{ for each } t \geq 0\}$.

The second version of the renewal theorem is due to Feller. Rather than require F to be spread-out, it is demanded only that F be non-arithmetic (i.e. there exists no positive number η such that all the points of increase of F appears in the set $\{k\eta : k \geq 0\}$). On the other hand, the conditions on b are now strengthened somewhat. Let

$$\bar{\sigma}_n(b, h) = \sup\{b(s) : nh \leq s < (n+1)h\}$$

$$\underline{\sigma}_n(b, h) = \inf\{b(s) : nh \leq s < (n+1)h\}.$$

The function b is said to be **directly Riemann integrable** (d.R.i.) if $\sum_{n=0}^{\infty} \bar{\sigma}_n(|b|, h) < \infty$ for all $h > 0$ and if

$$\lim_{h \downarrow 0} h \cdot \sum_{n=0}^{\infty} (\bar{\sigma}_n(b, h) - \underline{\sigma}_n(b, h)) = 0.$$

Note that every directly Riemann integrable function b is necessarily bounded, Lebesgue integrable over $[0, \infty)$, and convergent to zero at infinity.

(2.10) THEOREM. Suppose that $F \in \mathcal{F}$ and is non-arithmetic. If b is d.R.i., then $(U_F * b)(t) \rightarrow \lambda \int_0^{\infty} b(s)ds$ as $t \rightarrow \infty$.

For a proof, see Feller (1970). With this strong hypothesis on b , it is well known that the non-arithmetic hypothesis on F is necessary and sufficient for $(U_F * b)(t)$ to converge to $\lambda \int_0^{\infty} b(s)ds$. By this, we mean that if $\lambda > 0$, then it is necessary and sufficient that F be non-arithmetic in order that $(U_F * b)(t) \rightarrow \lambda \int_0^{\infty} b(s)ds$ for all d.R.i. b . (For the necessity, note that if F is arithmetic with span η and $b(t) = I(0 \leq t < \eta/2)$, then $(U_F * b)(t) = U_F(t) - U_F(t - \eta/2)$, so that $(U_F * b)(n\eta) \rightarrow \lambda\eta$ by the discrete renewal theorem, whereas $(U_F * b)(n\eta + \frac{3}{4}\eta) \rightarrow 0$ as $n \rightarrow \infty$.)

3. LIMIT THEOREMS FOR EQUILIBRIUM PROCESSES

Our objective here is to discuss the implications for equilibrium processes of the renewal theorems stated in Section 2. An equilibrium process is a generalization of regenerative process that was introduced by Smith (1955).

Specifically, let $X = (X(t) : t \geq 0)$ be a (measurable) S -valued stochastic process. Given an increasing sequence of random times $0 \leq T(0) < T(1) < \dots$, set $T(-1) = 0$ and let $N(t) = \max\{n \geq -1 : T(n) \leq t\}$. Put $\tilde{X}(t) = (X(t), t - T(N(t)))$. X is said to be an equilibrium process (with respect to $(T(n) : n \geq 0)$) if:

(3.1) i) $(\tilde{X}(T(n) + t) : t \geq 0) \stackrel{\mathcal{D}}{=} (\tilde{X}(T(0) + t) : t \geq 0)$ for $n \geq 0$ ($\stackrel{\mathcal{D}}{=}$ denotes equality in distribution).

ii) $T(n)$ is independent of $(\tilde{X}(T(n) + t) : t \geq 0)$ for $n \geq 0$.

X is a **non-delayed** equilibrium process if $T(0) = 0$ and is otherwise said to be **delayed**. Note that we do not demand that $(\tilde{X}(t) : t < T(n))$ be independent of $(\tilde{X}(t) : t \geq T(n))$, as would be the case for a regenerative process.

The principal application to equilibrium processes of renewal methods involves the study of the long-run behavior of the system. Specifically, let $\theta_t X = (X(t + s) : s \geq 0)$ and let f be a real-valued (suitably measurable) function. We will discuss the asymptotic behavior of $Ef(\theta_t X)$; of course, $Ef(X(t))$ is but a special case of this.

Let $\tau_i = T(i) - T(i - 1)$ for $i \geq 0$. Set $a_f(t) = Ef(\theta_t X)$, $b_f(t) = E\{f(\theta_{T(0)+t} X); \tau_1 > t\}$, $c_f(t) = E\{f(\theta_t X); \tau_0 > t\}$, $G(t) = P\{\tau_0 \leq t\}$ and $F(t) = P\{\tau_1 \leq t\}$. Finally, put $V_i(f) = \sup\{f(\theta_t X) : T(i - 1) \leq t < T(i)\}$ and $Y_i(f) = \int_{T(i-1)}^{T(i)} f(\theta_s X) ds$ for $i \geq 0$.

