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Probability in the Engineering and Informational Sciences, **2**, 1988, 377–382. Printed in the U.S.A.

CONDITIONS UNDER WHICH A MARKOV CHAIN CONVERGES TO ITS STEADY STATE IN FINITE TIME

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Analysis of the initial transient problem of Monte Carlo steady-state simulation motivates the following question for Markov chains: when does there exist a deterministic T such that $P\{X(T) = y | X(0) = x\} = \pi(y)$, where π is the stationary distribution of X? We show that this can essentially never happen for a continuous-time Markov chain; in discrete time, such processes are i.i.d. provided the transition matrix is diagonalizable.

1. INTRODUCTION

Let $X = \{X(t): t \geq 0\}$ be a real-valued stochastic process, which is ergodic in the sense that there exists a (deterministic) constant α such that

$$\bar{X}(t) \equiv \frac{1}{t}\int_0^t X(s)ds \to \alpha \text{ a.s.}$$

P.W. Glynn's research was supported by the National Science Foundation Grant ECS 840-4809 and by Army Research Office Contract DAAG29-84-K-0030. D.L. Iglehart's research was supported by the National Science Foundation Grant DCR 8509668 and by Army Research Office Contract DAAG29-84-K-0030.

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as $t \to \infty$. A major problem in the area of computational probability is the determination of the steady-state constant α.

One frequently used numerical strategy involves simulating the process X up to time t; $\bar{X}(t)$ is then employed as an estimator of α. This Monte Carlo approach to numerically calculating α suffers from the defect that $\bar{X}(t)$ is generally biased due to the influence of the initial condition used for initiating the simulation. A commonly recommended method for dealing with the bias due to the initial condition is to delete some appropriately chosen initial segment of the simulation.

It has been suggested that for Markov chains, boundedness of the associated regeneration time (by M, say) implies that the chain is in steady state at time M. Consequently, for chains with bounded regeneration times, one could deal with the initial transient effects by deleting the first M time units of the simulation.

In this note, we investigate the class of Markov chains which achieve steady state in finite time. We show, by example, that boundedness of the regeneration times does not imply that the chain is in steady state in finite time. We then prove that a Markov chain can achieve steady state in finite time only in a very trivial case. If the transition matrix is diagonalizable, the only such processes are discrete-time chains describing a sequence of independent and identically distributed random variables (i.i.d.r.v.'s).

Related work appears in a paper by Subelman [6]; the emphasis there is on determining the class of initial distributions from which a Markov chain can achieve steady state in finite time. In this paper, we are more concerned with classifying the structure of transition matrices for which steady state is achieved in finite time for *all* initial distributions simultaneously. Furthermore, we study the relationship of this property to the regenerative structure of the process.

2. THE RESULTS

We say that a discrete-time Markov chain X on state space S has bounded regeneration times if there exists $y \in S$ and a deterministic constant $M(y)$ such that $P\{T(y) \le M(y) | X_0 = y\} = 1$, where $T(y) = \inf\{n \ge 1: X_n = y\}$. Given a Markov chain X with unique stationary distribution $\pi = (\pi(y): y \in S)$, the chain X is in steady state after finite (deterministic) time if there exists a deterministic constant T such that

$$P\{X(T) = y | X(0) = x\} = \pi(y)$$

for all $x, y \in S$.

Our first goal is to show that if X has bounded regeneration times, it does not follow that X will be in steady state after finite time. Consider the chain $X = \{X_n: n \ge 0\}$ on state space $S = \{0,1,2\}$ having the transition matrix

$$P = \begin{bmatrix} 0 & 3/16 & 13/16 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

It is clear that $T(0) \le 3$ a.s. (regardless of the initial state); hence X has bounded regeneration times.

On the other hand, the stationary distribution of P is given by $\pi = (16/35, 3/35, 16/35)$. We will now show that $P^n(0,0) = P\{X_n = 0 | X_0 = 0\} \ne \pi(0) = 16/35$ for any $n \ge 0$, so that X is not in steady state after finite time. Observe that $P^n(0,0) = u_n$, where $\{u_n: n \ge 0\}$ is the renewal sequence corresponding to the mass function $\{f_n: n \ge 1\}$, and $f_n = P\{T(0) = n | X_0 = 0\}$. The generating function of $\{f_n: n \ge 0\}$ is given by $f(z) = (13z^2 + 3z^3)/16$ and is related to the generating function $u(z)$ associated with $\{u_n: n \ge 0\}$ via the relation $u(z) = (1 - f(z))^{-1}$ (see p. 330 of Feller [3]). Using the method of partial fractions, it follows that

$$u(z) = \frac{16}{35}(1 - z)^{-1} + \frac{9}{14}\left(1 + \frac{3}{4}z\right)^{-1} - \frac{1}{10}\left(1 + \frac{z}{4}\right)^{-1}.$$

By expanding $u(z)$ in powers of z, we find that

$$u_n = P^n(0,0) = \frac{16}{35} + \frac{9}{14}\left(-\frac{3}{4}\right)^n - \frac{1}{10}\left(-\frac{1}{4}\right)^n.$$

Since $(9/14)(3/4)^n > (1/10)(1/4)^n$ for $n \ge 0$, this proves our claim.

