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A JOINT CENTRAL LIMIT THEOREM FOR THE SAMPLE MEAN AND REGENERATIVE VARIANCE ESTIMATOR*

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Abstract

Let $\{V(k) : k \geqslant 1\}$ be a sequence of independent, identically distributed random vectors in $\mathbb{R}^d$ with mean vector μ. The mapping g is a twice differentiable mapping from $\mathbb{R}^d$ to $\mathbb{R}^1$. Set $r = g(\mu)$. A bivariate central limit theorem is proved involving a point estimator for r and the asymptotic variance of this point estimate. This result can be applied immediately to the ratio estimation problem that arises in regenerative simulation. Numerical examples show that the variance of the regenerative variance estimator is not necessarily minimized by using the "return state" with the smallest expected cycle length.

Keywords and phrases

Bivariate central limit theorem, joint limit distribution, ratio estimation, regenerative simulation, simulation output analysis.

1. Introduction

Let $X = \{X(t) : t \geqslant 0\}$ be a (possibly) delayed regenerative process with regeneration times $0 = T(-1) \leqslant T(0) < T(1) < T(2) < \ldots$. To incorporate regenerative sequences $\{X_n : n \geqslant 0\}$, we pass to the continuous time process $X = \{X(t) : t \geqslant 0\}$, where $X(t) = X_{[t]}$ and $[t]$ is the greatest integer less than or equal to t. Under quite general conditions (see Smith [7]),

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$$r_t \equiv \frac{1}{t}\int_0^t f(X(s))\,\mathrm{d}s \to r, \quad \text{a.s.} \tag{1.1}$$

as $t \to \infty$. The task of estimating r via a simulation of the process X is known as the *regenerative steady-state simulation problem.*

The relation (1.1) states that the sample mean r_t is a strongly consistent estimator for r. It turns out that the regenerative structure of X can be fruitfully applied to obtain an estimator s_t for measuring the variability of r_t; the estimator s_t lies at the heart of the regenerative method of steady-state simulation.

It has been suggested, however, that correlation between r_t and s_t can lead to degradation in the performance of procedures based on the regenerative method; see, for example, Bratley et al. [1], p. 113. A related problem concerns the fact that many processes are regenerative with respect to more than one sequence of regenerative epochs (for example, discrete state Markov chains). In such a setting, one would like some guidelines for how to choose a sequence which is optimal in the sense of statistical efficiency.

We begin in sect. 2 by discussing a more general estimation problem and associated central limit theorem (CLT). In sect. 3, we specialize this CLT to the case of regenerative processes. This section contains a limit theorem which describes the asymptotic correlation structure of r_t and s_t. The result gives an explicit (computable) formula for the asymptotic covariance and variances of r_t and s_t. We conclude in sect. 4 with a set of examples which illustrate the application of the limit theorem. One of our examples shows that the statistical efficiency of the regenerative method is not necessarily increased by using shorter regenerative cycles, contrary to the folklore.

2. A general central limit theorem

Let $\{(V(k), \alpha(k)) : k \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) random vectors (r.v.'s taking values in $\mathbb{R}^{d+1}$; we view $V(k)$ as a column vector with components $V_1(k), \ldots, V_d(k)$. We assume that:

$$\begin{aligned} &\text{(i)} \quad P\{\alpha(1) > 0\} = 1 \\ &\text{(ii)} \quad E(V_1(1)^4 + \ldots + V_d(1)^4 + \alpha^2(1)) < \infty. \end{aligned} \tag{2.1}$$

Let $\mu = EV(1)$; for a given function $g: \mathbb{R}^d \to \mathbb{R}^1$, our goal is to estimate $r = g(\mu)$. We interpret the quantity $\alpha(k)$ as a random variable which measures the "effort" required to generate $V(k)$.

A number of important estimation problems can be formulated as special cases of the above.