(3.2) PROPOSITION. i) Suppose that $EV_0(|f|) < \infty$ and $EV_1(|f|) < \infty$. Then, for all $t \geq 0$, $a_f(t)$ is finite-valued and satisfies, for all $t \geq 0$,

$$(3.3) \quad a_f(t) = c_f(t) + (G * U_F * b_f)(t).$$

ii) Suppose that $EY_0(|f|) < \infty$, $EY_1(|f|) < \infty$. Then, for a.e. $t \geq 0$ (a.e. with respect to Lebesgue measure), $a_f(t)$ is finite-valued. Furthermore, a_f satisfies (3.3) at a.e. $t \geq 0$.

PROOF. A standard argument shows that when f is bounded, (3.3) holds. (Theorem 2.2 can be applied directly in this case.) An easy approximation argument then shows that (3.3) holds for all non-negative f . To extend to the case where f is unbounded and of mixed sign, we use another approximation argument to show that (3.3) holds at each $t \geq 0$ for which $a_{|f|}(t) < \infty$. For i), note that $c_{|f|}(t) \leq EV_0(|f|)$, $b_{|f|}(t) \leq EV_1(|f|)$, and

it follows that $a_{|f|}(t) \leq EV_0(|f|) + U_F(t) \cdot EV_1(|f|) < \infty$. For ii), observe that $\int_0^\infty c_{|f|}(s) ds = EY_0(|f|)$, $\int_0^\infty b_{|f|}(s) ds = EY_1(|f|)$, and it follows that $\int_0^t a_{|f|}(s) ds \leq EY_0(|f|) + U_F(t) \cdot EY_1(|f|) < \infty$ for all $t \geq 0$.

Note that the convolution $U_F * b_f$ appears in (3.3), thereby suggesting that the renewal theorem will play a central role in the analysis of the asymptotic behavior of $a_f(t)$. The following theorem is the translation of Theorem 2.9 to our current setting. (Observe that Fubini's theorem implies that $\int_0^\infty b_f(t) dt = EY_1(f)$.)

(3.4) THEOREM. Suppose that $F \in \mathcal{F}$ and is spread-out. If $E(V_0(|g|) + V_1(|g|) + Y_1(|g|)) < \infty$, then

$$\lim_{t \rightarrow \infty} \sup_{|f| \leq |g|} |Ef(\theta_t X) - \lambda EY_1(f)| = 0.$$

PROOF. The proof is basically a consequence of Theorem 2.9. Note that $EV_0(|g|) < \infty$ implies that $c_f(t) \rightarrow 0$ uniformly in f satisfying $|f| \leq |g|$. To handle $G * U_F * b_f$, we apply a slight extension of Theorem 2.9, stated as Theorem 1 of Arjas, Nummelin, and Tweedie (1978).

In terms of the hypotheses that enter Theorem 3.4, the moment conditions on $V_0(|g|)$ and $V_1(|g|)$ might, at first glance, seem unnecessary. They are basically imposed in order to guarantee that $c_{|g|}(t)$ and $b_{|g|}(t)$ be bounded functions that vanish at infinity. As pointed out in Examples 2.7 and 2.8, some condition of this kind is necessary.

Suppose that $E\tau_1 < \infty$ and let $\pi(\cdot)$ be the probability distribution on S defined by $\pi(\cdot) = \lambda E\{\int_{T(0)}^{T(1)} I(X(s) \in \cdot) ds\}$. An important corollary to Theorem 3.4 is that if F is spread-out, then $P\{X(t) \in \cdot\}$ converges to $\pi(\cdot)$ in total variation norm.

(3.5) COROLLARY. Suppose that $F \in \mathcal{F}$ and is spread-out. If $E\tau_1 < \infty$, then

$$\|P\{X(t) \in \cdot\} - \pi(\cdot)\| \rightarrow 0 \text{ as } t \rightarrow \infty.$$

where $\|\eta(\cdot)\| \stackrel{\Delta}{=} \sup\{|\eta(A)| : A \text{ is Borel measurable}\}$.

