

UPPER BOUNDS ON POISSON TAIL PROBABILITIES

Peter W. GLYNN

Mathematics Research Center, University of Wisconsin-Madison, WI 53706, USA

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Upper bounds on the left and right tails of the Poisson distribution are given. These bounds can be easily computed in a numerically stable way, even when the Poisson parameter is large. Such bounds can be applied to variate generation schemes and to numerical algorithms for computing terminal rewards of uniformizable continuous-time Markov chains.

bounds * Poisson distribution

1. Introduction

Let $p(\lambda, k) = \exp(-\lambda) \cdot \lambda^k / k!$ ($k = 0, 1, \dots$) be the Poisson mass function. Our goal in this paper is to provide upper bounds on the left and right tail probabilities, which are defined, respectively, by the formulae

$$P(\lambda, n) = \sum_{k=0}^n p(\lambda, k),$$

$$\bar{P}(\lambda, n) = \sum_{k=n}^{\infty} p(\lambda, k).$$

Such bounds have application in several numerical methods arising from the analysis of stochastic systems.

Application 1. To generate variates from the clipped Poisson distribution, it is common to compute a table of the Poisson distribution function (see Bratley, Fox and Schrage [2, pp. 170-171, 334-335]). The numerical computation of the table requires truncating the Poisson tail, thereby introducing numerical error. In order to bound the numerical error, it is necessary to have a priori bounds on the probability mass of the truncated tails (see Fox and Glynn [5]).

Application 2. Let $X = \{X(t): t \geq 0\}$ be a uniformizable continuous-time Markov chain with generator Q . Given that f is a real-valued function defined on the state space S of X , one is often interested in determining numerically the

terminal reward $r = Ef(X(T))$, where T is deterministic. The parameter r can be computed as

$$r = \sum_{k=0}^{\infty} e^{-\alpha T} \frac{(\alpha T)^k}{k!} Ef(Y(k)), \tag{1}$$

for $\alpha \geq \Lambda = \sup\{-Q_{xx}: x \in S\}$, where $Y = \{Y(k): k \geq 0\}$ is an appropriately defined discrete-time Markov chain living on S . Gross and Miller [6] have suggested numerical algorithms based on the representation (1). Of course, from a numerical standpoint, it is necessary to truncate the infinite sum appearing in (1) at some finite quantity, say m . The absolute error introduced by truncating (1) at m is given by

$$\epsilon(\alpha, f) = \left| \sum_{k=m+1}^{\infty} e^{-\alpha T} \frac{(\alpha T)^k}{k!} Ef(Y(k)) \right|.$$

Let $\|f\| = \sup |f(x)|: x \in S$. By observing that $|Ef(Y(k))| \leq \|f\|$ for $k \geq 0$, it follows that $\epsilon(\alpha, f) \leq \|f\| \cdot \bar{P}(\alpha T, m+1)$; thus, for bounded functions, an explicit a priori error bound can be calculated, provided that $\bar{P}(\alpha T, m+1)$ can be bounded.

Incidentally, by letting $f(\cdot) = K$, one finds that $\sup\{\epsilon(\alpha, f): \|f\| \leq K\} = K \cdot \bar{P}(\alpha T, m+1)$;

this suggests that one should choose α so as to minimize $\bar{P}(\alpha T, m+1)$. Recall that $\bar{P}(\alpha T, m+1) = P\{N(\alpha T) > m\}$, where $N(\cdot)$ is a unit intensity Poisson process. Since $N(\cdot)$ has non-decreasing

paths, $\bar{P}(\cdot, m + 1)$ must be non-decreasing in α . Hence, the best choice of α for minimizing $\bar{P}(\alpha T, m + 1)$ is $\alpha = \Lambda$.

Application 3. The computation of the exponential of a matrix A arises (implicitly) whenever it is necessary to solve a system of linear differential equations. Certain numerical methods associated with the computation of the matrix exponential require truncation of the exponential power series. If $\|B\| = \max_j \{\sum_i |B_{ij}|\}$, then it is easy to show that

$$\left\| \exp(A) - \sum_{j=0}^{n-1} A^j/j! \right\| \leq \exp(\|A\|) \cdot \bar{P}(\|A\|, n).$$

Poisson tail bounds are therefore useful in bounding truncation error in this general numerical setting.