(2.2) EXAMPLE

If $d = 1$ and $g(x) = x$, the estimation problem is to find the mean of $V(1)$; this is, of course, the *classical estimation problem.*

(2.3) EXAMPLE

If $d = 2$ and $g(x, y) = xy^{-1}$, the goal is to estimate the ratio of two means $r = EV_1(1)/EV_2(1)$; this is known as the *ratio expectation problem.*

As is well known in the simulation literature, the regenerative steady-state simulation problem is a special case of the ratio estimation problem. To be precise, we set

$$V_1(k) = \int_{T(k-1)}^{T(k)} f(X(s))\,\mathrm{d}s\,,$$

$$V_2(k) = T(k) - T(k-1)\,.$$

If f is non-negative, then $r_t \to EV_1(1)/EV_2(1)$ a.s. as $t \to \infty$ under (2.1), so that the steady-state parameter r can be estimated as the ratio of two expectations. It is customary in the regenerative setting to assume that the effort of generating $V(k)$ is equal to the length of the kth regenerative cycle, namely $\alpha(k) = \tau(k) \equiv T(k) - T(k-1)$.

For further applications of ratio estimation in simulation, see section 8.10 in the book by Law and Kelton [6].

(2.4) EXAMPLE

Let f be a non-negative function defined on the state space of a delayed regenerative process X, and for $\gamma > 0$, set

$$r = E\left(\int_0^\infty \mathrm{e}^{-\gamma t} f(X(t))\,\mathrm{d}t\right).$$

In Fox and Glynn [2], it is shown that r can be expressed as $r = g(\mu)$, where

$$V_1(1) = \int_0^{T(0)} e^{-\gamma t} f(X(t))\, dt$$

$$V_2(1) = e^{-\gamma T(0)}$$

$$V_3(1) = \int_{T(0)}^{T(1)} e^{-\gamma t} f(X(t))\, dt$$

$$V_4(1) = \exp[-\gamma(T(1) - T(0))]$$

$$g(y_1, \ldots, y_4) = y_1 + (1 - y_4)^{-1} y_2 y_3 .$$

Throughout this paper, we assume that g is twice continuously differentiable in a neighborhood A of μ.

Let $U(-1) = 0 \leqslant U(0) < U(1) < \ldots$, where $U(k) = U(k-1) + \alpha(k)$ for $k \geqslant 1$; set

$$N(t) = \max\{k : U(k) \leqslant t\} .$$

Then, $N(t)$ is the number of observations $V(1), \ldots, V(k)$ generated by t units of effort.

In many applications, $U(0) = U(-1) = 0$; however, in the case that the simulation involves a delayed regenerative process, it is convenient to let $U(k) = T(k)$, so that $U(0) - U(-1)$ represents the length of the first cycle.

For $n \geqslant 1$, set

$$\bar{V}(n) = \frac{1}{n} \sum_{j=1}^{n} V(j) ;$$

then, for $t \geqslant 0$, we let

$$\mu(t) = \begin{cases} \bar{V}(N(t)) ; & N(t) \geqslant 1 \\ 0 ; & N(t) < 1 \end{cases}$$

and put $\beta(t) = g(\mu(t))$. To prepare for a CLT for $\beta(t)$, we need the following notation. Let $C = EV(1)\ V(1)^T - EV(1)\ EV(1)^T$ be the covariance matrix of $V(1)$. (B^T denotes the transpose of the matrix B.) The next two results, which are proved by Glynn and Iglehart [4], constitute the strong law and CLT for $\beta(t)$.

(2.5) PROPOSITION

Under (2.1), $\beta(t) \to r$ a.s. as $t \to \infty$.

(2.6) THEOREM

Under (2.1), $t^{1/2}(\beta(t) - r) \Rightarrow N(0, E\alpha(1) \cdot \nabla g(\mu)^T C \nabla g(\mu))$ as $t \to \infty$, where $\nabla g(x)$ is the gradient of g evaluated at x (written as a column vector in $\mathbb{R}^d$).

To use the CLT (2.6) for confidence intervals, we need a method for consistently estimating $\sigma^2 = E\alpha(1) \cdot \nabla g(\mu)^T C \nabla g(\mu)$. Let $\bar{M}(n)$ be the moment estimator

$$\bar{M}(n) = \frac{1}{n} \sum_{k=1}^{n} V(k)\ V^T(k)$$

and set

$$C(t) = \bar{M}(N(t)) - \mu(t)\ \mu(t)^T ,$$

for $N(t) \geq 1$ and $C(t) = 0$ otherwise. Finally, put

$$v(t) = \bar{\alpha}(t) \cdot \nabla g(\mu(t))^T\ C(t) \nabla g(\mu(t)),$$

where

$$\bar{\alpha}(t) = \sum_{k=1}^{N(t)} \alpha(k)/N(t).$$

The strong law of large numbers yields:

(2.7) PROPOSITION

Under (2.1), $v(t) \to \sigma^2$ a.s. as $t \to \infty$.