Miller (1972) points out that, in some sense, the requirement that F be spread-out is necessary. Suppose that $F \in \mathcal{F}$ has finite mean and yet is not

spread-out. For such an F , Miller showed that there exists a real-valued regenerative process X for which τ_1 has distribution F and yet $X(t)$ does not even converge in distribution as $t \rightarrow \infty$. Thus, in order to weaken the requirement on F to an assumption that F is non-arithmetic, it is necessary to include a hypothesis on X .

Suppose that S is a metric space. Let $D_S[0, \infty)$ be the Skorohod space of right-continuous functions $x : [0, \infty) \rightarrow S$ having left limits at every $t > 0$. The hypothesis on X involves assuming that X has paths in $D_S[0, \infty)$. The next result is basically Miller's extension of Feller's version of the renewal theorem to the equilibrium process setting.

(3.6) THEOREM. Suppose that $F \in \mathcal{F}$ is non-arithmetic and has finite mean. If S is a metric space and $X \in D_S[0, \infty)$, then $P\{X(t) \in \cdot\} \Rightarrow \pi(\cdot)$ as $t \rightarrow \infty$. (\Rightarrow denotes weak convergence).

PROOF. This result was established by Miller (1972) in the special case that $S = \mathbb{R}$ (this argument is stated in terms of regenerative processes but easily extends to equilibrium processes). To extend to the general case, let $f : S \rightarrow \mathbb{R}$ be a continuous map. Then, $(f(X(t)) : t \geq 0)$ is an equilibrium process having paths in $D_{\mathbb{R}}[0, \infty)$. We can now apply Theorem 3.1 of Miller (1972) to conclude that $f(X(t)) \Rightarrow f(X(\infty))$ as $t \rightarrow \infty$, where $X(\infty)$ has distribution π . It follows from Theorem 5.2 ii) of Billingsley (1968) that $X(t) \Rightarrow X(\infty)$ as $t \rightarrow \infty$.

We note that Theorem 3.6, in contrast to Corollary 3.5, requires topological assumptions on S and path regularity of X . Furthermore, the total variation convergence of Corollary 3.5 is much stronger than the weak convergence of Theorem 3.6. In addition, note that Theorem 3.6 permits us only to assert that $Ef(X(t)) \rightarrow Ef(X(\infty))$ for all bounded continuous f ; if f is either unbounded or discontinuous, Theorem 3.6 offers no conclusion, in contrast to Theorem 3.4. On the other hand, the non-arithmetic assumption on τ_1 is considerably weaker than the demand that τ_1 be spread-out, as appeared in Theorem 3.4 and Corollary 3.5. Thus, both the Smith and Feller versions described above are somewhat unsatisfactory.

4. A SMOOTHED VERSION OF THE RENEWAL THEOREM

In Sections 4 and 5, we state and prove the main results of this paper. We start in this section by describing a smoothed variant of the renewal theorem. This theorem will basically establish "smoothed convergence" of $(U_F * b)(t)$ to $\lambda \int_0^\infty b(s)ds$ under the weakest possible conditions, namely integrability of b over $[0, \infty)$ and non-arithmeticity of F .

Let K be an absolutely continuous distribution in \mathcal{F} which has a Lebesgue density k with support contained in $[0, T]$ ($0 < T < \infty$); we further require that k be continuous on $[0, T]$. We will use K to smooth out the solution $\mathbf{a} = U_F * b$ to the renewal equation. Specifically, we will consider the quantity $(K * \mathbf{a})(t) = \int_0^T k(s)\mathbf{a}(t-s)ds$ as $t \rightarrow \infty$. Thus, K is used to smooth the solution \mathbf{a} in the region $[t-T, t]$. Since convolution with K ought to enhance the regularity of \mathbf{a} , we expect that some of the pathologies which we discussed in Section 2 will cease to be a problem. The next theorem is an illustration of this point (compare with Theorem 2.9).

(4.1) THEOREM. Suppose that $F \in \mathcal{F}$ is non-arithmetic. If c is a non-negative Lebesgue integrable function on $[0, \infty)$, then

$$\lim_{t \rightarrow \infty} \sup_{|b| \leq c} |(K * U_F * b)(t) - \lambda \int_0^\infty b(s)ds| = 0.$$