In the above applications, an upper bound on the tails is used to determine $n_1(\epsilon)$, $n_2(\epsilon)$ for which $P\{n_1(\epsilon) \leq N(\lambda) \leq n_2(\epsilon)\} \geq 1 - \epsilon$, where ϵ is a prescribed error tolerance. (In Application 2, one would generally take $n_1(\epsilon) = 0$.) One can argue that a straightforward method exists for choosing such an $(n_1(\epsilon), n_2(\epsilon))$ pair. Choose $n_1(\epsilon) = 0$ and let $n_2(\epsilon)$ be the first m for which $P(\lambda, m) \geq 1 - \epsilon$; the latter operation can be done numerically by successively adding the mass probabilities $p(\lambda, k)$. This method has two disadvantages. First, the computation of $n_2(\epsilon)$ requires programming set-up time and expense. Secondly, for large λ , the computation of the mass probabilities $p(\lambda, k)$ involves significant numerical underflow-overflow problems; overcoming these difficulties is non-trivial (see Fox and Glynn [5]).

On the other hand, for large λ , the central limit theorem (CLT) applies to yield

$$P\{\lambda + z_1\lambda^{1/2} \leq N(\lambda) \leq \lambda + z_2\lambda^{1/2}\} \rightarrow \Phi(z_2) - \Phi(z_1) \tag{2}$$

as $\lambda \rightarrow \infty$, where $\Phi(x) = \int_{-\infty}^x \phi(t) dt$ and $\phi(x) = (2\pi)^{-1/2} \exp(-x^2/2)$. The problem in applying (2) (together with a table of the normal distribution function) to obtain $n_1(\epsilon)$ and $n_2(\epsilon)$ is that (2) is only true in the limit. For finite λ , (2) gives no usable information. An obvious refinement would be to combine (2) with the Berry-Esseen theorem (see Feller [4, p. 542]). This, however, leads nowhere since the Berry-Esseen error bound is independent of z_1 and z_2 and consequently gives

no usable upper bound on the Poisson tail probabilities.

The basic difficulty in applying the normal approximation (or variants thereof) to determination of upper bounds on the tail of the Poisson is that the tail of the Poisson distribution is much fatter than that of the approximating normal. Specifically,

$$\lim_{n \rightarrow \infty} \frac{\bar{P}(\lambda, n)}{\bar{\Phi}((n - \lambda)/\lambda^{1/2})} = \infty, \tag{3}$$

where $\bar{\Phi}(x) = 1 - \Phi(x)$. See the discussion following Corollary 1 for further elaboration on this point.

Our error bounds are phrased in terms of the normal tail probabilities Φ and $\bar{\Phi}$. Thus, one can view the bounds here as giving information on the range of n , relative to λ , over which the normal approximation can be 'corrected' to give suitable bounds on the Poisson.

2. Results

Our first result bounds the tail probabilities in terms of the mass function.

Proposition 1. Assume $\lambda > 0$. (i) If $0 \leq n < \lambda$, then

$$P(\lambda, n) \leq p(\lambda, n) \cdot (1 - (n/\lambda))^{-1}.$$

(ii) If $n > \lambda - 1$ and $m \geq 1$, then

$$\bar{P}(\lambda, n) \leq \left(1 - \left(\frac{\lambda}{n+1}\right)^m\right)^{-1} \cdot \sum_{k=n}^{n+m-1} p(\lambda, k).$$

The following corollary is immediate.

Corollary 1. (i) $\lim_{\lambda \rightarrow \infty} P(\lambda, n)/p(\lambda, n) = 1$,
 (ii) $\lim_{n \rightarrow \infty} \bar{P}(\lambda, n)/p(\lambda, n) = 1$.

According to the second part of our corollary, virtually all the mass of the Poisson right tail $\bar{P}(\lambda, n)$ sits at the point n . Corollary 1 (ii) can also be used to obtain an asymptotic relationship for the quality of the normal tail approximation for the Poisson distribution. Recall, from (2), that the CLT suggests approximating $\bar{P}(\lambda, n)$ by $\bar{\Phi}((n - \lambda)/\lambda^{1/2})$. Now, by Corollary 1 (ii) and a standard tail asymptotic for the normal (see Feller [3, p.