This example also proves that boundedness of regeneration times does not imply that the mixing constants for the process vanish for n sufficiently large (see pp. 345–350 of Ethier and Kurtz [2] for a discussion of mixing); allusions to such a result have appeared in the simulation literature (see p. 1093 of Schruben [5])

It is perhaps interesting to note that if one strengthens the above bounded regeneration time condition to uniform boundedness, the class of Markov chains so obtained is trivial in nature. To be precise, we say that a Markov chain on state space S has uniformly bounded regeneration times if there exists a deterministic constant M such that $P\{T(x) \le M | X_0 = x\} = 1$ for all $x \in S$. We claim that any irreducible countable-state Markov chain having uniformly bounded regeneration times has a corresponding transition matrix P for which every element $P(x,y)$ is either one or zero.

To see this, suppose that there exist $x,y,z \in S$ such that $P(x,y) > 0$, $P(x,z) > 0$ with $y \ne z$. We assume that these points are distinct: a similar argument works if x equals y or z. Since x is recurrent by assumption and $P(x,z) > 0$, it follows that $P\{T(x) < \infty | X_0 = z\} = 1$. This obviously implies that $P\{T(x) < T(z) | X_0 = z\} > 0$; a symmetric argument proves $P\{T(x) <$

$T(y)|X_0 = y\} > 0$. This implies that $P\{T(x) < T(z)|X_0 = y\}$, $P\{T(x) < T(y)|X_0 = z\}$ cannot both be equal to zero. For if they were, then

$$\begin{aligned} P\{T(x) < T(z)|X_0 = z\} &= P\{T(x) < T(z);\ T(x) < T(y)|X_0 = z\} \\ &\quad + P\{T(z) > T(y)|X_0 = z\} \cdot P\{T(x) < T(z)|X_0 = y\} \\ &\leq P\{T(x) < T(y)|X_0 = z\} \\ &\quad + P\{T(x) > T(y)|X_0 = z\} \cdot P\{T(x) < T(z)|X_0 = y\} = 0, \end{aligned}$$

which is a contradiction. Suppose then, without loss of generality, that $P\{T(x) < T(z)|X_0 = y\} > 0$. Thus, we conclude that there is a path of positive probability from z to x, from x to y in one step, and from y back to x which does not revisit z anywhere on the path. By iterating the loop from x through y back to x indefinitely, we see that there are paths of arbitrary length emanating from z which do not revisit z, contradicting $T(z) \leq M$ a.s. on $\{X_0 = z\}$.

We now turn to characterizing those Markov chains which achieve steady state after finite time. First, observe that for such a process, there exists T such that

$$P\{X(t) = y|X(0) = x\} = \pi(y) \tag{2.1}$$

for all $x, y \in S$, $t \geq T$. Consequently, the following result proves that no irreducible finite-state continuous-time Markov chain (with more than one state) can ever be in steady state in finite time.

Proposition 1: There exists no irreducible uniformizable continuous-time Markov chain having more than one state such that: there exists $x \in S$ and $t_1 < t_2$ for which

$$P\{X(t_2) = \cdot\,|X(0) = x\} = P\{X(t_1) = \cdot\,|X(0) = x\}. \tag{2.2}$$

PROOF: For $t \geq 0$, let $P(t) = (P(t,x,y): x,y \in S)$, where $P(t,x,y) = P\{X(t) = y|X(0) = x\}$. If η is the row vector in which $\eta(y) = P(t_1,x,y)$, then Eq. (2.2) implies that $\eta P(t_2 - t_1) = \eta$. Since η is a stochastic solution to the stationarity equations for $P(t_2 - t_1)$, it follows that $\{X(n(t_2 - t_1)): n \geq 0\}$ is positive recurrent. Consequently, $X = \{X(t): t \geq 0\}$ is positive recurrent and irreducible so there exists a stochastic vector π such that $P(t,z,y) \to \pi(y)$ as $t \to \infty$; furthermore, π is the unique stochastic solution of $\pi Q = 0$, where Q is the generator of X. The stationarity of η for $P(t_2 - t_1)$ implies that $P(n(t_2 - t_1),x,y) \to \eta(y)$ as $n \to \infty$, from which it is evident that $\eta = \pi$. (See pp. 152–168, 261–270 of Cinlar [1] for the relevant theory.)