PROOF. First we extend b and k to $(-\infty, \infty)$ by setting $b(t) = k(t) = 0$ for $t < 0$. Let $U'_F(t) = U_F(t) - \lambda t$ and $\varphi_b(t) = (K * b)(t)$. Observe that $K * U_F = U_F * K$ and $\int_0^\infty \varphi_b(s)ds = \int_0^\infty b(s)ds$ for b integrable. Hence,

$$\begin{aligned} & \sup_{|b| \leq c} |(K * U_F * b)(t) - \lambda \int_0^\infty b(s)ds| \\ (4.2) \quad &= \sup_{|b| \leq c} |(U'_F * \varphi_b)(t) - \lambda \int_0^\infty \varphi_b(s)ds| \\ &\leq \sup_{|b| \leq c} |(U'_F * \varphi_b)(t)| + \lambda \int_t^\infty \varphi_c(s)ds. \end{aligned}$$

Since φ_c is integrable over $[0, \infty)$, the second term tends to zero as $t \rightarrow \infty$. For the first term, we approximate φ_b by a suitably chosen piecewise

constant function φ_b^h . Set $\varphi_b^h(t) = \varphi_b(h\lfloor t/h \rfloor)$ for $h > 0$ and write $U'_F * \varphi_b = U'_F * \varphi_b^h + U'_F * (\varphi_b - \varphi_b^h)$. Let $\ell = \lfloor t/h \rfloor$ and $r = t - h\lfloor t/h \rfloor$. It may be verified that

$$(4.3) \quad (U'_F * \varphi_b^h)(t) = \varphi_b(\ell h) \cdot U'_F(r) + \sum_{i=0}^{\ell-1} \varphi_b((\ell - i - 1)h) \cdot (U'_F(r + (i+1)h) - U'_F(r + ih)).$$

It follows that

$$(4.4) \quad \sup_{|b| \leq c} |(U'_F * \varphi_b^h)(t)| \leq \varphi_c(h\lfloor t/h \rfloor)(U_F(h) + \lambda h) + \sum_{j=0}^{\ell-1} \varphi_c(jh) |U_F(t - jh) - U_F(t - (j+1)h) - \lambda h|.$$

For the first term on the right-hand side of (4.4), we note that $\varphi_c(t) = \int_0^t c(s)k(t-s)ds \rightarrow 0$ by the bounded convergence theorem (k is bounded because of continuity over its support). So the first term vanishes as $t \rightarrow \infty$. To deal with the second term, recall that $U_F(t - jh) - U_F(t - (j+1)h) \rightarrow \lambda h$ as $t \rightarrow \infty$ by Blackwell's theorem (Feller (1970), p. 360). Furthermore, these differences are bounded uniformly in t and j . Hence, the second term in (4.4) vanishes as $t \rightarrow \infty$, provided that we show $\sum_{j=0}^{\infty} \varphi_c(jh) < \infty$.

To prove this, let $M = \sup\{|k(s)| : 0 \leq s \leq T\}$ and let $\varepsilon(h) = \sup\{|k(t) - k(t-h)| : h \leq t \leq T\}$. Note that $\varepsilon(h) \downarrow 0$ as $h \downarrow 0$ by uniform continuity of k . For $|b| \leq c$ and $0 < t' = h\lfloor t/h \rfloor < t$,

$$\begin{aligned} |\varphi_b(t) - \varphi_b(t')| &= \left| \int_{t-T}^t k(t-s)b(s)ds - \int_{t'-T}^{t'} k(t'-s)b(s)ds \right| \\ &\leq \int_{t'}^t k(t-s)|b(s)|ds + \int_{t-T}^{t'} |k(t-s) - k(t'-s)||b(s)|ds \\ &\quad + \int_{t'-T}^{t-T} k(t'-s)|b(s)|ds \\ &\leq M \int_{t-h}^t c(s)ds + \int_h^T |k(u) - k(u-t+t')|c(t-u)du \end{aligned}$$

(4.5)

$$\begin{aligned} &+ \int_{t-t'}^h k(u)c(t-u)du + \int_{t-t'}^h k(u-t+t')c(t-u)du \\ &+ M \int_{t-T-h}^{t-T} c(s)ds \\ &\leq M \left(\int_{t-h}^t c(s)ds + 2 \int_{t-h}^{t+h} c(s)ds + \int_{t-T-h}^{t-T} c(s)ds \right) \\ &+ \varepsilon(h) \int_{t-T}^t c(s)ds. \end{aligned}$$

Since $\int_{t-\alpha}^t c(s)ds$ is a Lebesgue integrable function of t over $[0, \infty)$ for any finite α , it is evident from (4.5) that

$$\begin{aligned} h \sum_{j=0}^{\infty} \varphi_c(jh) &= \int_0^{\infty} \varphi_c(h\lfloor t/h \rfloor)dt \\ &\leq \int_0^{\infty} \varphi_c(t)dt + \int_0^{\infty} |\varphi_c(t) - \varphi_c(h\lfloor t/h \rfloor)|dt < \infty. \end{aligned}$$