175)),

$$\frac{\bar{P}(\lambda, n)}{\bar{\Phi}((n-\lambda)/\lambda^{\frac{1}{2}})} - \frac{np(\lambda, n)}{\lambda^{\frac{1}{2}}\phi((n-\lambda)/\lambda^{\frac{1}{2}})}$$

as $n \rightarrow \infty$. By Stirling's formula [3, p. 52], we obtain

$$\frac{\bar{P}(\lambda, n)}{\bar{\Phi}((n-\lambda)/\lambda^{\frac{1}{2}})} \sim \exp\left[(n-\lambda)^2/2\lambda - (n+\frac{1}{2})\ln(n) + n \ln(\lambda e)\right] \cdot e^{-\lambda}/\lambda^{\frac{1}{2}}$$

as $n \rightarrow \infty$; this immediately yields (3). Note that the exponential appearing on the right-hand side converges to infinity (very) rapidly as $n \rightarrow \infty$, indicating that one needs to take great care in applying the normal approximation to the Poisson tail.

A further consequence of Proposition 1 (ii) (with $m = 1$) is that for $\alpha > 0$,

$$1 \leq \frac{\bar{P}(\lambda, \lfloor \lambda(1+\alpha) \rfloor)}{p(\lambda, \lfloor \lambda(1+\alpha) \rfloor)} \leq 1 + 1/\alpha, \tag{4}$$

which implies that

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \frac{1}{\lambda} \ln \bar{P}(\lambda, \lfloor \lambda(1+\alpha) \rfloor) &= \lim_{\lambda \rightarrow \infty} \frac{1}{\lambda} \ln p(\lambda, \lfloor \lambda(1+\alpha) \rfloor) \\ &= \alpha - (1+\alpha) \ln(1+\alpha); \end{aligned} \tag{5}$$

(5) is a statement of Chernoff's large deviations theorem (see Bahadur [1, pp. 6-9]) specialized to the Poisson distribution. Thus, (4) may be viewed as a refinement of Chernoff's result.

The bounds given by Proposition 1 can be used directly when λ is small, since $p(\lambda, k)$ can then be evaluated without numerical difficulties arising. If λ is large, it is useful to have error bounds on the mass function itself. Let $a = \lfloor \lambda \rfloor$.

Proposition 2. Let $\lambda, n \geq 1$. Then, (i) $p(\lambda, a-n) \leq (2\pi a)^{-\frac{1}{2}} \exp(-n(n-1)/2\lambda)$,

(ii) $p(\lambda, a+n) \leq (2\pi a)^{-\frac{1}{2}} \exp(-n(n-1)/2\lambda + (n-1)n(2n-1)/12\lambda^2)$.

The normal approximation (2) indicates that the choices $n_1(\epsilon), n_2(\epsilon)$ will be of order $\lambda^{\frac{1}{2}}$ from λ . Thus, the values of n appearing in Proposition 1, which are of most concern, are those within

order $\lambda^{\frac{1}{2}}$ from λ . Consequently, if λ is large, the factors n/λ and $\lambda/(n+1)$ appearing in Proposition 1 (i) and (ii), respectively, will be close to one, causing the bounds to blow up. Thus, more refined bounds are needed when λ is large.

Theorem 1. Suppose $\lambda \geq 2$. Then, (i) if $n \geq 2$,

$$P(\lambda, a-n) \leq \exp(1/8\lambda) \cdot \left(1 + \frac{1}{\lambda}\right) \times \bar{\Phi}\left(\frac{n-3/2}{\lambda^{\frac{1}{2}}}\right),$$

(ii) if $2 \leq n \leq (\lambda+3)/2$,

$$\begin{aligned} \bar{P}(\lambda, a+n) &\leq \exp(1/8\lambda) \cdot \left(1 + \frac{1}{\lambda}\right) \cdot \sqrt{2} \\ &\times \left(1 - \left(\frac{\lambda}{a+n+1}\right)^a\right)^{-1} \\ &\times \bar{\Phi}\left(\frac{n-3/2}{(2\lambda)^{\frac{1}{2}}}\right). \end{aligned}$$

An obvious limitation of Theorem 1 (ii) is the restriction on the size of n . However, since one expects to use these bounds only for large λ (in which case n will tend to be of order $\lambda^{\frac{1}{2}}$), this restriction is not serious. To avoid the computation of the numerically hard-to-compute reciprocal factor in Theorem 1 (ii), we also offer the following bound.