If $z(\alpha)$ is selected so that $P\{-z(\alpha) \leq N(0,1) \leq z(\alpha)\} = 1 - \alpha$, theorem (2.6) and proposition (2.7) imply that under (2.1)

$$[\beta(t) - z(\alpha)s(t)t^{-1/2},\ \beta(t) + z(\alpha)s(t)t^{-1/2}]$$

is an asymptotic $100(1-\alpha)\%$ confidence interval for r, where $s(t) = v(t)^{1/2}$.

We turn now to the task of obtaining a joint central limit theorem for our point estimator for r, namely $\beta(t)$, and our estimator $s(t)$ for the asymptotic standard deviation of $t^{1/2}(\beta(t) - r)$.

Recall that the function g is twice continuously differentiable in a neighborhood A of μ. Let $H(x)$ be the Hessian of second derivatives of g evaluated at x; i.e. $H(x)$ is a $d \times d$ matrix with

$$H_{ij}(x) = \frac{\partial^2}{\partial x_i \partial x_j} g(x).$$

Set $V_c(k) = V(k) - EV(1)$, $\alpha_c(k) = \alpha(k) - E\alpha(1)$,

$$R_1(k) = \nabla g(\mu)^T [V_c(k)\, V_c(k)^T - C]\, \nabla g(\mu) E\alpha(1),$$

$$R_2(k) = \nabla g(\mu)^T C\, H(\mu)\, V_c(k) E\alpha(1),$$

$$R_3(k) = \nabla g(\mu)^T C\, \nabla g(\mu)\, \alpha_c(k),$$

$$R(k) = R_1(k) + 2R_2(k) + R_3(k),$$

and

$$S(k) = \nabla g(\mu)^T\, V_c(k).$$

The following general CLT is proved by Glynn and Iglehart in [4].

(2.8) THEOREM

Under (2.1), $t^{1/2}(\beta(t) - r, s(t) - \sigma) \Rightarrow N(0, D)$ as $t \to \infty$, where

$$D = E\alpha(1) \cdot \begin{pmatrix} ES^2(1) & ER(1)S(1)/2\sigma \\ ER(1)S(1)/2\sigma & ER^2(1)/4\sigma^2 \end{pmatrix}.$$

While this expression is complicated, we can nevertheless compute it explicitly in the regenerative process case of discrete or continuous time Markov chains (see sect. 4).

3. The regenerative process case

In this section, we specialize the set-up of sect. 2 to the regenerative process case; see examples (2.3) and (2.4). For ease of exposition, we shall restrict our attention to non-delayed regenerative processes, that is $T(-1) = T(0) = 0$. As is the usual case, all limit theorems also hold for delayed regenerative processes. We treat only the ratio expectation problem, example (2.3).

Using the standard regenerative notation, we set

$$Y(k) = \int_{T(k-1)}^{T(k)} f(X(s))\,\mathrm{d}s = V_1(k)$$

$$\tau(k) = T(k) - T(k-1) = V_2(k);$$

assumption (2.1) translates as $E(Y(1)^4 + \tau(1)^4) < \infty$. The matrix C is given by

$$C = \begin{pmatrix} \operatorname{var} Y(1) & \operatorname{cov}(Y(1), \tau(1)) \\ \operatorname{cov}(Y(1), \tau(1)) & \operatorname{var} \tau(1) \end{pmatrix}.$$