To complete the proof of the theorem, it is sufficient to prove that $\overline{\lim}_{t \rightarrow \infty} \sup_{|b| \leq c} |(U'_F * (\varphi_b - \varphi_b^h))(t)| \equiv \beta(h)$, with $\beta(h) \downarrow 0$ as $h \downarrow 0$. It is easily seen that (4.5) proves that $|\varphi_b(t) - \varphi_b^h(t)| \leq \psi_h(t)$, where

$$\psi_h(kh) = M \left[3 \int_{h(k-1)}^{h(k+2)} c(s)ds + \int_{h(k-1)-T}^{h(k+1)-T} c(s)ds \right] + \varepsilon(h) \int_{hk-T}^{h(k+1)} c(s)ds$$

and $\psi_h(t) = \psi_h(h\lfloor t/h \rfloor)$. A routine calculation shows that $\psi_h(t) \rightarrow 0$ as $t \rightarrow \infty$ and that $\int_0^{\infty} \psi_h(t)dt \equiv \gamma(h)$, with $\gamma(h) \downarrow 0$ as $h \downarrow 0$. Now,

$$(4.6) \quad \sup_{|b| \leq c} |(U'_F * (\varphi_b - \varphi_b^h))(t)| \leq \int_{[0, t]} (U_F(ds) + \lambda ds) \psi_h(t-s).$$

Using the piecewise-constant structure of ψ_h , we can argue, as we did for (4.3), that

$$\begin{aligned} (U_F * \psi_h)(t) &\leq \psi_h(h\lfloor t/h \rfloor)U_F(h) \\ &\quad + \sum_{j=0}^{\lfloor t/h \rfloor - 1} \psi_h(jh)(U_F(t - jh) - U_F(t - (j-1)h)). \end{aligned}$$

Again, Blackwell's theorem asserts that $U_F(t - jh) - U_F(t - (j - 1)h) \rightarrow \lambda h$ as $t \rightarrow \infty$. Using the aforementioned properties of ψ_h , we find that $\overline{\lim}_{t \rightarrow \infty} (U_F * \psi_h)(t) \leq \lambda \int_0^\infty \psi_h(s) ds$. It follows from (4.6) that

$$\overline{\lim}_{t \rightarrow \infty} \sup_{|b| \leq c} |(U'_F * (\varphi_b - \varphi_b^h))(t)| \leq 2\lambda\gamma(h),$$

completing the proof.

The following "averaging" result is an immediate consequence of Theorem 4.1.

(4.7) COROLLARY. Suppose that $F \in \mathcal{F}$ is non-arithmetic and b is Lebesgue integrable on $[0, \infty)$. Then, for $h > 0$,

$$\frac{1}{h} \int_{t-h}^t (U_F * b)(s) ds \rightarrow \lambda \int_0^\infty b(s) ds \text{ as } t \rightarrow \infty.$$

A noteworthy feature of this result is that it holds for any positive h . Thus, if the solution $U_F * b$ is averaged in an arbitrarily small neighborhood of t , it behaves nicely.

5. SMOOTHED LIMIT THEOREMS FOR EQUILIBRIUM PROCESSES

In this section, we apply Theorem 4.1 to equilibrium processes. Our first result is obtained by combining Proposition 3.2 and Theorem 4.1.

(5.1) THEOREM. Suppose that $F \in \mathcal{F}$ is non-arithmetic and $E(V_0(|g|) + Y_1(|g|)) < \infty$. Assume G has a Lebesgue density with support contained in $[0, T]$ ($T < \infty$). If the density is continuous on $[0, T]$, then

$$\lim_{t \rightarrow \infty} \sup_{|f| \leq |g|} |Ef(\theta_t X) - \lambda EY_1(f)| = 0.$$

PROOF. This result follows immediately from Proposition 3.2 and Theorem 4.1, upon showing that $c_f(t)$ converges to zero uniformly in f . (Actually, Proposition 3.2 does not strictly apply here; the proof can be modified to show that (3.3) holds at all $t \geq 0$, however.) But $|c_f(t)| \leq c_g(t) \leq E\{V_0(|g|); \tau_0 > t\} \rightarrow 0$.

Total variation convergence is an easy corollary of Theorem 5.1.