If $\tau = \inf\{t > 0: P(t,x,\cdot) = \pi(\cdot)\}$, we have just shown that $0 \leq \tau \leq t_1 < \infty$. Since X is uniformizable, for any real t and Λ,

$$P(t) = \exp(Q\Lambda) \cdot \sum_{k=0}^{\infty} Q^k (t - \Lambda)^k / k! \tag{2.3}$$

where the series has an infinite radius of convergence. Set $\Lambda = \tau + \epsilon$ for $\epsilon > 0$ and observe that the kth derivative of (2.3) at $t = \Lambda$ satisfies

$$P^{(k)}(\tau + \epsilon) = \exp(Q(\tau + \epsilon))Q^k.$$

Since it is clear that $P(t,x,\cdot) = \pi(\cdot)$ for $t > \tau$, it is obvious that the xth row of $P^{(k)}(\tau + \epsilon)$ vanishes for $k \geq 1$. We may therefore conclude from Eq. (2.3) that $P(t,x,\cdot) = \pi(\cdot)$ for all t. But this violates $P(0) = I$, since $\pi(x) < 1$ by irreducibility. ∎

We will now discuss the discrete-time case. Let P be the transition matrix of the chain and Π be the matrix in which all rows are identical to π. Note that if Eq. (2.1) is in force for $T = n$, then $P^{n+1} = P^n$.

Proposition 2: Let P be a diagonalizable finite-state stochastic matrix. If $P^{n+1} = P^n$ for some $n \geq 1$, then $P^m = P$ for all $m \geq 1$.

PROOF: If $P^{n+1} = P^n$, then $f(P) = 0$, where $f(x)$ is the polynomial $x^n(x - 1)$. It follows that the minimal polynomial of P takes the form $x^r(x - 1)$, where $r \leq n$ (p. 133 of Lancaster [4]). Thus, 0 and 1 are the only possible eigenvalues of P. Since P is diagonalizable, $P = R^{-1}DR$ where D is a diagonal matrix whose diagonal elements belong to $\{0,1\}$, and R is the associated similarity transformation. So, $D^m = D$ for $m \geq 1$, and hence $P^m = R^{-1}D^mR = P$ for $m \geq 1$. ∎

Recall that if P has a single closed communicating class, then $m^{-1}(I + P + \cdots + P^{m-1}) \to \Pi$ as $m \to \infty$, where π is the unique stationary probability vector of P. If $P^{n+1} = P^n$ for some $n \geq 1$, it follows that $P = \Pi$, where Π has all rows identical. In other words, if P is a diagonalizable matrix having a single closed communicating class, which achieves steady state in finite time, then P describes a sequence of i.i.d.r.v.'s. A related result appears in Subelman [6, p. 257].

Consider for $0 < p_1, p_2 < 1$

$$P = \begin{pmatrix} p_1 & p_2 & 1 - p_1 - p_2 \\ p_1 & 0 & 1 - p_1 \\ p_1 & 0 & 1 - p_1 \end{pmatrix}.$$

It is easily verified that P^2 has identical rows equal to $(p_1, p_1p_2, 1 - p_1 - p_1p_2)$, and hence, $P^2 = \Pi$ without $P = \Pi$. Evidently, the above transition matrix is not diagonalizable.

Despite the above example, we believe that the discussion of this section shows that the class of chains achieving steady state in finite time is rather limited in scope.

Acknowledgments

The authors wish to thank the referee for noticing an error in an example given in the original version of this paper, as well as pointing out the Subelman paper.

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Probability in the Engineering and Informational Sciences, 2, 1988, 383–385. Printed in the U.S.A.

A SIMPLE PROOF OF INSTABILITY OF A RANDOM-ACCESS COMMUNICATION CHANNEL

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We give an extremely simple argument to prove that in infinite user-communication channels, under the Aloha protocol, the number of successful transmissions is finite with probability 1. The same result is then shown to hold for those back-off protocols whose transmission probabilities are bounded away from 0.

1. INTRODUCTION

Suppose that the numbers of messages arriving during each of the time periods $n = 1,2,\ldots$ are independent and identically distributed random variables. Let $a_i = P\{i \text{ arrivals}\}$, and suppose that $a_0 + a_1 < 1$. Each arriving message will transmit at the end of the period in which it arrives. If exactly 1 message is transmitted then the transmission is successful and the message leaves the system. However, if at any time 2 or more messages simultaneously transmit then a collision is deemed to occur and these messages remain in the system. Once a message is involved in a collision it will, independently of all else, transmit at the end of each additional period with probability p—the so-called Aloha protocol. We now show that such a system is asymptotically unstable in the

This research was supported by the U.S. Air Force Office of Scientific Research under Grant AFOSR-86-0153.

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