As indicated earlier, $g(x_1, x_2) = x_1/x_2$ so $\nabla g(x_1, x_2) = (1/x_2, -x_1/x_2^2)$ for $x_2 \neq 0$, and it is easily calculated that

$$\nabla g(EY(1), E\tau(1))^T C\, \nabla g(EY(1), E\tau(1))$$

$$= (E\tau(1))^{-2}\, EZ^2(1),$$

where $Z(k) = Y(k) - r\tau(k)$. Since $\alpha(k) = \tau(k)$, it follows from theorem (2.6) that

$$t^{1/2}(\beta(t) - r) \Rightarrow N(0, EZ^2(1)/E\tau(1)) \tag{3.1}$$

as $t \to \infty$. It also follows from (3.1), by a standard argument, that

$$t^{1/2}(r_t - r) \Rightarrow N(0, EZ^2(1)/E\tau(1))$$

as $t \to \infty$. Thus, the CLT for regenerative processes is a special case of theorem (2.6) (see also corollary 9.1 in the paper by Smith [7]). Recall that the quantity $EZ^2(1)/E\tau(1)$

is independent of the "return state" used to form the regenerative cycles; see, for example, Iglehart [5].

As for the variance estimator $v(t)$, this reduces to

$$v(t) = \frac{1}{N(t)} \sum_{k=1}^{N(t)} (Y(k) - \beta(t)\,\tau(k))^2 / \bar{\tau}(N(t))$$

in the current setting; $v(t)$ is the well-known regenerative variance estimator (see Iglehart [5] for details).

We will specialize theorem (2.8) to the regenerative setting in order to obtain a joint limit distribution for $\beta(t)$ and the estimator $s(t)$.

Let $Y_c(k) = Y(k) - EY(1)$, $\tau_c(k) = \tau(k) - E\tau(1)$. Here,

$$V_c(k)\, V_c(k)^T = \begin{pmatrix} Y_c(k)^2 & Y_c(k)\,\tau_c(k) \\ Y_c(k)\,\tau_c(k) & \tau_c(k)^2 \end{pmatrix},$$

$$C = \begin{pmatrix} EY_c^2(1) & EY_c(1)\,\tau_c(1) \\ EY_c(1)\,\tau_c(1) & E\tau_c^2(1) \end{pmatrix},$$

and $\nabla g(\mu)^T = (1, -r)/E\tau(1)$. Then,

$$R_1(k) = \nabla g(\mu)^T\, [V_c(k)\, V_c(k)^T - C]\, \nabla g(\mu)\, E\alpha(1)$$

$$= \frac{1}{E\tau(1)} \cdot (Z^2(k) - EZ^2(1)).$$

(Recall that for the regenerative case we take $\alpha(k) = \tau(k)$.) Furthermore, the Hessian $H(\mu)$ is equal to

$$H(\mu) = (E\tau(1))^{-2} \begin{pmatrix} 0 & -1 \\ -1 & 2r \end{pmatrix}$$

so

$$CH(\mu) = (E\tau(1))^{-2} \begin{pmatrix} -EY_c(1)\,\tau_c(1) & -EY_c(1)\,Z(1) + rEY_c(1)\,\tau_c(1) \\ -E\tau_c^2(1) & -E\tau_c(1)\,Z(1) + rE\tau_c^2(1) \end{pmatrix}$$

and thus,

$$\begin{aligned} R_2(k) &= \nabla g(\mu)^T C\,H(\mu)\,V_c(k)\,E\alpha(1) \\ &= (E\tau(1))^{-2}\,(-Z(k)\cdot(EZ(1)\,\tau(1)) - \tau_c(k)\cdot(EZ^2(1)))\,. \end{aligned}$$

Finally,

$$\begin{aligned} R_3(k) &= \nabla g(\mu)^T C\,\nabla g(\mu)\,\alpha_c(k) \\ &= \frac{\sigma^2}{E\tau(1)}\,\alpha_c(k) = (EZ^2(1))\cdot(E\tau(1))^{-2}\,\tau_c(k) \end{aligned}$$

so that

$$\begin{aligned} R(k) &= (E\tau(1))^{-1}\,(Z^2(k) - EZ^2(1)) \\ &\quad - 2(E\tau(1))^{-2}\,(Z(k)\cdot(EZ(1)\,\tau(1)) + \tau(k)\,EZ^3(1) - E\tau(1)\,EZ^2(1)) \\ &\quad + EZ^2(1)\,(E\tau(1))^{-2}\,(\tau(k) - E\tau(1)) \\ &= (E\tau(1))^{-1}\,(Z^2(k) - \sigma^2\tau(k) - 2Z(k))\cdot(E(Z(1)\,\tau(1))\cdot(E\tau(1))^{-2}) \end{aligned}$$