(5.2) COROLLARY. Suppose that $F \in \mathcal{F}$ is non-arithmetic with finite mean. Assume G satisfies the conditions of Theorem 5.1. If π is defined as in Section 3, then

$$\|P\{X(t)\varepsilon\cdot\} - \pi(\cdot)\| \rightarrow 0 \text{ as } t \rightarrow \infty.$$

The interesting point here is that the "delay cycle" can smooth the equilibrium process enough to obtain total variation convergence. Recall, from Section 3, that Miller's example shows that, without the smoothing of the delay cycle, we can not even expect convergence in distribution (without making further assumptions about the paths of X).

The results of Section 4 can also be used to obtain smoothed limit theorems that are more reminiscent of Corollary 4.7.

(5.3) THEOREM. Suppose that $F \in \mathcal{F}$ is non-arithmetic and $E(Y_0(|g|) + Y_1(|g|)) < \infty$. If K is as described in Section 4, then

$$\lim_{t \rightarrow \infty} \sup_{|f| \leq |g|} \left| \int_0^T k(s) Ef(\theta_{t-s} X) ds - \lambda EY_1(f) \right| = 0.$$

PROOF. To prove this result, we note that $\int_0^T k(s) Ef(\theta_{t-s} X) ds = (K * c_f)(t) + (G * K * U_f * b_f)(t)$ for $t \geq T$. Since $(K * U_F * b_f)(t) \rightarrow \lambda EY_1(f)$ uniformly in f (and can be bounded uniformly in both f and t), the bounded convergence theorem shows that $(G * K * U_F * b_f)(t) \rightarrow \lambda EY_1(f)$ uniformly in f . For the other term observe that $|(K * c_f)(t)| \leq (K * c_{|g|})(t) \leq M \int_{t-T}^t c_{|g|}(s) ds \rightarrow 0$. Here, M is a bound on k and we use the fact that $EY_0(|g|) < \infty$ to conclude that $c_{|g|}$ is integrable on $[0, \infty)$ (This implies that $\int_{t-T}^t c_{|g|}(s) ds \rightarrow 0$).

Again, we can write down a total variation convergence corollary to this result; it is the equilibrium process analog of the averaging result given by Corollary 4.7.

(5.4) COROLLARY. Suppose that $F \in \mathcal{F}$ is non-arithmetic with finite mean. Let π be defined as in Section 3. Then for any $h > 0$,

$$\left\| \frac{1}{h} \int_{t-h}^t P\{X(s)\varepsilon\cdot\} ds - \pi(\cdot) \right\| \rightarrow 0 \text{ as } t \rightarrow \infty.$$

A common feature of the smoothed limit theorems of this section is that convergence is proved under minimal hypotheses. In addition to its mathematical virtues, this should make hypothesis validation easier from an applications viewpoint. Furthermore, it is our opinion that these smoothed versions provide information that is, from a practical perspective, essentially equivalent to that obtained from an unsmoothed version.

REFERENCES

- Arjas, E., Nummelin, E., and R. L. Tweedie (1974), Uniform limit theorems for non-singular renewal and Markov renewal processes, *Journal of Applied Probability*, **15**, 112–125.
- Billingsley, P. (1964), *Convergence of Probability Measures*, John Wiley, New York.
- Feller, W. (1970), *An Introduction to Probability Theory and Its Applications*, Vol. 2, John Wiley, New York.
- Glynn, P. W. (1988), A low bias steady-state estimator for equilibrium processes, Technical Report, Department of Operations Research, Stanford University, Stanford, CA.
- Karlin, S., and H. M. Taylor (1975), *A First Course in Stochastic Processes*, Academic Press, New York.
- Miller, D. R. (1972), Existence of limits in regenerative processes, *Annals of Mathematical Statistics*, **43**, 1275–1242.
- Smith, W. L. (1955), Regenerative stochastic processes, *Proceedings of the Royal Society London, Series A*, **232**, 6–31.
- Smith, W. L. (1960), Remarks on the paper 'Regenerative stochastic processes,' *Proceedings of the Royal Society London, Series A*, **256**, 496–501.

Supercritical Branching Processes with Countably Many Types and the Size of Random Cantor Sets

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1. INTRODUCTION

Sam Karlin has made many fundamental contributions to branching processes. (See Athreya and Ney (1972) and its bibliography for descriptions of some of these papers.) Given his interest in the use of branching processes in population genetics, it is natural that most of these papers deal with branching processes of finitely many types. Here we wish to present a convergence theorem for a supercritical branching process with countably many types.

Branching processes with infinitely many types have, of course, been considered before. Moy (1967) considered countably many types, but it is

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