Proposition 3. Suppose $\lambda \geq 2$. For $2 \leq n \leq (\lambda+3)/2$,

$$\begin{aligned} \bar{P}(\lambda, a+n) &\leq \exp(1/16) \cdot \left(1 + \frac{1}{\lambda}\right) \cdot \sqrt{2} \\ &\times \left(1 - \exp\left(-\frac{2n}{9}\right)\right)^{-1} \\ &\times \bar{\Phi}\left(\frac{n-3/2}{(2\lambda)^{\frac{1}{2}}}\right). \end{aligned}$$

Note that all the terms appearing in the upper bounds of Theorem 1 (i) and Proposition 3 can be easily computed in a numerically stable fashion. These bounds should therefore be suitable for the applications previously described.

As for the quality of the bounds, some sharpness was sacrificed in the derivation in order to obtain bounds that are easily computable. In con-

nection with the sharpness issue, lower bounds on $p(\lambda, n)$ (see [5]) in terms of the normal density show that lower bounds in terms of the normal tail are also possible. Thus, the Poisson tail (for values of n indicated here) can essentially be bounded above and below by normal tails. This suggests that the bounds obtained here describe (modulo some terms that grow 'slowly' in n) the dominant behavior of the Poisson tail.

3. Proofs

Proof of Proposition 1. Observe that

$$\begin{aligned}
 P(\lambda, n) &= p(\lambda, n) \cdot \left(1 + \sum_{k=0}^{n-1} \frac{n(n-1) \cdots (k+1)}{\lambda^{n-k}} \right) \\
 &\leq p(\lambda, n) \cdot \left(1 + \sum_{k=0}^{n-1} \left(\frac{n}{\lambda} \right)^{n-k} \right) \\
 &\leq p(\lambda, n) \cdot \sum_{k=0}^{\infty} \left(\frac{n}{\lambda} \right)^k = p(\lambda, n) \cdot \left(1 - \frac{n}{\lambda} \right)^{-1}.
 \end{aligned}$$

For $\bar{P}(n)$, it is evident that

$$\begin{aligned}
 \bar{P}(\lambda, n) &= \sum_{k=n}^{n+m-1} p(\lambda, k) + \lambda^m \cdot \sum_{k=n}^{\infty} p(\lambda, k) \\
 &\quad \times \frac{k!}{(k+m)!} \\
 &\leq \sum_{k=n}^{n+m-1} p(\lambda, k) + \lambda^m \cdot \sum_{k=n}^{\infty} p(\lambda, k) \\
 &\quad \times (n+1)^{-m} \\
 &= \sum_{k=n}^{n+m-1} p(\lambda, k) + \left(\frac{\lambda}{n+1} \right)^m \bar{P}(\lambda, n).
 \end{aligned}$$

Solving for $\bar{P}(\lambda, n)$ yields Proposition 1 (ii).

The following lemma collects a series of inequalities which we shall need for the remainder of our proofs.

- Lemma 1.** (i) For $y > -1$, $\ln(1+y) \leq y$,
 (ii) for $y \geq 0$, $-\ln(1+y) \leq -y + y^2/2$,
 (iii) for $y \geq 0$, $(1+y)^{1/2} \leq 1 + y/2$.

Proof. For (i), let $g(y) = y - \ln(1+y)$ and observe that $g(0) = 0$ and $g'(y) < 0$ for $-1 < y < 0$, whereas $g'(y) > 0$ for $y > 0$. Hence, 0 is a global

minimizer of $g(\cdot)$, so that $g(y) \geq g(0) = 0$ for $y > -1$. The proofs of (ii) and (iii) are similar.