(recall that $\sigma^2 = EZ^2(1)/E\tau(1)$), and $S(k) = (E\tau(1))^{-1}\,Z(k)$). Setting

$$A(k) = Z^2(k) - \sigma^2\,\tau(k)$$

and

$$\lambda = 2EZ(1)\,\tau(1)/E\tau(1)\,,$$

we find from theorem (2.8) that

$$D = (E\tau(1))^{-1} \begin{bmatrix} EZ^2(1) & \dfrac{EA(1)Z(1) - \lambda EZ^2(1)}{2\sigma} \\ \dfrac{EA(1)Z(1) - \lambda EZ^2(1)}{2\sigma} & \dfrac{EA^2(1) - 2\lambda EA(1)Z(1) + \lambda^2 EZ^2(1)}{4\sigma^2} \end{bmatrix} \qquad (3.2)$$

The matrix D describes the dependence structure of the bivariate normal distribution characterizing the limit behavior of $\beta(t)$ and $s(t)$.

As indicated in sect. 2, the estimator

$$r_t = \frac{1}{t} \int_0^t f(X(s))\, ds$$

is closely related to $\beta(t)$ and a standard argument and theorem (2.8) imply that

$$t^{1/2}(r_t - r, s(t) - \sigma) \Rightarrow N(0, D) \qquad (3.3)$$

as $t \to \infty$, where D is given by (3.2).

With the limit theorem (3.3) at our disposal, we can now investigate the influence of the estimator $\beta(t)$ (or equivalently, to small order, r_t) on the standard deviation estimate $s(t)$. Suppose, for example, that μ were known. In this case, $r = g(\mu)$ is also known. How much does this improve our estimate of σ?

The appropriate variance estimate $v_\mu(t)$, with $\mu = (EY(1), E\tau(1))$ known, is now

$$v_\mu(t) = \nabla g(\mu) \left\{ \frac{1}{N(t)} \sum_{k=1}^{N(t)} V(k)\, V(k)^T - \mu\mu^T \right\} \nabla g(\mu)^T \cdot E\tau(1)$$

$$= (E\tau(1))^{-1} \frac{1}{N(t)} \sum_{k=1}^{N(t)} Z^2(k).$$

Reasoning analogous to that used above (but simpler in its execution) shows that

$$t^{1/2}(r_t - r, s_\mu(t) - \sigma) \Rightarrow N(0, D_\mu), \qquad (3.4)$$

at $t \to \infty$, where $s_\mu(t) = v_\mu(t)^{1/2}$ and

$$D_\mu = (E\tau(1))^{-1} \begin{bmatrix} EZ^2(1) & \dfrac{EZ^3(1)}{2\sigma} \\ \dfrac{EZ^3(1)}{2\sigma} & \dfrac{EZ^4(1) - (EZ^2(1))^2}{4\sigma^2} \end{bmatrix}. \tag{3.5}$$

The difference between D_μ and D is the "penalty" induced by having to estimate the mean vector μ in the regenerative case. We note that in the classical estimation problem where $\tau(k) \equiv 1$ a.s., $D_\mu = D$ and so, asymptotically, there is no "penalty" associated with the estimation of μ.

The difference in the asymptotic variance of $s(t)$ and $s_\mu(t)$ is given by

$$\Delta = (4\sigma^2)^{-1} (-2\sigma^2 EZ(1)\tau(1) + \sigma^4 E\tau^2(1) - 2\lambda EZ^3(1)$$
$$+ 2\lambda\sigma^2 EZ(1)\tau(1) + \lambda^2 EZ^2(1) + (EZ^2(1))^2).$$

Perhaps surprisingly, Δ can be either positive or negative, as will be shown below. In other words, it is possible that one can improve the standard deviation estimate $s(t)$ by estimating μ, even when μ is known; in this case, $\mu(t)$ is basically being used as a control variate for $s(t)$ (see [1], pp. 59–61). In spite of the counter-example below, in most practical examples we would expect that $d(2, 2) > d_\mu(22)$. This fact is borne out in the numerical examples of sect. 4.