Proof of Proposition 2. For $1 \leq n \leq a$, we use the fact that $a \leq \lambda$ to obtain

$$\begin{aligned}
 p(\lambda, a-n) &= p(\lambda, a) \cdot \left(\frac{a}{\lambda} \right) \left(\frac{a-1}{\lambda} \right) \cdots \left(\frac{a-n+1}{\lambda} \right) \\
 &\leq p(\lambda, a) \cdot (1) \cdot \left(1 - \frac{1}{\lambda} \right) \cdots \left(1 - \frac{(n-1)}{\lambda} \right) \\
 &= p(\lambda, a) \cdot \exp \left(\sum_{k=0}^{n-1} \ln \left(1 - \frac{k}{\lambda} \right) \right).
 \end{aligned}$$

By Lemma 1 (i), $\ln(1 - k/\lambda) \leq -k/\lambda$, so

$$\begin{aligned}
 p(\lambda, a-n) &\leq p(a) \cdot \exp \left(- \sum_{k=0}^{n-1} \frac{k}{\lambda} \right) \\
 &= p(\lambda, a) \cdot \exp(-n(n-1)/2\lambda),
 \end{aligned}$$

by a standard summation formula (see Knuth [7, p. 55]). To bound $p(\lambda, a)$, we use a Stirling formula-type inequality (see Feller [3, p. 54]),

$$a! > (2\pi a)^{1/2} a^a \cdot e^{-a} \cdot \exp(1/(12a+1)),$$

which yields $(b = \lambda - a)$

$$\begin{aligned}
 p(\lambda, a) &\leq \frac{e^{-\lambda} \cdot \lambda^a}{\sqrt{2\pi a} a^a \cdot e^{-a}} \\
 &= \frac{1}{\sqrt{2\pi a}} e^{-b} \left(1 + \frac{b}{a} \right)^a \\
 &\leq \frac{1}{\sqrt{2\pi a}} e^{-b} \cdot e^b = (2\pi a)^{-1/2}; \tag{6}
 \end{aligned}$$

the last inequality is obtained by exponentiating both sides of $\ln(1 + b/a) \leq b/a$ (see Lemma 1 (i)). This proves Proposition 2 (i) for $n \leq a$; for $n > a$, the inequality is trite. As for $p(\lambda, a+n)$, use $a \geq \lambda - 1$ to obtain

$$\begin{aligned}
 p(\lambda, a+n) &= p(\lambda, a) \cdot \frac{\lambda^n}{(a+1)(a+2) \cdots (a+n)} \\
 &\leq p(\lambda, a) \cdot \frac{\lambda^n}{\lambda(\lambda+1) \cdots (\lambda+n-1)} \\
 &= p(\lambda, a) \cdot \exp \left(- \sum_{k=0}^{n-1} \ln \left(1 + \frac{k}{\lambda} \right) \right) \\
 &\leq p(\lambda, a) \cdot \exp \left(- \sum_{k=0}^{n-1} \frac{k}{\lambda} + \sum_{k=0}^{n-1} \frac{k^2}{2\lambda^2} \right),
 \end{aligned}$$

the latter inequality by Lemma 1 (ii). Using standard summation formulae (see Knuth [7, p. 55]) and (6) gives Proposition 2 (ii).

Proof of Theorem 1. By Proposition 2 (i),

$$P(\lambda, a-n) \leq (2\pi a)^{-\frac{1}{2}} \cdot \sum_{k=n}^a \exp\left(-\frac{k(k-1)}{2\lambda}\right).$$

Since $g(x) = -x(x-1)/2\lambda$ is non-increasing on $[1/2, \infty)$,

$$\begin{aligned} &\exp(-x(x-1)/2\lambda) \\ &\leq \int_{x-1}^x \exp(-u(u-1)/2\lambda) du \end{aligned}$$

for $x \geq 3/2$. Thus, if $n \geq 2$,

$$\begin{aligned} &P(\lambda, a-n) \\ &\leq (2\pi a)^{-\frac{1}{2}} \exp(1/8\lambda) \\ &\quad \times \int_{n-1}^{\infty} \exp(-(u-1/2)^2/2\lambda) du \\ &\leq \exp(1/8\lambda) (\lambda/a)^{\frac{1}{2}} \bar{\Phi}((n-3/2)/\lambda^{\frac{1}{2}}). \end{aligned} \quad (7)$$

By Lemma 1 (iii), $(\lambda/a)^{\frac{1}{2}} \leq 1 + b/2a$. For $\lambda \geq 2$, $\lambda \leq |\lambda| + 1 \leq |\lambda| + 2 - 2/\lambda = |\lambda| + (2/\lambda) \cdot (\lambda - 1) \leq |\lambda| + (2/\lambda) \cdot |\lambda|$ so that $b/2a \leq 1/\lambda$; substituting into (7) yields Theorem 1 (i).

For Theorem 1 (ii), we first use Proposition 1 (ii) with $m = a$ to obtain

$$\begin{aligned} \bar{P}(\lambda, a+n) &\leq \left(\left(1 - \frac{\lambda}{a+n+1} \right)^a \right)^{-1} \\ &\quad \times \sum_{k=n}^{n+a-1} p(\lambda, k). \end{aligned}$$

For $n \leq k \leq n+a-1$, we have

$$\begin{aligned} &-\frac{k(k-1)}{2\lambda} + \frac{(k-1)k(2k-1)}{12\lambda^2} \\ &\leq -\frac{k(k-1)}{2\lambda} \cdot \beta, \end{aligned}$$

where $\beta = 1 - (n+a)/(3\lambda) + (2\lambda)^{-1}$. By Proposition 2 (ii), it follows that

$$\begin{aligned} \bar{P}(\lambda, a+n) &\leq \left(1 - \left(\frac{\lambda}{a+n+1} \right)^a \right)^{-1} \cdot (2\pi a)^{-\frac{1}{2}} \\ &\quad \times \sum_{k=n}^{n+a-1} \exp(-k(k-1)\beta/2\lambda). \end{aligned} \quad (8)$$

As in the bound for $P(\lambda, a-n)$, the latter sum is

dominated by

$$\begin{aligned} &\int_{n-1}^{\infty} \exp(-u(u-1)\beta/2\lambda) du \\ &= \exp(\beta/8\lambda) \cdot \sqrt{\frac{\lambda}{\beta}} \cdot (2\pi)^{\frac{1}{2}} \\ &\quad \times \bar{\Phi}\left(\frac{(n-3)}{2} \sqrt{\frac{\beta}{\lambda}}\right). \end{aligned} \quad (9)$$

Now, $\beta \leq 1$ since $n+a \geq n \geq 2$; furthermore, since $n \leq (\lambda+3)/2$, it follows that $\beta \geq 1 - (\lambda/2 + 3/2 + \lambda)/3\lambda + (2\lambda)^{-1} = 1/2$. Hence, $\beta^{-\frac{1}{2}} \leq \sqrt{2}$, $\exp(\beta/8\lambda) \leq \exp(1/8\lambda)$, and

$$\bar{\Phi}\left(\frac{(n-3)}{2} \sqrt{\frac{\beta}{\lambda}}\right) \leq \bar{\Phi}\left(\frac{(n-3)}{2} \cdot (2\lambda)^{-\frac{1}{2}}\right).$$

Combining these inequalities and the previously obtained $(\lambda/a)^{\frac{1}{2}} \leq (1 + 1/\lambda)$ with (8) and (9), we get Theorem 1 (ii).

Proof of Proposition 3. For $\lambda \geq 2$, $\exp(1/8\lambda) \leq \exp(1/16)$, so it remains only to show that

$$\left(1 - \left(\frac{\lambda}{a+n+1} \right)^a \right)^{-1} \leq \left(1 - \exp\left(\frac{-2n}{a}\right) \right)^{-1}. \quad (10)$$

Since $a+1 \geq \lambda$, it follows that $\lambda/(a+n+1) \leq 1 - (n/(\lambda+n))$. Now,

$$1 - \frac{n}{\lambda+n} \leq \exp(-n/(\lambda+n))$$

(exponentiate both sides of Lemma 1 (i)), so

$$\left(\frac{\lambda}{a+n+1} \right)^a \leq \exp\left(\frac{-na}{\lambda+n}\right). \quad (11)$$

The function $f(x) = (x-1)/(x+1)$ is non-decreasing on $[0, \infty)$ so $f(x) \geq f(2) = 1/3$ for $x \geq 2$. Thus, for $\lambda \geq 2$, $(\lambda-1)/(\lambda+1) \geq 1/3$, proving that $a \geq (1/3)(\lambda+1)$. Hence, $a(\lambda+n)^{-1} \geq a(\lambda + \lambda/2 + 3/2)^{-1} \geq (\lambda+1)/(3 \cdot (\lambda+1) \cdot 3/2) = 2/9$. Relation (11) then yields

$$\left(\frac{\lambda}{a+n+1} \right)^a \leq \exp(-2n/9),$$

from which (10) follows immediately.

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