(3.6) EXAMPLE

Suppose the conditional distribution function $P\{Z(1) \leqslant x | \tau(1)\}$ is a.s. a symmetric distribution about zero. Then

$$\Delta = (4\sigma^2)^{-1} (\sigma^4 E\tau^2(1) + (EZ^2(1))^2) \geqslant 0.$$

(3.7) EXAMPLE

Suppose $Y(k)$ takes the form $Y(k) = (a(k) + r)\tau(k) - b(k)$, where $a(k)$, $b(k)$, $\tau(k)$ are independent r.v.'s with $E(a(k)^4 + b(k)^4 + \tau(k)^4) < \infty$ and $Eb(k) = Ea(k) \cdot E\tau(k)$. Note that

$$Z(k) = Y(k) - r\tau(k)$$
$$= a(k)\tau(k) - b(k)$$

and

$$\lambda = 2EZ(1)\,\tau(1)/E\tau(1)$$

$$= 2Ea(1)\cdot \operatorname{var}\tau(1)/E\tau(1).$$

Choose the distributions of $a(k)$, $\tau(k)$ so that $a(k) > 0$, $\operatorname{var}\tau(k) > 0$; consequently, $\lambda > 0$. For the distribution of $b(k)$, let $0 < \epsilon < 1$ and set

$$P\{b(k) = E\alpha(1)\cdot E\tau(1) + \epsilon^{3/5}(1-\epsilon)^{-1}\} = 1 - \epsilon,$$

$$P\{b(k) = E\alpha(1)\cdot E\tau(1) - \epsilon^{-2/5}\} = \epsilon.$$

Note that

$$Eb(1) = E\alpha(1)\cdot E\tau(1)$$

$$E(b(1) - Eb(1))^2 = \epsilon^{6/5}(1-\epsilon)^{-1} + \epsilon^{1/5}$$

$$E(b(1) - Eb(1))^3 = \epsilon^{9/5}(1-\epsilon)^{-2} - \epsilon^{-1/5}.$$

By letting $\epsilon \downarrow 0$, $E(b(1)-Eb(1))^2 \to 0$ and $E(b(1)-Eb(1))^3 \to -\infty$. Thus, choosing ϵ sufficiently small yields a $Z(1)$ for which $EZ(1)^3$ is so large that the term $-2\lambda EZ^3(1)$ dominates everything else in the expression for Δ; this leads, of course, to a negative Δ.

4. Some numerical examples

In this section, we present some preliminary numerical examples which are intended to illustrate several features of the above results. All of these examples are continuous time Markov chains, in fact birth-death processes. The first example is the queue-length process $\{X(t): t \geqslant 0\}$, for an $M/M/5$ queue (Poisson arrivals, exponential service, and 5 servers) with truncated state-space $E = \{0, 1, 2, \ldots, 14\}$. Thus, the system can accommodate at most 14 customers, 5 in service and 9 in queue. Since the state-space is finite and irreducible, we know that $X(t) \Rightarrow X$ as $t \to \infty$. The arrival rate $\lambda = 10$ and the service rate $\mu = 4$. For this model the birth parameters are $\lambda_i = 10$ and the death parameters are $\mu_i = 4(i \wedge 5)$. Table 4.1 contains values associated with estimating the expected stationary queue-length, that is, $r = EX = 2.6287$. Values are given for three return states: 0, 5, and 10. Elements of the two covariance matrices are denoted as follows: $D = \|d(i, j)\|$ and $D_\mu = \|d_\mu(i, j)\|$. All moments are computed using formulae from Glynn and Iglehart [4].

Table 4.1
M/M/s queue, $r = EX$

Parameter	Return state 0	5	10
$E\{\tau(1)\}$	1.24828	0.51130	16.36170
$d(1, 1)$	2.18920	2.18920	2.18919
$d(1, 2)$	2.84292	2.84292	2.84290
$d(2, 2)$	68.78821	30.18836	1432033.12500
$d_\mu(1, 1)$	2.18920	2.18920	2.18919
$d_\mu(1, 2)$	5.60163	−0.02955	−13.44083
$d_\mu(2, 2)$	24.76746	10.04333	190.10800

A number of comments are in order with respect to table 4.1. Return state 5 yields the shortest expected cycle length of the three return states used. The $d(1, 1)$ and $d_\mu(1, 1)$ elements are invariant (except for round-off errors) with respect to return state. This is expected since $d(1, 1) = EZ^2(1)/E\tau(1)$, the invariance constant in the CLT (3.1). What is unexpected is that $d(1, 2)$ also appears to be independent of return state. This invariance appeared in all the models and f values we considered. Additional theoretical work is being pursued to establish our conjecture that $d(1, 2)$ is independent of return state. Observe that the variability of $d(2, 2)$ and $d_\mu(2, 2)$ over return states is enormous. Also, $d_\mu(2, 2)$ is much smaller than $d(2, 2)$. For this example, $d(2, 2)$ and $d_\mu(2, 2)$ are both minimal for return state 5, the one with the shortest expected cycle length. This is not true in general, as a counter-example given below indicates.

Our next example is the truncated $M/M/\infty$ queue in which $X(t)$ is the number of jobs in the system at time t. Again, $E = \{0, 1, \ldots, 14\}$ and $\lambda = 35$, $\mu = 5$. So the system can accommodate at most 14 customers, all of whom will be in service. The quantity estimated here is $r = P\{X \leqslant 10\} = 0.90666$. Numerical values are given in table 4.2. Similar comments are relevant for table 4.2 as were for table 4.1. In addition, note that $d(2, 2)$ is not minimal for state 5, the return state with the shortest expected cycle length.

Our last example is the classical repairman problem with $n = 10$ machines, $m = 4$ spares, machine failure rate $\lambda = 1$, service rate $\mu = 4$, and $s = 3$ servers. For this model, the birth-death process $X(t)$ denotes the number of machines down at time t.

Table 4.2

$M/M/\infty$ queue, $r = P\{X \leqslant 10\}$

Parameter	Return state 0	5	10
$E\{\tau(1)\}$	31.15363	0.12975	0.16479
$d(1,1)$	0.01702	0.01702	0.01702
$d(1,2)$	−0.01612	−0.01612	−0.01612
$d(2,2)$	5436.38379	0.03313	0.03253
$d_\mu(1,1)$	0.01702	0.01702	0.01702
$d_\mu(1,2)$	−0.03234	−0.02532	−0.00122
$d_\mu(2,2)$	0.90090	0.05396	0.01007

Table 4.3

Repairman problem, $r = P\{X \leqslant 2\}$

Parameter	Return state 0	5	10
$E\{\tau(1)\}$	1.64168	0.43228	9.33730
$d(1,1)$	0.17208	0.17208	0.17208
$d(1,2)$	0.01104	0.01104	0.01104
$d(2,2)$	16.98646	0.97848	11340.19727
$d_\mu(1,1)$	0.17208	0.17208	0.17208
$d_\mu(1,2)$	−0.32921	0.21195	0.59497
$d_\mu(2,2)$	1.14822	0.50353	7.05369

The parameter to be estimated in this example is $P\{X \leqslant 2\} = 0.4035$. Table 4.3 contains values of the relevant quantities. Again we see that $d(1, 1)$ and $d(1, 2)$ are independent of return state and that $d(2, 2)$ varies enormously for different return states. Finally, note that the elements $d_\mu(2, 2)$ are much smaller than $d(2, 2)$.

In conclusion, we list the main lessons learned from these three numerical examples.

1. The variance of the point estimate of r, $d(1, 1)$, is independent of the return state. As mentioned above, this fact was previously established.

2. The covariance between the point estimates of r and σ, $d(1, 2)$, appears to also be independent of the return state. This fact came as a surprise and deserves further study.

3. The ratio $d(2, 2)/d(1, 1)$ is enormous. This large variability in the point estimate of σ may contribute to the sometimes observed under-coverage of regenerative confidence intervals.

4. In general, the variance of the point estimate of σ with r unknown, $d(2, 2)$, is much larger than the corresponding estimate when r is assumed known, $d_{\mu}(2, 2)$. Recall that a counter-example in sect. 3 shows that the reverse can hold.

5. While the variance of the estimate of σ, $d(2, 2)$, is not always minimized by taking the return state with the shortest expected cycle length, using this return state appears to be an excellent procedure in practice.

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